THERMODYNAMICS
TUTORIAL 16
STEADY FLOW OF COMPRESSIBLE FLUIDS

This tutorial is set at QCF Levels 5 and 6

On completion of this tutorial you should be able to

- Determine one-dimensional steady flow of gases and vapours through nozzles
- Analyse and solve problems involving adiabatic flow through long pipes.
- Identify stagnation properties at a point in a fluid stream
- Analyse and solve problems involving simple jet propulsion systems

Contents

You should make sure that you have studied the contents of tutorial 5 on Entropy changes during various expansion processes.

1. Steady Adiabatic Flow in Pipes

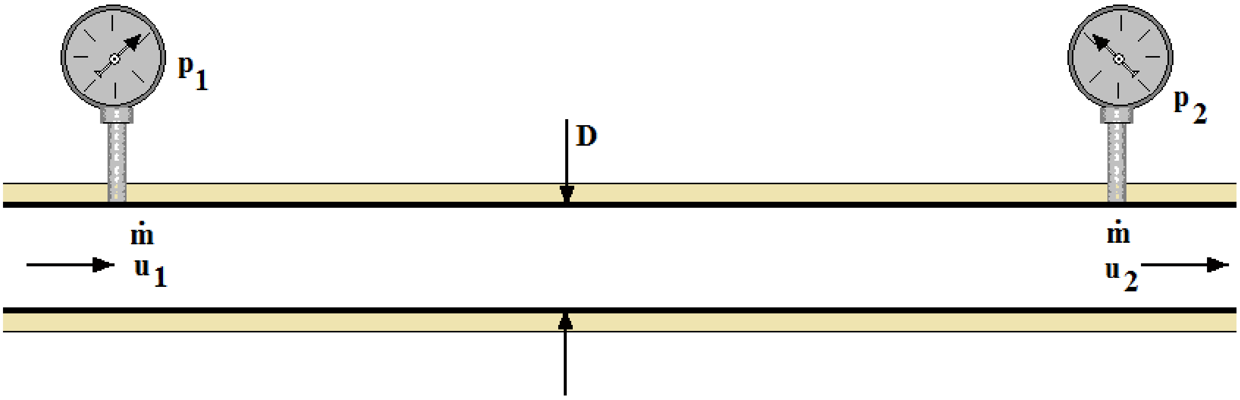

Figure 1

Adiabatic flow is flow with no heat lost or gained from the surroundings. In a long pipe it means that the pipe must be insulated. There will be friction with the pipe wall so some internal heat will be produced that will affect the temperature. It is more usual to think of long pipes as having total cooling with the surroundings so that the temperature does not change and this is Isothermal Flow that is usually covered in fluid mechanics rather than thermodynamics studies.

Since the pressure drops along the pipe the volume would be expected to expand so the velocity u would be expected to increase. For an expansion between point (1) and point (2) the law of expansion would be:

$$pV^{\gamma} = C$$

V is the volume flow rate γ is the adiabatic index for the gas. $V = Au$

Between (1) and (2) $p_1V_1^{\gamma} = p_2V_2^{\gamma}$

$$\frac{V_2}{V_1} = \frac{\dot{m}V_2}{\dot{m}V_1} = \frac{\rho_2}{\rho_1} = \left(\frac{p_1}{p_2}\right)^{\frac{1}{\gamma}} \qquad \rho_2 = \rho_1\left(\frac{p_1}{p_2}\right)^{\frac{1}{\gamma}}$$

$\dot{m}$ = mass flow rate = ρAu

$$u = \frac{\dot{m}}{\rho A} \qquad \rho = \frac{\dot{m}}{Au} \qquad \frac{\dot{m}}{Au_1} = \frac{\dot{m}}{Au_2}\left(\frac{p_1}{p_2}\right)^{\frac{1}{\gamma}} \qquad u_2 = u_1\left(\frac{p_1}{p_2}\right)^{\frac{1}{\gamma}}$$

These equations may help you calculate pressures and velocity but take no account of friction. The friction in a pipe is related to the velocity of the fluid and in this case the velocity is increasing so this complicates the problem. If we know the value of the adiabatic index γ the problem is simpler.

2. *Steady Isentropic Flow in Ducts of Varying Cross Section*

Isentropic means constant entropy. In this case we will consider the flow to be *Adiabatic* also, that is, with no heat transfer.

Consider gas flowing in a duct which varies in size. The pressure and temperature of the gas may change.

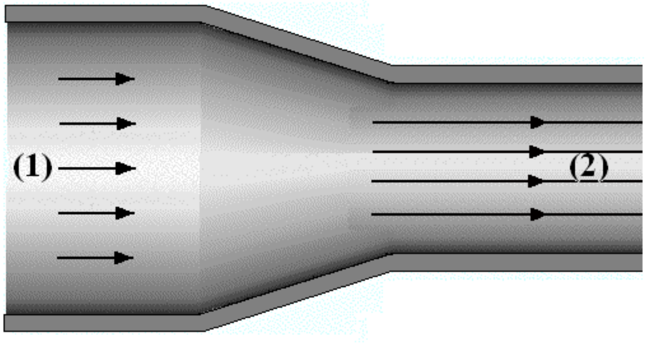

Figure 2

Applying the steady flow energy equation between (1) and (2) we have :

$$\Phi - P = \Delta U + \Delta F.E. + \Delta K.E. + \Delta P.E.$$

For Adiabatic Flow, $\Phi = 0$ and if no work is done then $P = 0$

$$\Delta U + \Delta F.E. = \Delta H$$

hence

$$0 = \Delta H + \Delta K.E. + \Delta P.E.$$

In specific energy terms this becomes:

$$0 = \Delta h + \Delta\, k.e. + \Delta\, p.e.$$

Rewriting we get:

$$h_1 + \frac{u_1^2}{2} + gz_1 = h_2 + \frac{u_2^2}{2} + gz_2$$

For a gas, $h = C_p T$ so we get Bernoulli's equation for gas which is:

$$c_p T_1 + \frac{u_1^2}{2} + gz_1 = c_p T_2 + \frac{u_2^2}{2} + gz_2$$

Note that T is absolute temperature in Kelvins $T = {}^oC + 273$

2.1 Stagnation Conditions

If a stream of gas is brought to rest, it is said to STAGNATE. This occurs on leading edges of any obstacle placed in the flow and in instruments such as a Pitot Tube. Consider such a case for horizontal flow in which P.E. may be neglected.

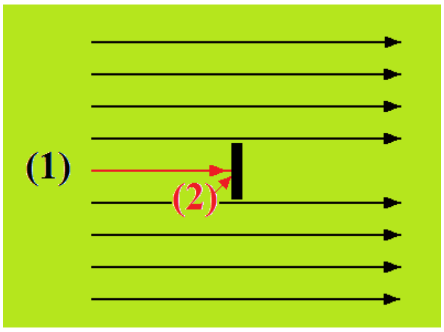

Figure 3

$u_2 = 0$ and $z_1 = z_2$ so

$$c_p T_1 + \frac{u_1^2}{2} + 0 = c_p T_2 + 0 + 0 \qquad \mathbf{T_2 = \frac{u_1^2}{2C_p} + T_1}$$

T_2 is the stagnation temperature for this case

$$\text{Let } T_2 - T_1 = \Delta T = \frac{u_1^2}{2C_p}$$

Now $C_p - C_v = R$ and $\frac{C_p}{C_v} = \gamma$ the adiabatic index . It follows that

$$C_p = \frac{R}{\gamma - 1} \qquad \Delta T = \frac{u_1^2(\gamma - 1)}{2\gamma R}$$

It can be shown elsewhere that the speed of sound a is given by $a = \sqrt{(\gamma RT)}$ or $a^2 = (\gamma RT)$
Hence at point 1

$$\frac{\Delta T}{T_1} = \frac{u_1^2(\gamma - 1)}{2\gamma R T_1} = \frac{u_1^2(\gamma - 1)}{2a_1^2}$$

The ratio u/a is the Mach Number M_a so this may be written as:

$$\frac{\Delta T}{T_1} = \frac{M_a^2(\gamma - 1)}{2}$$

If M_a is less than 0.2 then M_a^2 is less than 0.04 and so $\Delta T/T_1$ is less than 0.008. It follows that for low velocities, the rise in temperature is negligible under stagnation conditions.

The equation may be written as:

$$\frac{T_2 - T_1}{T_1} = \frac{M_a^2(\gamma - 1)}{2} \quad \text{so} \quad \frac{T_2}{T_1} = \frac{M_a^2(\gamma - 1)}{2} + 1$$

Combine the gas law and the law for an adiabatic expansion

$$\frac{pV}{T} = \text{constant and } pV^{\gamma} = \text{constant} \qquad \frac{T_2}{T_1} = \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}$$

Hence

$$\left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}} = \frac{M_a^2(\gamma-1)}{2} + 1$$

$$\left(\frac{p_2}{p_1}\right) = \left[\frac{M_a^2(\gamma-1)}{2} + 1\right]^{\frac{\gamma}{\gamma-1}}$$

p_2 is the stagnation pressure. If we now expand the equation using the binomial theorem we get

$$\left(\frac{p_2}{p_1}\right) = 1 + \frac{\gamma M_a^2}{2}\left\{1 + \frac{M_a^2}{4} + \frac{M_a^2}{8} + \cdots \ldots\right\}$$

If M_a is less than 0.4 then

$$\left(\frac{p_2}{p_1}\right) = 1 + \frac{\gamma M_a^2}{2}$$

Now compare the equations for gas and liquids

LIQUIDS

$$u = \sqrt{\frac{2\Delta p}{\rho}}$$

GAS

$$\left(\frac{p_2}{p_1}\right) = 1 + \frac{\gamma M_a^2}{2}$$

Put $p_2 = p_1 + \Delta p$

$$\Delta p = \frac{\gamma M_a^2}{2} \qquad p_1 = \frac{\gamma u_1^2 p_1}{2\gamma RT} = \frac{\rho_1 u_1^2}{2}$$

$$u = \sqrt{\frac{2\Delta p}{\rho}} \quad \text{the same as for a liquid}$$

Note

$$\rho_1 = \frac{p_1}{RT} \text{ and } M_a^2 = \frac{u_1^2}{\gamma RT}$$

SELF ASSESSMENT EXERCISE No. 1

Take γ = 1.4 and R = 283 J/kg K in all the following questions.

1. An aeroplane flies at Mach 0.8 in air at 15^{o} C and 100 kPa pressure. Calculate the stagnation pressure and temperature. (Answers 324.9 K and 152.4 kPa)

2. Repeat problem 1 if the aeroplane flies at Mach 2. (Answers 518.4 K and 782.4 kPa)

3. The pressure on the leading edges of an aircraft is 4.52 kPa more than the surrounding atmosphere. The aeroplane flies at an altitude of 5 000 metres. Calculate the speed of the aeroplane.
(Answer 109.186 m/s)

   Note from fluids tables, find that a = 320.5 m/s p_1 = 54.05 kPa γ = 1.4

4. An air compressor delivers air with a stagnation temperature 5 K above the ambient temperature. Determine the velocity of the air. (Answer 100.2 m/s)

Let's now extend the work to Pitot tubes.

3. Pitot Static Tube

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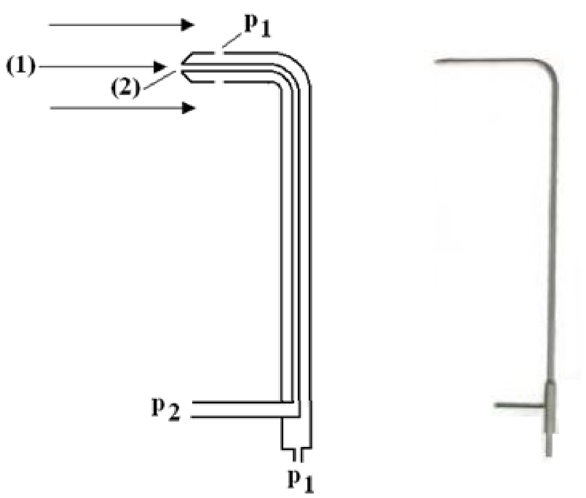

Figure4

$$p_2 = p_1 + \Delta p$$

Using the formula in the last section, the velocity v may be found.

WORKED EXAMPLE No. 1

A pitot tube is pointed into an air stream which has a pressure of 105 kPa. The differential pressure is 20 kPa and the air temperature is 20°C. Calculate the air speed.

SOLUTION

$p_o = p_1 + \Delta p = 105 + 20 = 125$ kPa Note the use of p_o for stagnation pressure)

$$\left(\frac{p_o}{p_1}\right) = \left[\frac{M_a^2(\gamma - 1)}{2} + 1\right]^{\frac{\gamma}{\gamma-1}}$$

$$\left(\frac{125}{105}\right) = \left[\frac{M_a^2(1.4 - 1)}{2} + 1\right]^{\frac{1.4}{1.4-1}} = [0.2M_a^2 + 1]^{3.5}$$

$$1.190^{\frac{1}{3.5}} = 1.051 = 0.2M_a^2 + 1$$

$$Ma = 0.505$$

$$a = (\gamma RT)^{0.5} = (1.4 \times 287 \times 293)^{0.5} = 343 \text{ m/s}$$

$$M_a = u/a \quad \text{hence } u = 173.2 \text{ m/s}$$

Let's further extend the work now to venturi meters and nozzles.

4. Venturi Meters and Nozzles

Consider the diagrams below and apply Isentropic theory between the inlet and the throat.

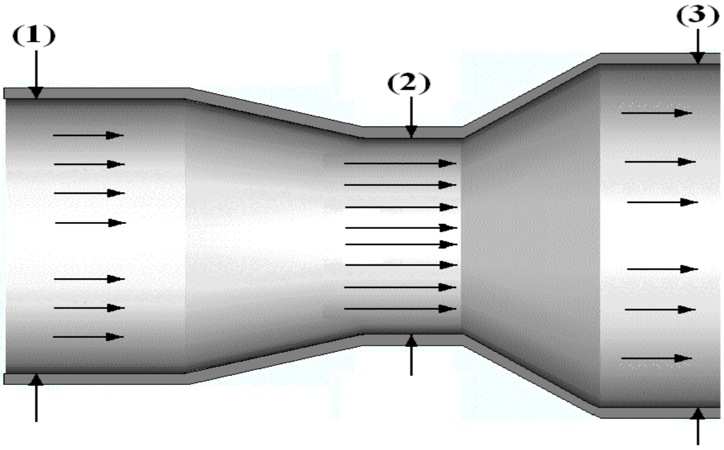

Figure 5

Apply the Steady Flow Energy Equation

$$u_2^2 - u_1^2 = h_1 - h_2$$

If the Kinetic energy at inlet is ignored this gives us $u_2^2 = h_1 - h_2$

For a gas $h = C_pT$ so

$$u_2^2 = C_p[T_1 - T_2]$$

Using

$$C_p = \frac{\gamma R}{\gamma - 1} \text{ we get } u_2^2 = \frac{2\gamma R}{\gamma - 1}[T_1 - T_2]$$

$RT = pV/m = p/\rho$ so

$$u_2^2 = \frac{2\gamma}{\gamma - 1}\left[\frac{p_1}{\rho_1} - \frac{p_2}{\rho_2}\right]$$

For isentropic flow

$$p_1V_1^\gamma = p_2V_2^\gamma \text{ so } \frac{p_1}{\rho_1^\gamma} = \frac{p_2}{\rho_2^\gamma}$$

$$u_2^2 = \frac{2\gamma}{\gamma - 1}\left(\frac{p_1}{\rho_1}\right)\left[1 - \frac{p_2\rho_1}{p_1\rho_2}\right] = \frac{2\gamma}{\gamma - 1}\left(\frac{p_1}{\rho_1}\right)\left[1 - \left(\frac{p_2}{p_1}\right)^{1-\frac{1}{\gamma}}\right]$$

The mass flow rate $\dot{m} = \rho_2 A_2 u_2 C_d$ where C_d is the coefficient of discharge which for a well designed nozzle or Venturi is the same as the coefficient of velocity since there is no contraction and only friction reduces the velocity.

$$\rho_2 = \rho_1\left(\frac{p_2}{p_1}\right)^{\frac{1}{\gamma}}$$

$$\dot{m} = C_dA_2\sqrt{\frac{2\gamma}{\gamma - 1}\left(\frac{p_1}{\rho_1}\right)\left[\left(\frac{p_2}{p_1}\right)^{\frac{2}{\gamma}} - \left(\frac{p_2}{p_1}\right)^{1+\frac{1}{\gamma}}\right]}$$

If a graph of mass flow rate is plotted against pressure ratio (p_2/p_1) we get:

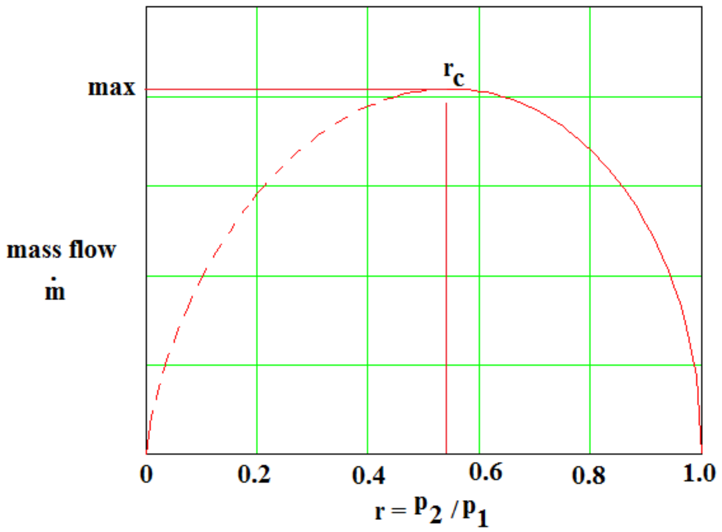

Figure 6

Apparently the mass flow rate starts from zero and reached a maximum and then declined to zero. The left half of the graph is not possible as this contravenes the 2nd law and in reality the mass flow rate stays constant over this half.

What this means is that if you started with a pressure ratio of 1, no flow would occur. If you gradually lowered the pressure p_2, the flow rate would increase up to a maximum and not beyond. The pressure ratio at which this occurs is the ***Critical Ratio*** (r_c) and the nozzle or Venturi is said to be choked when passing maximum flow rate. Let

$$\frac{p_2}{p_1} = r$$

For maximum flow rate,

$$\frac{d\dot{m}}{dr} = 0$$

The student should differentiate the mass formula above and show that at the maximum condition the critical pressure ratio is

$$r = \left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma}{\gamma-1}}$$

4.1 Maximum Velocity

If the formula for the critical pressure ratio is substituted into the formula for velocity, then the velocity at the throat of a choked nozzle/Venturi is :

$$u_2^2 = \left(\frac{\gamma p_2}{\rho_2}\right) = \gamma RT = a^2$$

Hence the maximum velocity obtainable at the throat is the local speed of sound.

4.2 Corrections for Inlet Velocity

In the preceding derivations, the inlet velocity was assumed negligible. This is not always the case and especially in Venturi Meters, the inlet and throat diameters are not very different and the inlet velocity should not be neglected. The student should go through the derivation again from the beginning but this time keep v_1 in the formula and show that the mass flow rate is

$$\dot{m} = \frac{C_d A_2 \sqrt{\frac{2\gamma}{\gamma - 1}\left(\frac{p_1}{\rho_1}\right)\left[\left(\frac{p_2}{p_1}\right)^{\frac{2}{\gamma}} - \left(\frac{p_2}{p_1}\right)^{1+\frac{1}{\gamma}}\right]}}{\sqrt{1 - \left(\frac{A_2}{A_1}\right)^2 \left(\frac{p_2}{p_1}\right)^{\frac{2}{\gamma}}}}$$

The critical pressure ratio can be shown to be the same as before.

4.3 More on Isentropic Flow

When flow is isentropic it can be shown that all the stagnation properties are constant. Consider the conservation of energy for a horizontal duct

$$h + ½ u^2 = \text{constant} \qquad h = \text{specific enthalpy}$$

If the fluid is brought to rest the total energy must stay the same so the stagnation enthalpy h_o is given by :

$h_o = h + ½u^2$ and will have the same value at any point in the duct.

Since $h_o = C_p T_o$ then T_o (the stagnation temperature) must be the same at all points. It follows that the stagnation pressure p_o is the same at all points also. This knowledge is very useful in solving questions.

4.4 Isentropic Efficiency (Nozzle Efficiency)

If there is friction present but the flow remains adiabatic, then the entropy is not constant and the nozzle efficiency is defined as

$$\eta = \frac{\text{actual enthalpy drop}}{\text{ideal enthalpy drop}} \quad \eta = \frac{T_1 - T_2}{T_1 - T_2'} \text{ for a gas}$$

T_2' is the ideal temperature following expansion. Now apply the conservation of energy between the two points for isentropic and non isentropic flow :

$$C_p T_1 + ½ u_1^2 = C_p T_2 + ½ u_2^2 \quad \text{....... for isentropic flow}$$

$$C_p T_1 + ½u_1^2 = C_p T_{2'} + ½u_{2'}^2 \quad \text{.........for non isentropic}$$

$$\text{hence} \quad \eta = \frac{T_1 - T_2}{T_1 - T_2'} = \frac{u_2^2 - u_1^2}{u_{2'}^2 - u_1^2}$$

If v_1 is zero (for example Rockets) then this becomes

$$\eta = \frac{u_2^2}{u_{2'}^2}$$

SELF ASSESSMENT EXERCISE No. 2

1. A Venturi Meter must pass 300 g/s of air. The inlet pressure is 2 bar and the inlet temperature is 120°C. Ignoring the inlet velocity, determine the throat area.
   Take C_d as 0.97. Take γ =1.4 and R = 287 J/kg K (assume choked flow)
   (Answer 0.000758 m^2)

2. Repeat problem 1 given that the inlet is 60 mm diameter and the inlet velocity must not be neglected.
   (Answer 0.000747 m^2)

3. A nozzle must pass 0.5 kg/s of steam with inlet conditions of 10 bar and 400°C. Calculate the throat diameter that causes choking at this condition. The density of the steam at inlet is 3.263 kg/m^3.
   Take γ for steam as 1.3 and C_d as 0.98.
   (Answer 23.2 mm)

4. A Venturi Meter has a throat area of 500 mm^2. Steam flows through it, and the inlet pressure is 7 bar and the throat pressure is 5 bar. The inlet temperature is 400°C. Calculate the flow rate. The density of the steam at inlet is 2.274 kg/m^3.
   Take γ= 1.3. R = 462 J/kg K. Cd=0.97.
   (Answer 383 g/s)

5. A pitot tube is pointed into an air stream which has an ambient pressure of 100 kPa and temperature of 20°C. The pressure rise measured is 23 kPa. Calculate the air velocity.
   Take γ = 1.4 and R = 287 J/kg K.
   (Answer 189.4 m/s)

6. A fast moving stream of gas has a temperature of 25°C. A thermometer is placed into it in front of a small barrier to record the stagnation temperature. The stagnation temperature is 28°C. Calculate the velocity of the gas. Take γ= 1.5 and R = 300 J/kg K.
   (Answer 73.5 m/s)

Let's do some further study of nozzles of venturi shapes now.

5. *Convergent - Divergent Nozzles*

A nozzle fitted with a divergent section is in effect a Venturi shape. The divergent section is known as a diffuser.

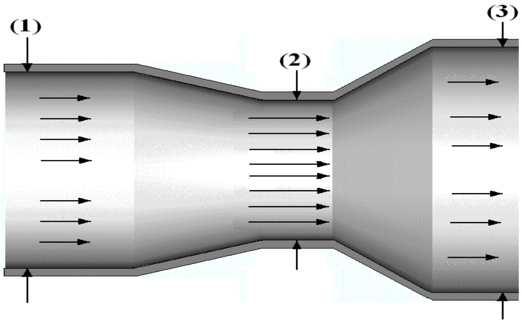

Figure 7

If p_1 is constant and p_3 is reduced in stages, at some point p_2 will reach the critical value which causes the nozzle to choke. At this point the velocity in the throat is sonic.

If p_3 is further reduced, p_2 will remain at the choked value but there will be a further pressure drop from the throat to the outlet. The pressure drop will cause the volume of the gas to expand. The increase in area will tend to slow down the velocity but the decrease in volume will tend to increase the velocity. If the nozzle is so designed, the velocity may increase and become supersonic at exit.

In rocket and jet designs, the diffuser is important to make the exit velocity supersonic and so increase the thrust of the engine.

5.1 Nozzle Areas

When the nozzle is choked, the velocity at the throat is the sonic velocity and the Mach number is 1. If the Mach number at exit is M_e then the ratio of the throat and exit area may be found easily as follows.

$u_t = (\gamma R T_t)^{0.5}$ $u_e = M_e(\gamma R T_e)^{0.5}$ mass/s = $\rho_t A_t v_t = \rho_e A_e v_e$.

$$\frac{A_t}{A_e} = \frac{\rho_e u_e}{\rho_t u_t}$$

It was shown earlier that

$$\frac{\rho_e}{\rho_t} = \left(\frac{p_e}{p_t}\right)^{\frac{1}{\gamma}}$$

$$\frac{A_t}{A_e} = \left(\frac{p_e}{p_t}\right)^{\frac{1}{\gamma}} M_e \sqrt{\left(\frac{p_e}{p_t}\right)^{1-\frac{1}{\gamma}}} = M_e \left(\frac{p_e}{p_t}\right)^{\frac{1+\gamma}{2\gamma}}$$

There is much more which can be said about nozzle design for gas and steam with implications to turbine designs.

WORKED EXAMPLE No. 2

Solve the exit velocity for the nozzle shown assuming isentropic flow:

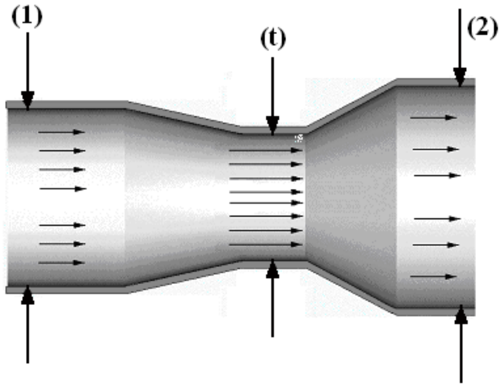

Figure 8

T_1 = 350 K P_1 = 1 MPa p_2 = 100 kPa

The nozzle is fully expanded (choked). Hence M_t = 1 (the Mach No.) The adiabatic index γ = 1.4

SOLUTION

The critical pressure

$$r = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}} = \left(\frac{2}{1.4+1}\right)^{\frac{1.4}{1.4-1}} = 0.528$$

$$p_t = p_1\, r = 1 \times 0.528 = 0.528 \text{ MPa}$$

$$\frac{T_t}{T_1} = \left(\frac{p_t}{p_1}\right)^{\frac{\gamma-1}{\gamma}} = (0.528)^{0.286} \qquad T_t = 350(0.528)^{0.286} = 291.6 \text{ K}$$

It was shown earlier that the stagnation temperature is given by

$$\frac{T_0}{T_t} = 1 + \frac{M^2(\gamma-1)}{2} = 1.2 \quad T_0 = 1.2 \times 291.6 = 350 \text{ K}$$

It makes sense that the initial pressure and temperature are the stagnation values since the initial velocity is zero.

$$T_2 = T_t\left(\frac{p_2}{p_t}\right)^{\frac{\gamma-1}{\gamma}} = 181.3 \text{ K}$$

$a_2 = (\gamma R T_2)^{0.5}$ = 270 m/s

$$\frac{T_0}{T_1} = \frac{350}{181.3} = \frac{M_a^2(\gamma-1)}{2} + 1 = 0.2M_a^2 + 1 \text{ hence } M_2 = 2.157$$

$$a = \sqrt{\gamma RT} = \sqrt{1.4 \times 287 \times 181.3} = 270 \text{ m/s}$$

$$u_2 = 2.157 \times 270 = 582.4 \text{ m/s}$$

SELF ASSESSMENT EXERCISE No. 3

1. A nozzle is used with a rocket propulsion system. The gas is expanded from complete stagnation conditions inside the combustion chamber of 20 bar and 3000K. Expansion is isentropic to 1 bar at exit. The molar mass of the gas is 33 kg/kmol. The adiabatic index is 1.2. The throat area is 0.1 m^2. Calculate the thrust and area at exit.

   (Answers 0.362 m^2 and 281.5 kN)

   Recalculate the thrust for an isentropic efficiency of 95%

   (Answer 274.3 kN)

   Note that expansion may not be complete at the exit area. You may assume

$$\frac{p}{p_o} = \left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma}{\gamma - 1}}$$

2. Dry saturated steam flows at 1 kg/s with a pressure of 14 bar. It is expanded in a convergent-divergent nozzle to 0.14 bar. Due to irreversibility's in the divergent section only, the isentropic efficiency 96%. The critical pressure ratio may be assumed to be 0.571. Calculate the following.

   The dryness fraction, specific volume and specific enthalpy at the throat
   (Answers 0.958, 0.23 m^2 and 2 683 kJ/kg)

   The velocity and cross sectional area at the throat and exit
   (Answers 462.6 m/s, 497mm^2 , 1 163 m/s and 73.2 cm^2)

   The overall isentropic efficiency
   (Answer 96.6%)

3. A jet engine is tested on a test bed. At inlet to the compressor the air is at 1 bar and 293 K and has negligible velocity. The air is compressed adiabatically to 4 bar with an isentropic efficiency of 85%. The compressed air is heated in a combustion chamber to 1 175 K. It is then expanded adiabatically in a turbine with an isentropic efficiency of 87%. The turbine drives the compressor. The gas leaving the turbine is expanded further reversibly and adiabatically through a convergent nozzle. The flow is choked at exit. The exit area is 0.1 m^2.

   Determine the following:

   The pressures at the outlets of the turbine and nozzle (Answers 2.137 bar and 1.129 bar)
   The mass flow rate (Answer 27.2 kg/s)
   The thrust produced. (Answer 17 kN)

   It may be assumed that

$$\frac{T_t}{T_o} = \frac{2}{\gamma + 1} \qquad a = \sqrt{\gamma R T}$$

4. Dry saturated steam expands through a convergent-divergent nozzle. The inlet and outlet pressures are 7 bar and 1 bar respectively at a rate of 2 kg/s. The overall isentropic efficiency is 90% with all the losses occurring in the divergent section. It may be assumed that $\gamma = 1.135$ and

$$\frac{p_t}{p_o} = \left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma}{\gamma - 1}}$$

   Calculate the areas at the throat and exit. (19.6 cm^2 and 38.8 cm^2)

   The nozzle is horizontal and the entry is connected directly to a large vessel containing steam at 7 bar. The vessel is connected to a vertical flexible tube and is free to move in all directions. Calculate the force required to hold the receiver static if the ambient pressure is 1.013 bar.
   (Answer 3.89 kN)