THERMODYNAMICS

TUTORIAL 7

COMPRESSIBLE FLOW

On completion of this tutorial you should be able to do the following.

- Define entropy
- Derive expressions for entropy changes in fluids
- Derive Bernoulli's equation for gas
- Derive equations for compressible ISENTROPIC flow
- Solve problems involving compressible flow

Note that more work on compressible flow may be found under ***FLUID MECHANICS.***

Let's start by revising entropy.

1. ENTROPY

1.1 DEFINITION

You should already be familiar with the theory of work laws in closed systems. You should know that the area under a pressure-volume diagram for a reversible expansion or compression gives the work done during the process.

In thermodynamics there are two forms of energy transfer, work (W) and heat (Q). By analogy to work, there should be a property which if plotted against temperature, then the area under the graph would give the heat transfer. This property is entropy and it is given the symbol S. Consider a p-V and T-s graph for a reversible expansion.

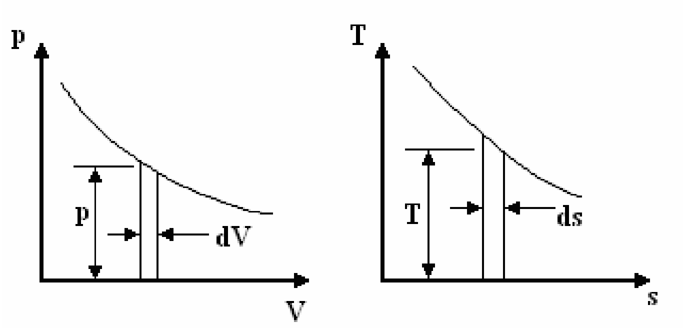

Figure 1

From the p-V graph we have $W = \int pdV$

From the T-S graph we have $Q = \int TdS$

This is the way entropy was developed for thermodynamics and from the above we get the definition

$$dS = dQ/T$$

The units of entropy are hence J/K.

Specific entropy has a symbol s and the units are J/kg K

It should be pointed out that there are other definitions of entropy but this one is the most meaningful for thermodynamics. A suitable integration will enable you to solve the entropy change for a fluid process.

2. <u>ISENTROPIC PROCESSES</u>

The word *Isentropic* means constant entropy and this is a very important thermodynamic process. It occurs in particular when a process is reversible and adiabatic. This means that there is no heat transfer to or from the fluid and no internal heat generation due to friction. In such a process it follows that if dQ is zero then dS must be zero. Since there is no area under the T-S graph, then the graph must be a vertical line as shown.

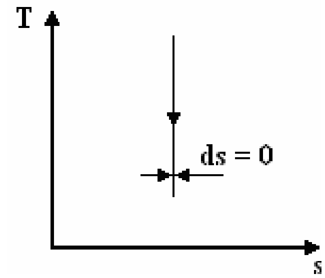

Figure 2

There are other cases where the entropy is constant. For example, if there is friction in the process generating heat but this is lost through cooling, then the nett result is zero heat transfer and constant entropy. You do not need to be concerned about this at this stage.

Entropy is used in the solution of gas and vapour problems. We should now look at practical applications of this property and study the entropy changes which occur in closed and steady flow systems for perfect gases and vapours. These derivations should be learned for the examination.

3. ENTROPY CHANGES FOR A PERFECT GAS IN A CLOSED SYSTEMS

Consider a closed system expansion of a fluid against a piston with heat and work transfer taking place.

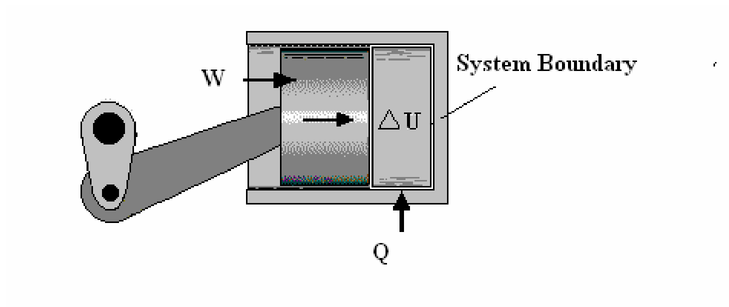

Figure 3

Applying the non-flow energy equation we have

$$Q + W = \Delta U$$

Differentiating we have $dQ + dW = dU$

Since dQ = TdS and dW = -pdV then $TdS - pdV = dU$

$$TdS = dU + pdV$$

This expression is the starting point for all derivations of entropy changes for any fluid (gas or vapour) in closed systems. It is normal to use specific properties so the equation becomes

$$Tds = du + pdv$$

but from the gas law pv = RT we may substitute for p and the equation becomes

$$Tds = du + RTdv/v$$

rearranging and substituting $du = c_V\, dT$ we have

$$ds = c_V\, dT/T + Rdv/v \text{...............(1)}$$

s is specific entropy

v is specific volume.

u is specific internal energy and later on is also used for velocity.

3.1 ISOTHERMAL PROCESS

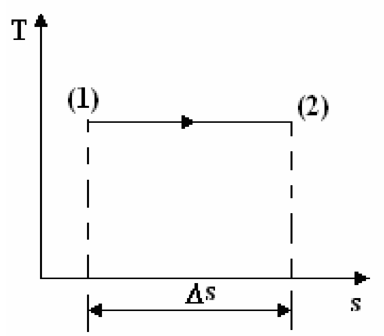

Figure 4

In this case temperature is constant. Starting with equation (1)

$$ds = c_V\, dT/T + R dv/v.$$

since dT = 0 then

$$s_2 - s_1 = \Delta s = R \ln(v_2/v_1)$$

A quicker alternative derivation for those familiar with the work laws is:

$Q + W = \Delta U$ but $\Delta U = 0$ then Q = - W and W $= -mRT \ln\dfrac{V_2}{V_1}$

$Q = \int T ds = T\Delta S$ but T is constant.

$$\Delta S = \frac{Q}{T} = -\frac{W}{T} = mR\ln\frac{V_2}{V_1}$$

$$\Delta S = mR\ln\frac{V_2}{V_1}$$

$\Delta s = R\ln\dfrac{v_2}{v_1}$ and since $\dfrac{v_2}{v_1} = \dfrac{p_1}{p_2}$

$$\Delta s = R\ln\frac{p_1}{p_2}$$

3.2 CONSTANT VOLUME PROCESS

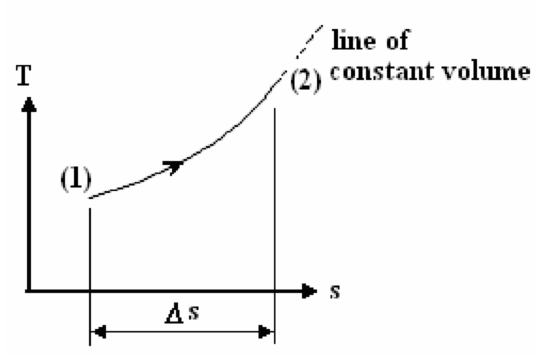

Figure 5

Starting again with equation (1) we have $ds = c_v dT/T + R dv/v$

In this case dv=0 so $ds = c_v dT/T$

Integrating between limits (1) and (2) $\Delta s = c_v \ln(T_2/T_1)$

3.3 CONSTANT PRESSURE PROCESS

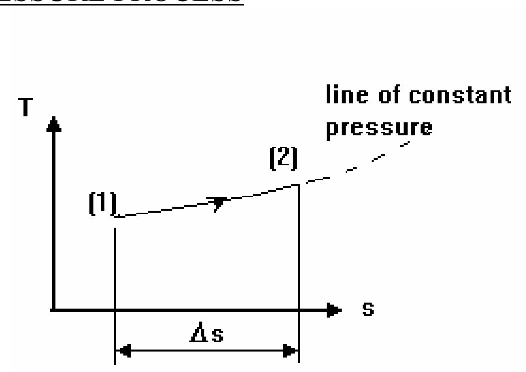

Figure 6

Starting again with equation (1) we have

$ds = C_v \dfrac{dT}{T} + R\dfrac{dv}{v}$ In this case we integrate and obtain

$\Delta s = C_v \ln\dfrac{T_2}{T_1} R \ln\dfrac{v_2}{v_1}$ For a constant pressure process, v/T = constant

$\dfrac{v_2}{v_1} = \dfrac{T_2}{T_1}$ so the expression becomes $\Delta s = C_v \ln\dfrac{T_2}{T_1} + R\ln\dfrac{T_2}{T_1} = (C_v + R)\ln\dfrac{T_2}{T_1}$

It was shown in an earlier tutorial that $R = c_p - c_v$ hence

$$\Delta s = C_p \ln\frac{T_2}{T_1}$$

3.4 POLYTROPIC PROCESS

This is the most difficult of all the derivations here. Since all the forgoing are particular examples of the polytropic process then the resulting formula should apply to them also.

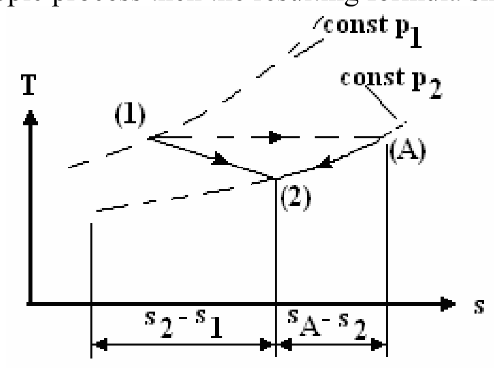

Figure 7

The polytropic expansion is from (1) to (2) on the T-s diagram with different pressures, volumes and temperatures at the two points. The derivation is done in two stages by supposing the change takes place first at constant temperature from (1) to (A) and then at constant pressure from (A) to (2). You could use a constant volume process instead of constant pressure if you wish.

$s_2-s_1 = (s_A-s_1) - (s_A-s_2)$

$s_2-s_1 = (s_A-s_1) + (s_2-s_A)$

For the constant temperature process

$(s_A-s_1) = R\ln(p_1/p_A)$

For the constant pressure process

$(s_2-s_A) = (c_p)\ln(T_2/T_A)$

Hence

$$\Delta s = R\ln\frac{p_1}{p_A} + C_p\ln\frac{T_2}{T_A} + s_2\text{-}s_1 \quad \text{Since } p_A = p_2 \text{ and } T_A = T_1$$

Then

$$\Delta s = s_2\text{-}s_1 = R\ln\frac{p_1}{p_2} + C_p\ln\frac{T_2}{T_1} \quad \text{Divide through by R}$$

$$\frac{\Delta s}{R} = \ln\frac{p_1}{p_2} + \frac{C_p}{R}\ln\frac{T_2}{T_1}$$

From the relationship between c_p, c_v, R and γ we have $c_p/R = \gamma/(\gamma-1)$

Hence $$\frac{\Delta s}{R} = \ln\frac{p_1}{p_2} + \frac{\gamma}{\gamma-1}\ln\frac{T_2}{T_1} \qquad \frac{\Delta s}{R} = \ln\frac{p_1}{p_2}\left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma-1}}$$

This formula is for a polytropic process and should work for isothermal, constant pressure, constant volume and adiabatic processes also. In other words, it must be the derivation for the entropy change of a perfect gas for any closed system process. This derivation is often requested in the exam.

WORKED EXAMPLE No. 1

A perfect gas is expanded from 5 bar to 1 bar by the law $pV^{1.2} = C$. The initial temperature is 200°C. Calculate the change in specific entropy.
R = 287 J/kg K $\gamma = 1.4$.

SOLUTION

$$T_2 = 473\left(\frac{1}{5}\right)^{1-\frac{1}{1.2}} = 361.7\text{K}$$

$$\frac{\Delta s}{R} = \ln\left(\frac{p_1}{p_2}\right)\left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma-1}}$$

$$\frac{\Delta s}{R} = (\ln 5)\left(\frac{361.7}{472}\right)^{3.5} = 0.671$$

$$\Delta s = 0.671 \text{ x } 287 = 192.5 \text{ J/kgK}$$

SELF ASSESSMENT EXERCISE No. 1

1. Calculate the specific entropy change when a perfect gas undergoes a reversible isothermal expansion from 500 kPa to 100 kPa. R = 287 J/kg K.
(Answer +461.9 J/kg K).

2. Calculate the total entropy change when 2 kg of perfect gas is compressed reversibly and isothermally from 9 dm^3 to 1 dm^3. R=300 J/kg K. (Answer -1.32 kJ/k)

3. Calculate the change in entropy when 2.5 kg of perfect gas is heated from 20°C to 100°C at constant volume. Take c_v= 780 J/kg K (Answer 470 J/K)

4. Calculate the total entropy change when 5 kg of gas is expanded at constant pressure from 30°C to 200°C. R = 300 J/kg K c_v= 800 J/kg K (Answer 2.45 kJ/K)

5. Derive the formula for the specific change in entropy during a polytropic process using a constant volume process from (A) to (2).

6. A perfect gas is expanded from 5 bar to 1 bar by the law $pV^{1.6} = C$. The initial temperature is 200°C. Calculate the change in specific entropy.
R = 287 J/kg K $\gamma = 1.4$. (Answer -144 J/kg K)

7. A perfect gas is expanded reversibly and adiabatically from 5 bar to 1 bar by the law $pV^{\gamma} = C$. The initial temperature is 200°C. Calculate the change in specific entropy using the formula for a polytropic process. R = 287 J/kg K $\gamma = 1.4$. (The answer should be zero since the process is constant entropy).

Let's go on to apply the knowledge of entropy to the flow of compressible fluids starting with isentropic flow.

4. ISENTROPIC FLOW

Isentropic means constant entropy. In this case we will consider the flow to be ADIABATIC also, that is, with no heat transfer.

Consider gas flowing in a duct which varies in size. The pressure and temperature of the gas may change.

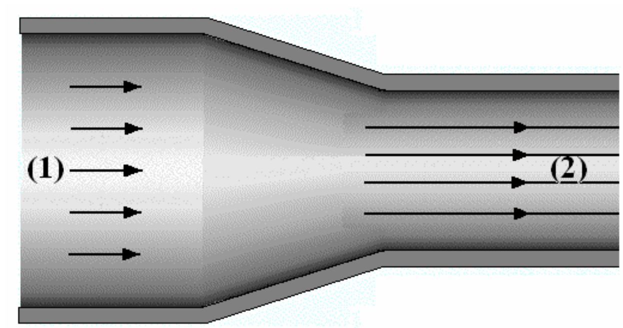

Figure 8

Applying the steady flow energy equation between (1) and (2) we have :

$$\Phi - P = \Delta U + \Delta F.E. + \Delta K.E. + \Delta P.E.$$

For Adiabatic Flow, $\Phi = 0$ and if no work is done then $P = 0$

$$\Delta U + \Delta F.E. = \Delta H$$

hence :

$$0 = \Delta H + \Delta K.E. + \Delta P.E.$$

In specific energy terms this becomes :

$$0 = \Delta h + \Delta k.e. + \Delta p.e.$$

rewriting we get:

$$h_1 + u_1^2/2 + g z_1 = h_2 + u_2^2/2 + g z_2$$

For a gas, $h = C_p T$ so we get Bernoulli's equation for gas which is :

$$C_p T_1 + u_1^2/2 + g z_1 = C_p T_2 + u_2^2/2 + g z_2$$

Note that T is absolute temperature in Kelvins $T = {}^oC + 273$

4.1 STAGNATION CONDITIONS

If a stream of gas is brought to rest, it is said to STAGNATE. This occurs on leading edges of any obstacle placed in the flow and in instruments such as a Pitot Tube. Consider such a case for horizontal flow in which P.E. may be neglected.

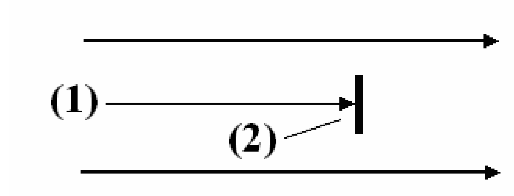

Figure 9

$u_2 = 0$ and $z_1 = z_2$ so $C_pT_1 + u_1^2/2 = C_pT_2 + 0$

$T_2 = u_1^2/2C_p + T_1$

T_2 is the stagnation temperature for this case.

$$\text{Let } T_2 - T_1 = \Delta T = u_1^2/2C_p$$

$$\Delta T = u_1^2/2C_p$$

Now $C_p - C_v = R$ and $C_p / C_v = \gamma$ γ is the adiabatic index .

hence $C_p = R / (\gamma - 1)$ and so :

$$\Delta T = u_1^2 (\gamma - 1) / (2\gamma R)$$

It can be shown elsewhere that the speed of sound a is given by :

$$a^2 = \gamma RT$$

hence at point 1:

$$\Delta T / T_1 = u_1^2 (\gamma - 1) / (2\gamma RT_1) = u_1^2 (\gamma - 1) / 2a_1^2$$

The ratio u/a is the Mach Number M_a so this may be written as :

$$\Delta T / T_1 = M_a^2 (\gamma - 1) / 2$$

If M_a is less than 0.2 then M_a^2 is less than 0.04 and so $\Delta T/T_1$ is less than 0.008. It follows that for low velocities, the rise in temperature is negligible under stagnation conditions.

The equation may be written as :

$$\frac{T_2 - T_1}{T_1} = \frac{M_a^2(\gamma - 1)}{2}$$

$$\frac{T_2}{T_1} = \left\{\frac{M_a^2(\gamma - 1)}{2}\right\} + 1$$

Since pV/T = constant and pV^γ = constant then :

$$\frac{T_2}{T_1} = \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}$$

Hence :

$$\left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}} = \frac{M_a^2(\gamma - 1)}{2} + 1$$

$$\left(\frac{p_2}{p_1}\right) = \left[\frac{M_a^2(\gamma - 1)}{2} + 1\right]^{\frac{\gamma}{\gamma-1}}$$

p_2 is the stagnation pressure. If we now expand the equation using the binomial theorem we get :

$$\frac{p_2}{p_1} = 1 + \frac{\gamma Ma^2}{2}\left\{1 + \frac{Ma^2}{4} + \frac{Ma^4}{8} + \ldots\ldots\ldots\right\}$$

If M_a is less than 0.4 then : $\frac{p_2}{p_1} = 1 + \frac{\gamma Ma^2}{2}$

Now compare the equations for gas and liquids :

LIQUIDS $u = (2\Delta p/\rho)^{0.5}$

GAS $\frac{p_2}{p_1} = 1 + \frac{\gamma Ma^2}{2}$

Put $p_2 = p_1 + \Delta p$ so : $\Delta p = \frac{\gamma Ma^2}{2} p_1 = \frac{\gamma v_1^2 p_1}{2\gamma RT} = \frac{\rho_1 u_1^2}{2}$

where $\rho_1 = p_1/RT$ and $M_a^2 = u_1^2/(\gamma RT)$

hence $u = (2\Delta p/\rho_1)^{0.5}$ which is the same as for liquids.

SELF ASSESSMENT EXERCISE No. 2

Take $\gamma = 1.4$ and R = 283 J/kg K in all the following questions.

1. An aeroplane flies at Mach 0.8 in air at 15^o C and 100 kPa pressure. Calculate the stagnation pressure and temperature. (Answers 324.9 K and 152.4 kPa)

2. Repeat problem 1 if the aeroplane flies at Mach 2. (Answers 518.4 K and 782.4 kPa)

3. The pressure on the leading edges of an aircraft is 4.52 kPa more than the surrounding atmosphere. The aeroplane flies at an altitude of 5 000 metres. Calculate the speed of the aeroplane.(Answer 109.186 m/s)

   Note from fluids tables, find that a = 320.5 m/s p_1 = 54.05 kPa γ = 1.4

4. An air compressor delivers air with a stagnation temperature 5 K above the ambient temperature. Determine the velocity of the air. (Answer 100.2 m/s)

Let's now extend the work to pitot tubes.

5. PITOT STATIC TUBE

A Pitot Static Tube is used to measure the velocity of a fluid. It is pointed into the stream and the differential pressure obtained gives the stagnation pressure.

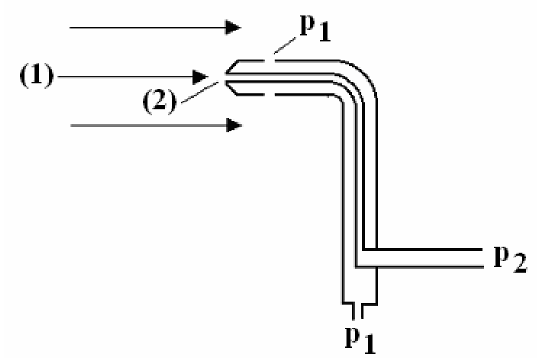

Figure 10

$p_2 = p_1 + \Delta p$

Using the formula in the last section, the velocity v may be found.

WORKED EXAMPLE No.2

A pitot tube is pointed into an air stream which has a pressure of 105 kPa. The differential pressure is 20 kPa and the air temperature is 20°C. Calculate the air speed.

SOLUTION

$p_2 = p_1 + \Delta p = 105 + 20 = 125$ kPa

$$\frac{p_2}{p_1} = \left[\left\{\frac{Ma^2(\gamma - 1)}{2}\right\} + 1\right]^{\frac{\gamma}{\gamma-1}}$$

$$\frac{125}{105} = \left[\left\{\frac{Ma^2(\gamma - 1)}{2}\right\} + 1\right] \text{hence } Ma = 0.634$$

$a = (\gamma RT)^{0.5} = (1.4 \times 287 \times 293)^{0.5} = 343$ m/s

$M_a = u/a$ hence u = 217.7 m/s

Let's further extend the work now to venturi meters and nozzles.

6. VENTURI METERS AND NOZZLES

Consider the diagrams below and apply Isentropic theory between the inlet and the throat.

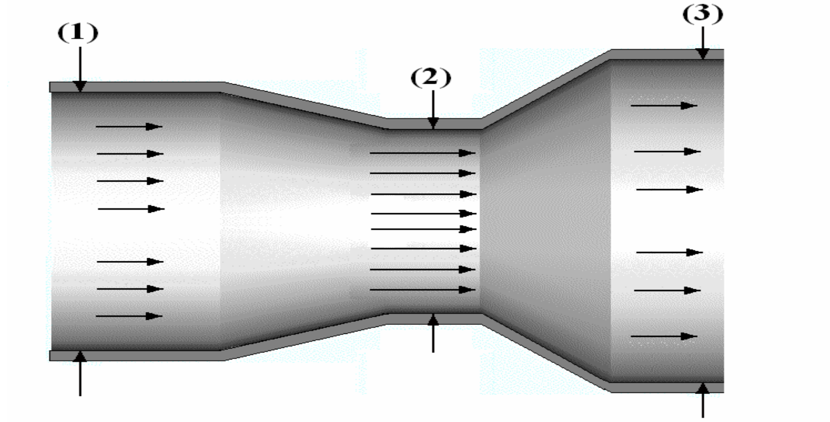

Figure 11

$u_2^2 - u_1^2 = h_1 - h_2$

If the Kinetic energy at inlet is ignored this gives us $u_2^2 = h_1 - h_2$

For a gas h = C_pT so: $u_2^2 = C_P[T_1 - T_2]$

Using $C_p = \gamma R/(\gamma-1)$ we get $u_2^2 = \frac{2\gamma R}{\gamma - 1}[T_1 - T_2]$

RT = pV/m = p/ρ so

$$u_2^2 = \frac{2\gamma}{\gamma-1}\left[\frac{p_1}{\rho_1} - \frac{p_2}{\rho_2}\right]$$

$$p_1V_1^\gamma = p_2V_2^\gamma \text{ so it follows that } \frac{p_1}{\rho_1^\gamma} = \frac{p_2}{\rho_2^\gamma}$$

$$u_2^2 = \frac{2\gamma}{\gamma-1}\left(\frac{p_1}{\rho_1}\right)\left[1 - \frac{p_2\rho_1}{p_1\rho_2}\right]$$

$$u_2^2 = \frac{2\gamma}{\gamma-1}\left(\frac{p_1}{\rho_1}\right)\left[1 - \left(\frac{p_2}{p_1}\right)^{1-\frac{1}{\gamma}}\right]$$

The mass flow rate m =$\rho_2 A_2 u_2 C_d$ where C_d is the coefficient of discharge which for a well designed nozzle or Venturi is the same as the coefficient of velocity since there is no contraction and only friction reduces the velocity.

$$\rho_2 = \rho_1\left(\frac{p_2}{p_1}\right)^{\frac{1}{\gamma}} \quad \text{hence} \quad m = C_d A_2\sqrt{\left[\frac{2\gamma}{\gamma-1}\right]\left\{[p_1\rho_1]\left[\left(\frac{p_2}{p_1}\right)^{\frac{2}{\gamma}} - \left(\frac{p_2}{p_1}\right)^{1+\frac{1}{\gamma}}\right]\right\}}$$

If a graph of mass flow rate is plotted against pressure ratio (p_2/p_1) we get:

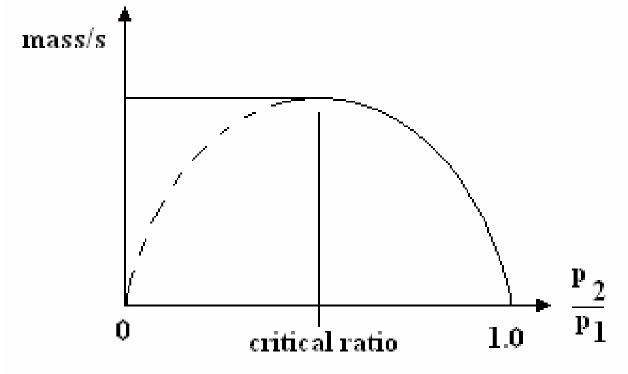

Figure 12

Apparently the mass flow rate starts from zero and reached a maximum and then declined to zero. The left half of the graph is not possible as this contravenes the 2nd law and in reality the mass flow rate stays constant over this half.

What this means is that if you started with a pressure ratio of 1, no flow would occur. If you gradually lowered the pressure p_2 , the flow rate would increase up to a maximum and not beyond. The pressure ratio at which this occurs is the CRITICAL RATIO and the nozzle or Venturi is said to be choked when passing maximum flow rate. Let $\frac{p_2}{p_1} = r$

For maximum flow rate, $\frac{dm}{dr} = 0$

The student should differentiate the mass formula above and show that at the maximum condition the critical pressure ratio is :

$$r = \left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma}{\gamma - 1}}$$

6.1 MAXIMUM VELOCITY

If the formula for the critical pressure ratio is substituted into the formula for velocity, then the velocity at the throat of a choked nozzle/Venturi is :

$$u_2^2 = \left\{\frac{\gamma p_2}{\rho_2}\right\} = \gamma RT = a^2$$

Hence the maximum velocity obtainable at the throat is the local speed of sound.

6.2 CORRECTION FOR INLET VELOCITY

In the preceding derivations, the inlet velocity was assumed negligible. This is not always the case and especially in Venturi Meters, the inlet and throat diameters are not very different and the inlet velocity should not be neglected. The student should go through the derivation again from the beginning but this time keep v_1 in the formula and show that the mass flow rate is

$$m = \frac{C_d A_2 \sqrt{\left[p_1 \rho_1 \frac{2\gamma}{\gamma - 1}\right]\left[\left(\frac{p_2}{p_1}\right)^{\frac{2}{\gamma}} - \left(\frac{p_2}{p_1}\right)^{1+\frac{1}{\gamma}}\right]}}{\sqrt{1 - \left(\frac{A_2}{A_1}\right)^2 \left(\frac{p_2}{p_1}\right)^{\frac{2}{\gamma}}}}$$

The critical pressure ratio can be shown to be the same as before.

6.3 MORE ON ISENTROPIC FLOW

When flow is isentropic it can be shown that all the stagnation properties are constant. Consider the conservation of energy for a horizontal duct :

$$h + u^2/2 = \text{constant} \qquad h = \text{specific enthalpy}$$

If the fluid is brought to rest the total energy must stay the same so the stagnation enthalpy h_o is given by :

$h_o = h + u^2/2$ and will have the same value at any point in the duct.

since $h_o = C_p T_o$ then To (the stagnation temperature) must be the same at all points. It follows that the stagnation pressure p_o is the same at all points also. This knowledge is very useful in solving questions.

6.4 ISENTROPIC EFFICIENCY (NOZZLE EFFICIENCY)

If there is friction present but the flow remains adiabatic, then the entropy is not constant and the nozzle efficiency is defined as :

$$\eta = \text{actual enthalpy drop/ideal enthalpy drop}$$

For a gas this becomes : $(T_1 - T_2)/(T_1 - T_2')$

T_2' is the ideal temperature following expansion. Now apply the conservation of energy between the two points for isentropic and non isentropic flow :

$$C_p T_1 + u_1^2/2 = C_p T_2 + u_2^2/2 \quad \text{....... for isentropic flow}$$

$$C_p T_1 + u_1^2/2 = C_p T_{2'} + u_{2'}^2/2 \quad \text{.........for non isentropic}$$

Hence

$$\eta = (T_1 - T_2)/(T_1 - T_2') = (u_2^2 - u_1^2)/(u_{2'}^2 - u_1^2)$$

If v_1 is zero (for example Rockets) then this becomes :

$$\eta = u_2^2 / u_{2'}^2$$

SELF ASSESSMENT EXERCISE No. 3

1. A Venturi Meter must pass 300g/s of air. The inlet pressure is 2 bar and the inlet temperature is 120°C. Ignoring the inlet velocity, determine the throat area. Take C_d as 0.97. Take γ =1.4 and R = 287 J/kg K (assume choked flow)
(Answer 0.000758 m^2)

2. Repeat problem 1 given that the inlet is 60 mm diameter and the inlet velocity must not be neglected. (Answer 0.000747 m^2)

3. A nozzle must pass 0.5 kg/s of steam with inlet conditions of 10 bar and 400°C. Calculate the throat diameter that causes choking at this condition. The density of the steam at inlet is 3.263 kg/m^3. Take γ for steam as 1.3 and C_d as 0.98.
(Answer 23.2 mm)

4. A Venturi Meter has a throat area of 500 mm^2. Steam flows through it, and the inlet pressure is 7 bar and the throat pressure is 5 bar. The inlet temperature is 400°C. Calculate the flow rate. The density of the steam at inlet is 2.274 kg/m^3.

   Take γ= 1.3. R = 462 J/kg K. Cd=0.97. (Answer 383 g/s)

5. A pitot tube is pointed into an air stream which has an ambient pressure of 100 kPa and temperature of 20°C. The pressure rise measured is 23 kPa. Calculate the air velocity. Take γ = 1.4 and R = 287 J/kg K. (Answer 189.4 m/s)

6. A fast moving stream of gas has a temperature of 25°C. A thermometer is placed into it in front of a small barrier to record the stagnation temperature. The stagnation temperature is 28°C. Calculate the velocity of the gas. Take γ= 1.5 and R = 300 J/kg K. (Answer 73.5 m/s)

Let's do some further study of nozzles of venturi shapes now.

7. CONVERGENT - DIVERGENT NOZZLES

A nozzle fitted with a divergent section is in effect a Venturi shape. The divergent section is known as a diffuser.

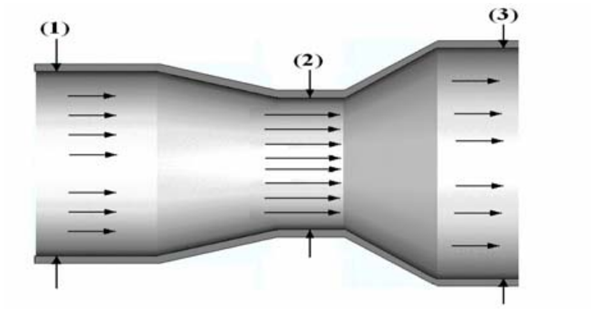

Figure 13

If p_1 is constant and p_3 is reduced in stages, at some point p_2 will reach the critical value which causes the nozzle to choke. At this point the velocity in the throat is sonic.

If p_3 is further reduced, p_2 will remain at the choked value but there will be a further pressure drop from the throat to the outlet. The pressure drop will cause the volume of the gas to expand. The increase in area will tend to slow down the velocity but the decrease in volume will tend to increase the velocity. If the nozzle is so designed, the velocity may increase and become supersonic at exit.

In rocket and jet designs, the diffuser is important to make the exit velocity supersonic and so increase the thrust of the engine.

7.1 NOZZLE AREAS

When the nozzle is choked, the velocity at the throat is the sonic velocity and the Mach number is 1. If the Mach number at exit is M_e then the ratio of the throat and exit area may be found easily as follows.

$u_t = (\gamma RT_t)^{0.5}$ $\quad$ $u_e = M_e(\gamma RT_e)^{0.5}$ $\quad$ mass/s = $\rho_t A_t v_t = \rho_e A_e v_e$.

$$\frac{A_t}{A_e} = \frac{\rho_e u_e}{\rho_t u_t} \quad \text{but earlier it was shown that } \frac{\rho_e}{\rho_t} = \left(\frac{p_e}{p_t}\right)^{\frac{1}{\gamma}}$$

$$\frac{A_t}{A_e} = \left(\frac{p_e}{p_t}\right)^{\frac{1}{\gamma}} \frac{M_e(\gamma RT_e)^{0.5}}{(\gamma RT_t)^{0.5}} \quad \text{It was also shown earlier that } \frac{T_e}{T_t} = \left(\frac{p_e}{p_t}\right)^{1-\frac{1}{\gamma}}$$

$$\frac{A_t}{A_e} = \left(\frac{p_e}{p_t}\right)^{\frac{1}{\gamma}} M_e \left\{\left(\frac{p_e}{p_t}\right)^{1-\frac{1}{\gamma}}\right\}^{0.5}$$

$$\frac{A_t}{A_e} = M_e \left(\frac{p_e}{p_t}\right)^{\frac{1+\gamma}{2\gamma}}$$

There is much more which can be said about nozzle design for gas and steam with implications to turbine designs. This should be studied in advanced text books.

WORKED EXAMPLE No.3

Solve the exit velocity for the nozzle shown assuming isentropic flow:

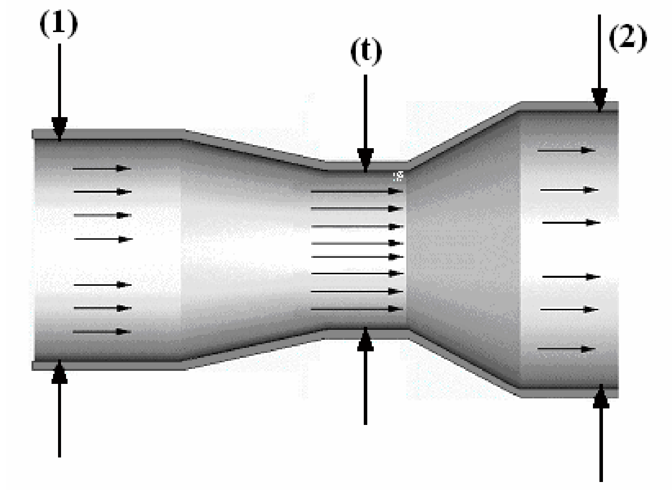

Figure 14

$T_1 = 350$ K $P_1 = 1$ MPa $p_2 = 100$ kPa

The nozzle is fully expanded (choked). Hence $M_t = 1$ (the Mach No.)
The adiabatic index $\gamma = 1.4$

SOLUTION

The critical pressure $p_t = p_1 \{2/(\gamma - 1)\}^{\gamma/(\gamma-1)} = 0.528$ MPa

$T_t/T_1 = (p_t/p_1)^{(\gamma-1)/\gamma}$ hence $T_t = 291.7$ K
$T_o/T_t = \{1 + M^2(\gamma - 1)/2\}$ hence $T_o = 350$ K

It makes sense that the initial pressure and temperature are the stagnation values since the initial velocity is zero.

$T_2 = T_t (p_2/p_t)^{(\gamma-1)/\gamma} = 181.3$ K $a_2 = (\gamma R T_2)^{0.5} = 270$ m/s

$p_o/p_2 = \{1 + M_2^2(\gamma - 1)/2\}^{\gamma/(\gamma-1)}$

Hence $M_2 = 2.157$ and $u_2 = 2.157 \times 270 = 582.4$ m/s

SELF ASSESSMENT EXERCISE No. 4

1. A nozzle is used with a rocket propulsion system. The gas is expanded from complete stagnation conditions inside the combustion chamber of 20 bar and 3000K. Expansion is isentropic to 1 bar at exit. The molar mass of the gas is 33 kg/kmol. The adiabatic index is 1.2. The throat area is 0.1 m^2. Calculate the thrust and area at exit.
(Answers 0.362 m^2 and 281.5 kN)

   Recalculate the thrust for an isentropic efficiency of 95%.
(Answer 274.3 kN)

   Note that expansion may not be complete at the exit area. You may assume

$$\frac{p_t}{p_o} = \left\{\frac{2}{\gamma+1}\right\}^{\frac{\gamma}{\gamma-1}}$$

2. A perfect gas flows through a convergent-divergent nozzle at 1 kg/s. At inlet the gas pressure is 7 bar, temperature 900 K and velocity 178 m/s. At exit the velocity is 820m/s. The overall isentropic efficiency is 85%. The flow may be assumed to be adiabatic with irreversibility's only in the divergent section.

   C_p = 1.13 kJ/kg K R = 287 J/kg K.

   Calculate the cross sectional areas at the inlet, throat and exit.
(Answers 20.8 cm^2, 10.22 cm^2 and 13.69 cm^2)

   Calculate the net force acting on the nozzle if it is stationary. The surrounding pressure is 1 bar. (-527 N) You may assume

$$\frac{p_t}{p_o} = \left\{\frac{2}{\gamma+1}\right\}^{\frac{\gamma}{\gamma-1}}$$

3. Dry saturated steam flows at 1 kg/s with a pressure of 14 bar. It is expanded in a convergent-divergent nozzle to 0.14 bar. Due to irreversibility's in the divergent section only, the isentropic efficiency 96%. The critical pressure ratio may be assumed to be 0.571. Calculate the following.

   The dryness fraction, specific volume and specific enthalpy at the throat.
(Answers 0.958, 0.23 m^3/kg and 2683 kJ/kg)

   The velocity and cross sectional area at the throat and exit.
(Answers 462.6 m/s , 497 mm^2, 1163 m/s and 73.2 cm^2.)

   The overall isentropic efficiency.
(Answer 96.6%)

4. A jet engine is tested on a test bed. At inlet to the compressor the air is at 1 bar and 293 K and has negligible velocity. The air is compressed adiabatically to 4 bar with an isentropic efficiency of 85%. The compressed air is heated in a combustion chamber to 1175 K. It is then expanded adiabatically in a turbine with an isentropic efficiency of 87%. The turbine drives the compressor. The gas leaving the turbine is expanded further reversibly and adiabatically through a convergent nozzle. The flow is choked at exit. The exit area is 0.1 m2.

   Determine the following.

   The pressures at the outlets of the turbine and nozzle.
   (Answers 2.38 bar and 1.129 bar)

   The mass flow rate. (Answer 27.2 kg/s)
   The thrust produced. (Answer 17 kN)

   It may be assumed that $\frac{T_t}{T_o} = \frac{2}{\gamma + 1}$ and a = $\sqrt{\gamma RT}$

5. Dry saturated steam expands through a convergent-divergent nozzle. The inlet and outlet pressures are 7 bar and 1 bar respectively at a rate of 2 kg/s. The overall isentropic efficiency is 90% with all the losses occurring in the divergent section. It may be assumed that $\gamma = 1.135$ and

$$\frac{p_t}{p_o} = \left\{\frac{2}{\gamma + 1}\right\}^{\frac{\gamma}{\gamma - 1}}$$

   Calculate the areas at the throat and exit. (Answers 19.6 cm^2 and 38.8 cm^2).

   The nozzle is horizontal and the entry is connected directly to a large vessel containing steam at 7 bar. The vessel is connected to a vertical flexible tube and is free to move in all directions. Calculate the force required to hold the receiver static if the ambient pressure is 1.013 bar.
   (Answer 3.868 kN)