

## **UNIT 62: STRENGTHS OF MATERIALS**

**Unit code: K/601/1409**

**QCF level: 5**

**Credit value: 15**

### **OUTCOME 2 - TUTORIAL 4**

#### **REINFORCED CONCRETE BEAMS**

**2 Be able to determine the behavioural characteristics of loaded beams, columns and struts**

*Simply supported beams:* use of Macaulay's method to determine the support reactions, slope and deflection due to bending in cantilevers and simply supported beams with combined concentrated and uniformly distributed loads

*Reinforced concrete beams:* theoretical assumptions; distribution of stress due to bending

*Columns:* stress due to asymmetrical bending; middle third rule for rectangular section columns and walls; middle quarter rule for circular section columns

*Struts:* end fixings; effective length; least radius of gyration of section; slenderness ratio; Euler and Rankine-Gordon formulae for determination of critical load

You should judge your progress by completing the self assessment exercises.

#### **CONTENTS**

1. **INTRODUCTION**
2. **ELASTIC BENDING**
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*It is assumed that students doing this tutorial are fully conversant with bending stresses in beams.*

## 1. INTRODUCTION

In order to understand the theory of reinforced concrete beams you would need to study a very wide range of material well beyond the scope of one part of one outcome of one module. For this reason the material offered here is greatly simplified.

## 2. ELASTIC BENDING

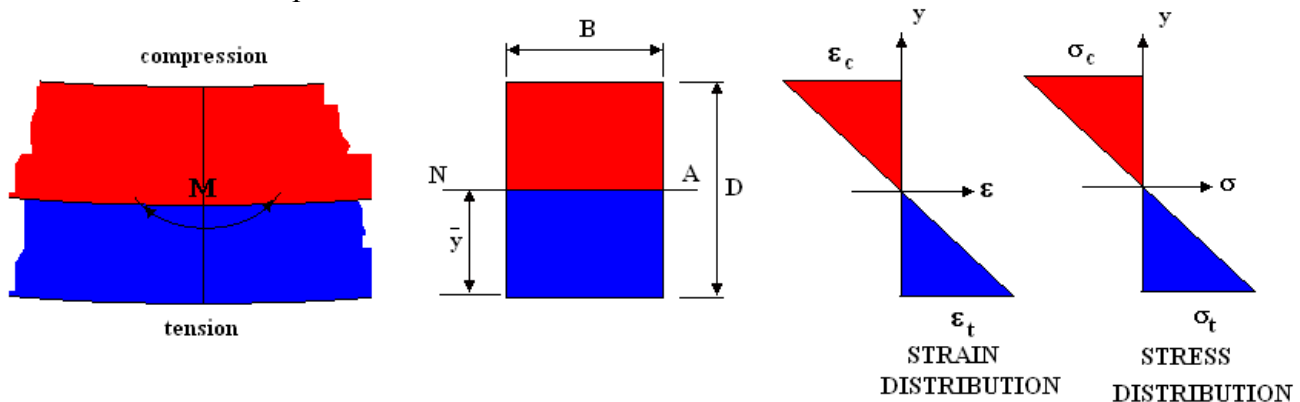
Consider a simple rectangular beam at a point where the bending moment is  $M$ .

The neutral axis passes through the centre of area.

The strain is directly proportional to distance  $y$ .

Since  $\sigma = E \epsilon$  the stress is also directly proportional to distance  $y$ .

Suffix 'c' refers to compression and 't' to tension.



From the bending formula we know that the stress at the outer edge is  $\sigma = M y/I = M/Z$

$y = D/2$  at the outer edges.

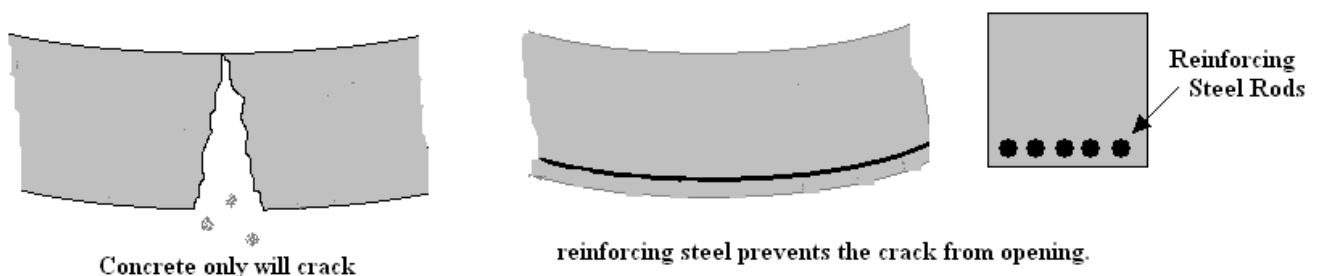
$I$  is the second moment of area about the centroid.

$Z = I/y = 2I/D$  is the elastic modulus. (This may be looked up in standard tables for commercial beams of various sections).

$M = \sigma Z$  where  $\sigma$  is the actual value of stress at the outer edge so long as it is within the elastic limit.

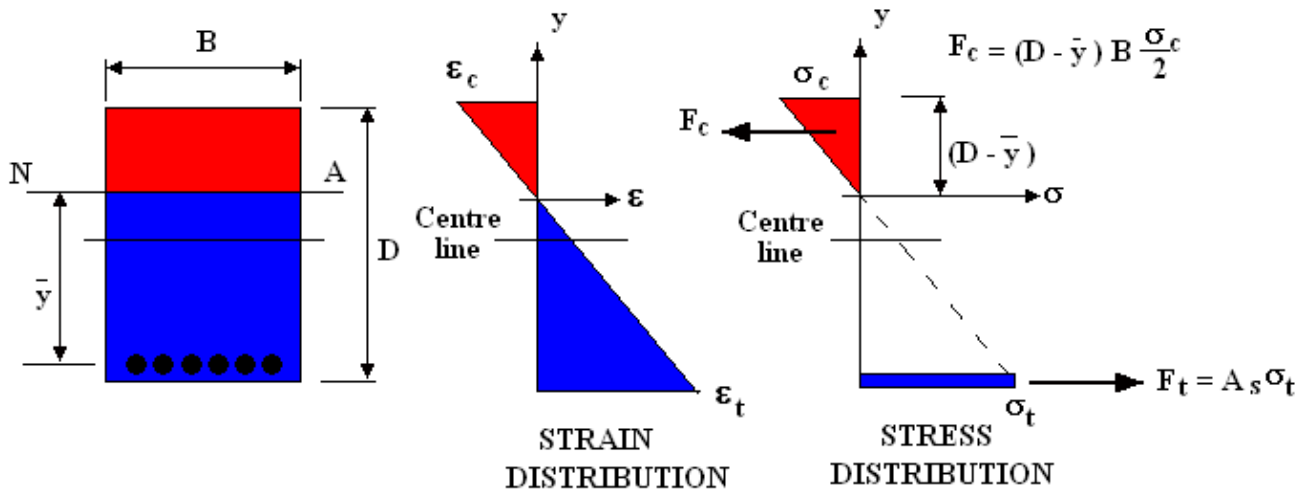
## 3. CONCRETE BEAMS

A beam made from pure concrete would fail very easily because concrete cannot withstand tension so a crack would open up on the tensile edge and spread through the section. To stop this happening, steel rods are embedded in the concrete near to the tensile edge. These are convoluted to help prevent them slipping in the concrete. We assume that the rods are so firmly embedded in the concrete that they become strained by the same amount as the concrete and although the concrete would break, the rods will bridge the gap and prevent the crack from opening



**The concrete under tension does not produce any force. The entire tensile force is carried by the steel rods.**

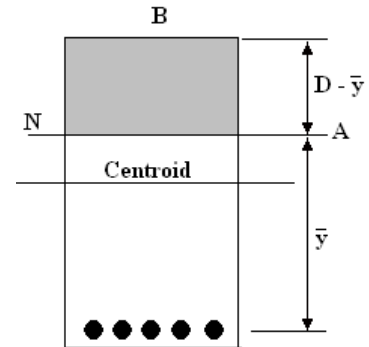
The diagram shows how the stress and strain are distributed in the concrete and steel reinforcement.  $A_s$  is the section area of the rods.



The problem now is that the neutral axis moves away from the centroid and to find  $\bar{y}$  we need to study how to find the second moment of area when we have two materials.

#### 4. SECOND MOMENT OF AREA

The concrete under compression is the shaded area and the compression force acting on this area must be equal to the tensile force in the steel rods. Because of this, the neutral axis is no longer at the middle and we need a way of determining  $\bar{y}$ .



For the concrete section only,  $I_c = B(D - \bar{y})^3/3$  about the neutral Axis. Assuming the steel is confined to one level and that  $\bar{y}$  is measured from this position,  $I_s = A_s \bar{y}^2$

This is steel and not concrete and we need to find the equivalent second moment of area for the equivalent area of concrete. Consider a section of steel on which the stress is  $\sigma$ .

The Force is  $F = A_s \sigma$  and since  $\sigma = \epsilon E$

The stress is  $\sigma_s = A_s \epsilon E_s$  for elastic materials. Suppose the steel was replaced with an area of concrete of equal strength and the same strain.

$$F = A_c \epsilon E_c = A_s \epsilon E_s \quad A_c = A_s \epsilon E_s / \epsilon E_c = A_s \quad E_s/E_c = n \quad n = E_s/E_c$$

This is the equivalent area of concrete and the equivalent second moment of area is  $I_s = n A_s \bar{y}^2$

The second moment of area for the section in equivalent concrete terms is

$$I_{NA} = \frac{B(D - \bar{y})^3}{3} + n A_s \bar{y}^2$$

Now consider the force balance again.

The tensile force in the rods is  $F_t = A_s \sigma_t = A_s E_t \epsilon_t$

The compressive force in the concrete is  $(D - \bar{y}) B \sigma_c/2 = (D - \bar{y}) B E_c \epsilon_c/2$

The tensile and compressive forces must be equal.  $A_s E_t \epsilon_t = (D - \bar{y}) B E_c \epsilon_c/2$

$$\frac{\epsilon_c}{\epsilon_t} = \frac{2A_s E_t}{B(D - \bar{y}) E_c} = \frac{2nA_s}{B(D - \bar{y})} \dots\dots\dots(1)$$

From the two similar triangles in the strain distribution we have

$$\frac{\epsilon_c}{D-\bar{y}} = \frac{\epsilon_t}{\bar{y}} \quad \text{hence} \quad \frac{\epsilon_c}{\epsilon_t} = \frac{D-\bar{y}}{\bar{y}} \dots\dots\dots(2)$$

Equate (1) and (2)  $\frac{D-\bar{y}}{\bar{y}} = \frac{2nA_s}{B(D-\bar{y})}$

This may be rearranged into a quadratic equation.

$$(D-\bar{y})^2 = \frac{2n\bar{y}A_s}{B} \quad D^2 + \bar{y}^2 - 2D\bar{y} - \frac{2n\bar{y}A_s}{B} = 0$$

$$\bar{y}^2 - 2\bar{y}\left(D + n\frac{A_s}{B}\right) + D^2 = 0$$

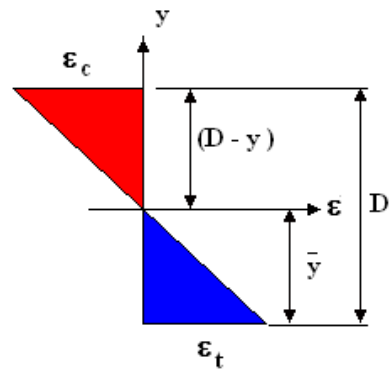
Now use the quadratic formula to solve

$$\bar{y} = \frac{-b \pm \sqrt{b^2 - 4ac^2}}{2a} \quad a=1 \quad b = -2\left(D + n\frac{A_s}{B}\right) \quad c = D^2$$

$$\bar{y} = \frac{2\left(D + n\frac{A_s}{B}\right) \pm \sqrt{4\left(D + n\frac{A_s}{B}\right)^2 - 4D^2}}{2} \quad \text{let} \left(D + n\frac{A_s}{B}\right) = \rho$$

$$\bar{y} = \frac{2\rho \pm \sqrt{4\rho^2 - 4D^2}}{2} = \rho \pm \sqrt{\rho^2 - D^2}$$

$$\bar{y} = \rho \pm \sqrt{(\rho - D)(\rho + D)}$$



**WORKED EXAMPLE No.1**

A rectangular reinforced concrete beam has the following data.

$D = 155 \text{ mm} \quad B = 120 \text{ mm} \quad n = 8 \quad A_s = 240 \text{ mm}^2$

Find the position of the neutral axis and the second moment of area about it.

**SOLUTION**

Suffix 'c' refers to concrete and 's' to steel.

$$\rho = \left(D + n\frac{A_s}{B}\right) = \left(155 + 8\frac{240}{120}\right) = 171$$

$$\bar{y} = 171 \pm \sqrt{(171-155)(171+155)} = 315 \pm \sqrt{(16)(326)} = 171 \pm 72.2$$

$\bar{y} = 98.8 \text{ mm}$  from the steel rods.

$$I_{NA} = \frac{B(D-\bar{y})^3}{3} + nA_s\bar{y}^2$$

$$I_{NA} = \frac{120(155-98.8)^3}{3} + 8 \times 240 \times 98.8^2 = 7.1 \times 10^6 + 18.742 \times 10^6 = 25.842 \times 10^6 \text{ mm}^4$$

### WORKED EXAMPLE No.2

For the same beam as example 1, the maximum strain in the concrete must not exceed 0.0003. Determine the maximum compressive stress in the concrete, the strain in the steel and the stress in the steel. Show that the tensile and compressive forces are equal. Assume the strain distribution is linear.

Take  $E_c = 25 \text{ GPa}$

### SOLUTION

$$E_s = n E_c = 8 \times 25 = 200 \text{ GPa}$$

$$\sigma_c = E_c \varepsilon_c = 25 \times 10^9 \times 0.0003 = 7.5 \text{ MPa}$$

$$\varepsilon_t = \frac{\bar{y} \varepsilon_c}{D - \bar{y}} = \frac{98.8 \times 10^{-3} \times 0.0003}{155 \times 10^{-3} - 98.8 \times 10^{-3}} = 0.000527$$

$$\sigma_s = E_s \varepsilon_s = 200 \times 10^9 \times 0.000527 = 105.4 \text{ MPa}$$

$$F_c = B(D - \bar{y})\sigma_c/2 = \{12(155 - 98.8)\} \times 10^{-6} \times 75 \times 10^6/2 = 252.9 \text{ kN (compressive)}$$

$$F_s = A_s \sigma_s = 240 \times 10^{-6} \times 105.4 \times 10^6 = 252.9 \text{ kN (Tensile)}$$

### SELF ASSESSMENT EXERCISE No.1

A rectangular reinforced concrete beam has the following data.

$$D = 394 \text{ mm} \quad B = 305 \text{ mm} \quad n = 8 \quad A_s = 1550 \text{ mm}^2$$

Find the position of the neutral axis and the second moment of area about it.

The maximum strain in the concrete must not exceed 0.0005. Determine the maximum compressive stress in the concrete, the strain in the steel and the stress in the steel. Assume the strain distribution is linear.

Take  $E_c = 25 \text{ GPa}$

#### Answers

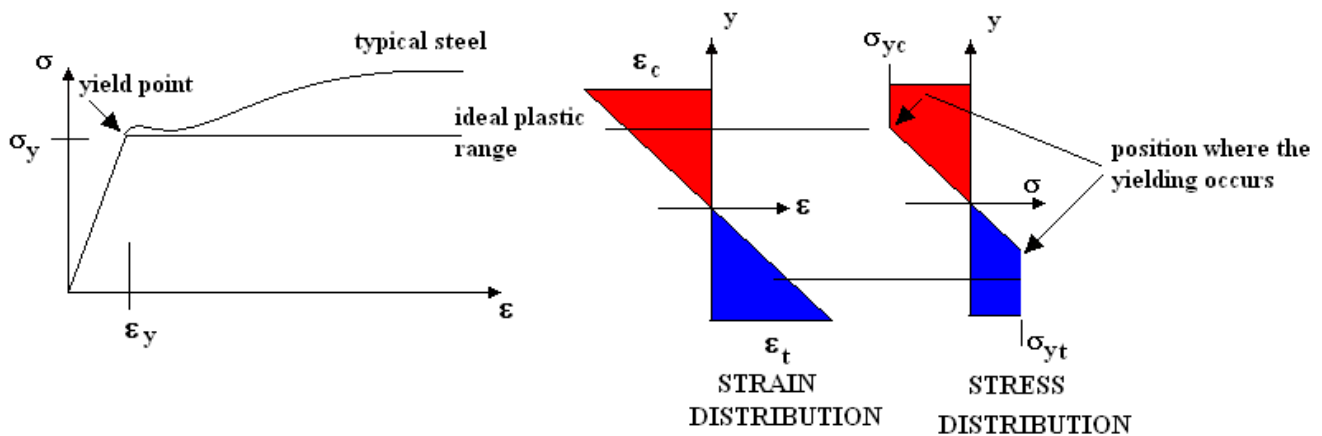
$$\bar{y} = 251 \text{ mm from the steel rods. } I_{NA} = 1.079 \times 10^9 \text{ mm}^4$$

$$\sigma_c = 12.5 \text{ MPa} \quad \varepsilon_s = 0.00088 \quad \sigma_s = 181.8 \text{ MPa}$$

The work covered so far is not a very realistic approach to reinforced beams and we should consider the theory of plastic failure and true stress distribution in order to make a realistic stab at solving beam problems. Most published text on reinforced concrete beams state that concrete fails in compression when the strain is 0.003. Further studies reveal that the steel will usually have passed its tensile yield point. The type of steel used is assumed to have a plastic range and so the stress in the steel will be the yield stress. Typical steel has a yield stress of around 415 MPa and an elastic modulus  $E$  of 200 GPa. The strain at the yield point is hence around 0.002. It is important to check that the steel has yielded if the yield stress value is used.

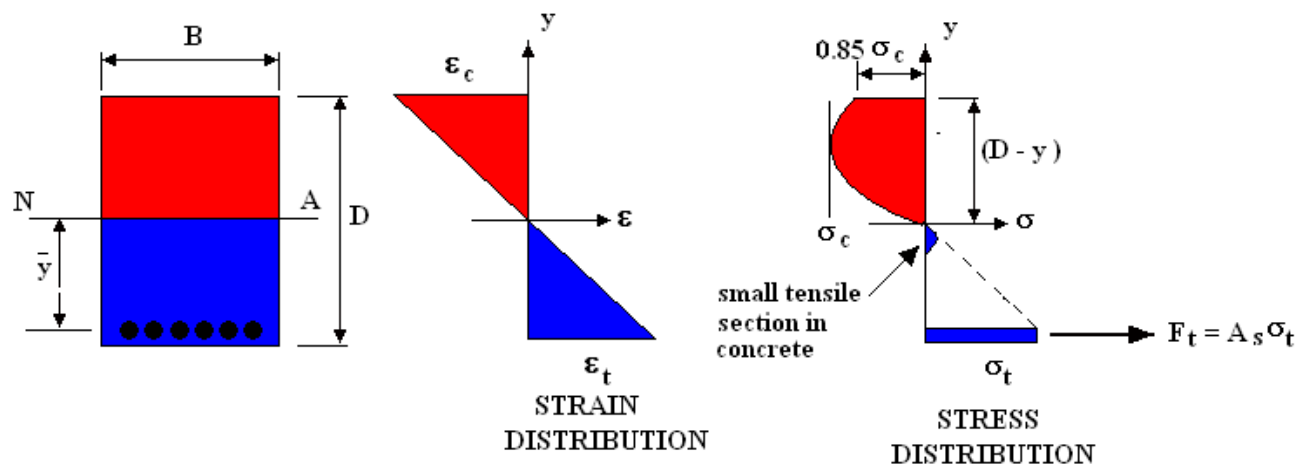
## 5. PLASTIC BENDING

If the material reaches its yield stress at some value of  $y$  less than  $D/2$ , the stress is no longer proportional to strain. To simplify matters let's assume that the stress stays constant at the yield value  $\sigma_{yt}$  for all strains beyond the yield point. The diagram compares pure plastic yielding with that of typical steel. The stress and strain distribution is now as shown. Note that the strain distribution is unchanged since strain is directly proportional to the distance from the neutral axis.

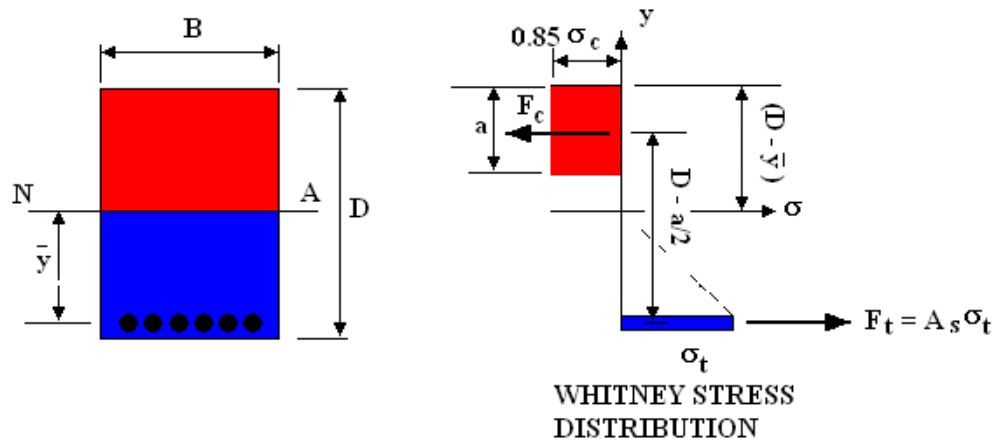


It may be of interest to note that for steel beams, yielding first occurs at the outer edge and the bending moment is  $M = \sigma_y Z_p$  where  $Z_p$  is the plastic modulus. This may also be found in tables for commercial sections but this is not much use for reinforced beams.

The relationship between stress and strain is not linear for the concrete. Although the strain distribution remains the same as before, the stress distribution is more like this.



In order to work out the compressive force in the concrete, the stress distribution is simplified to a box as shown. This is known as the Whitney box.



The compressive stress is taken as a rectangle of height  $a$  and length  $0.85 \sigma_c$

The dimension 'a' is usually given as  $a = \beta_1 (D - \bar{y})$

The compressive force  $F_c$  acts at the middle of the box at a distance  $(D - \bar{y} - a/2)$  from the neutral axis.

It is normal to take  $\beta_1 = 0.85$  for  $\sigma_c \leq 30$  MPa and 0.65 for larger stress values.

The compressive force is now given by the expression  $F_c = 0.85B a \sigma_c$

Equating tensile and compressive force we have

$$A_s \sigma_t = 0.85B a \sigma_c$$

$$a = \frac{A_s \sigma_t}{0.85B \sigma_c}$$

The bending moment is  $M = F_t (D - a/2) = F_c (D - a/2)$

$$M = F_t (D - a/2) = A_s \sigma_t (D - a/2)$$

Substitute for a and

$$M = A_s \sigma_t \left( D - \frac{A_s \sigma_t}{2 \times 0.85B \sigma_c} \right) = A_s \sigma_t \left( D - \frac{0.59 A_s \sigma_t}{B \sigma_c} \right)$$

Note the following points.

- The tensile strength in the concrete is always ignored.
- The steel rods may have yielded in which case the  $\sigma_t = \sigma_{yt}$
- Concrete is assumed to have failed in compression when the strain reaches 0.003.
- The strain distribution is assumed to be linear.
- There are limitations to the formulae developed here and more advanced text should be studied before applying them to real situations.

### **WORKED EXAMPLE No.3**

A rectangular concrete beam has the following dimensions.

$B = 305 \text{ mm}$        $D = 394 \text{ mm}$  (The distance from the top to the reinforcement).

Yield stress for concrete = 27.6 MPa

Yield stress for steel = 415 MPa

The steel rods have a cross sectional area of  $1550 \text{ mm}^2$

Assume that both the steel and concrete have yielded.

Find the neutral axis.

Find the moment capacity.

Given that the strain in the concrete is 0.003 at yield, verify that the steel has also yielded.

Take  $E_s = 200 \text{ GPa}$

### **SOLUTION**

$a = \frac{A_s \sigma_t}{0.85 B \sigma_c}$  and if both have yielded we put in the yield values.

$$a = \frac{1550 \times 10^{-6} \times 415 \times 10^6}{0.85 \times 305 \times 10^{-3} \times 27.6 \times 10^6} = 0.0899 \text{ m or } 89.9 \text{ mm}$$

Now find the neutral axis.

$$\beta_1(D - \bar{y}) = a \text{ and since } \sigma_c \leq 30 \text{ MPa } \beta_1 = 0.85$$

$$a = 89.9 = 0.85(394 - \bar{y}) \quad \bar{y} = 288.2 \text{ mm from the bottom edge.}$$

Based on the steel the moment capacity is:

$$M = A_s \sigma_s (D - a/2) = 1550 \times 10^{-6} \times 415 \times 10^6 (394 - 89.9/2) \times 10^{-3} = 224 \text{ kNm}$$

The strain in the concrete is found from the linear relationship.

$$E_s / \epsilon_c = \bar{y} / (D - \bar{y}) = 288.2 / (394 - 288.2) = 2.72$$

$$\epsilon_s = 0.003 \times 2.72 = 0.00817$$

$$\text{The strain in the steel at yield is } \epsilon_s = \sigma_y / E = 415 \times 10^6 / 200 \times 10^9 = 0.00207$$

The steel has exceeded the yield point.



## **SELF ASSESSMENT EXERCISE No.2**

1. A rectangular reinforced concrete beam has the following data.

$$D = 600 \text{ mm} \quad B = 400 \text{ mm}$$

Yield stress for concrete = 30 MPa

Yield stress for steel = 420 MPa

The steel rods have a cross sectional area of  $3000 \text{ mm}^2$

Assume that both the steel and concrete have yielded.

Find the neutral axis.

Find the moment capacity.

Given that the strain in the concrete is 0.003 at yield, verify that the steel has also yielded.

Take  $E_s = 200 \text{ Gpa}$

(Answers 455 mm from the reinforcement, 678 kNm)

2. A rectangular reinforced concrete beam has the following data.

$$D = 305 \text{ mm} \quad B = 533.4 \text{ mm}$$

Yield stress for concrete = 28 MPa

Yield stress for steel = 414 MPa

The steel rods have a cross sectional area of  $1936 \text{ mm}^2$

Assume that both the steel and concrete have yielded.

Find the neutral axis.

Find the moment capacity.

Given that the strain in the concrete is 0.003 at yield, calculate the actual strain and the yield point strain and verify that the steel has also yielded.

Take  $E_s = 200 \text{ GPa}$

(Answers 403 mm from the reinforcement, 383.3 kNm, 0.0093 and 0.0021)