

EDEXCEL MODULE

STRENGTHS OF MATERIALS - H2

OUTCOME 1 - TUTORIAL 1

COMPLEX STRESS AND STRAIN

You should judge your progress by completing the self assessment exercises. These may be sent for marking at a cost (see home page).

- On completion of this tutorial you should be able to do the following.
- Explain a complex stress situation.
- Derive formulae for complex stress situations.
- Analyse and solve stresses in a complex stress situation.
- Solve problems using a graphical method (Mohr's Circle)
- Explain the use of strain gauge rosettes to determine principal strains and stresses.
- Apply the theory to combined bending and torsion problems.

1. COMPLEX STRESS

Materials in a stressed component often have direct and shear stresses acting in two or more directions at the same time. This is a complex stress situation. The engineer must then find the maximum stress in the material. We will only consider stresses in two dimensions, x and y. The analysis leads on to a useful tool for solving complex stress problems called Mohr's' Circle of Stress.

1.1 DERIVATION OF EQUATIONS

Consider a rectangular part of the component material. Stress σ_x acts on the x plane and σ_y acts on the y plane. The shear stress acting on the plane on which σ_x acts is τ_x and τ_y acts on the plane on which σ_y acts. The shear stresses are complementary and so must have opposite rotation. We will take clockwise shear to be positive and anti-clockwise as negative.

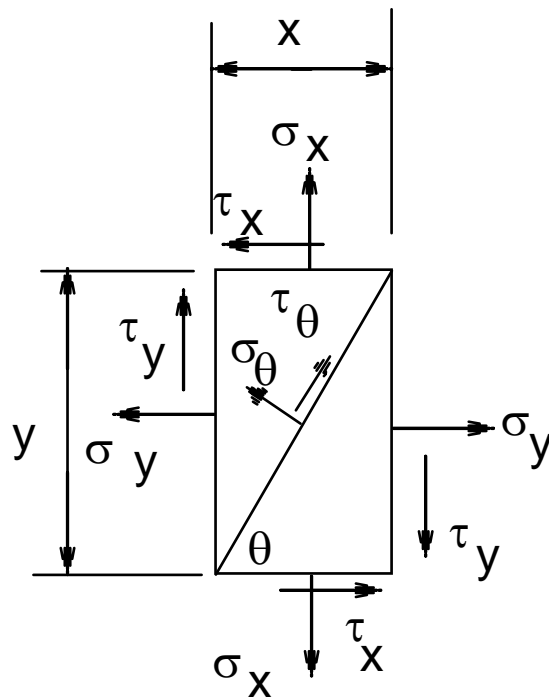


Figure 1

Now consider a plane at angle θ to the x plane. The plane is in equilibrium so all the forces and moments on the plane must add up to zero. The stresses must first be turned into forces.

If the material is 1 m thick normal to the paper then the areas are dx and dy on the sides and $y/\sin\theta$ or $x/\cos\theta$ on the sloping plane.

Consider the balance of forces on the lower corner. First the forces are resolved perpendicular and parallel to the plane.

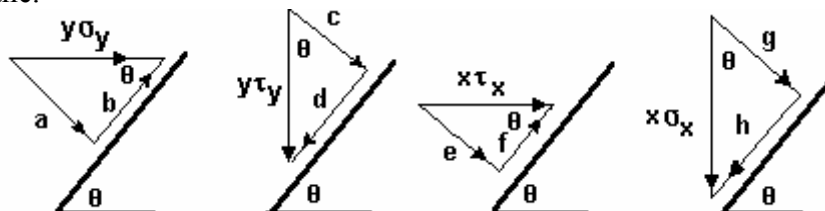


Figure 2

$$\begin{array}{ll}
 a = y\sigma_y \sin\theta. & b = y\sigma_y \cos\theta. \\
 c = y\tau_y \cos\theta. & d = y\tau_y \sin\theta. \\
 e = x\tau_x \sin\theta. & f = x\tau_x \cos\theta. \\
 g = x\sigma_x \cos\theta. & h = x\sigma_x \sin\theta.
 \end{array}$$

All the forces normal to the plane must add up to $(y/\sin\theta) \sigma_\theta$

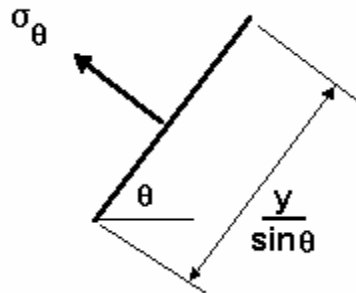


Figure 3

Balancing we have $a + c + e + g = (y/\sin\theta) \sigma_\theta$

All the forces parallel to the plane must add up to $(y/\sin\theta) \tau_\theta$

$$-f + h + b - d = (y/\sin\theta) \tau_\theta$$

Making the substitutions and conducting algebraic process will yield the following results.

$$\frac{y}{\sin\theta} \sigma_\theta = y\sigma_y \sin\theta + y\tau_y \cos\theta + x\tau_x \sin\theta + x\sigma_x \cos\theta$$

$$\sigma_\theta = \sigma_y \sin^2\theta + \tau_y \sin\theta \cos\theta + \frac{x}{y} \tau_x \sin^2\theta + \frac{x}{y} \sigma_x \sin\theta \cos\theta$$

$$\sigma_\theta = \sigma_y \frac{1 - \cos 2\theta}{2} + \tau_y \frac{\sin 2\theta}{2} + \tau_x \frac{\sin^2\theta}{\tan\theta} + \sigma_x \frac{\sin\theta \cos\theta}{\tan\theta}$$

$$\sigma_\theta = \sigma_y \frac{1 - \cos 2\theta}{2} + \tau_y \frac{\sin 2\theta}{2} + \tau_x \frac{\sin 2\theta}{2} + \sigma_x \frac{1 + \cos 2\theta}{2}$$

$$\sigma_\theta = \frac{\sigma_y}{2} - \sigma_y \frac{\cos 2\theta}{2} + \tau_y \frac{\sin 2\theta}{2} + \tau_x \frac{\sin 2\theta}{2} + \frac{\sigma_x}{2} + \sigma_x \frac{\cos 2\theta}{2}$$

The shear stress on both planes are equal so denote them both by $\tau_{xy} = \tau_x = \tau_y$

$$\sigma_\theta = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)\cos 2\theta}{2} + \tau_{xy} \sin 2\theta \quad \dots\dots\dots(1.1)$$

Repeating the process for the shear stress we get

$$\tau_\theta = \frac{(\sigma_x - \sigma_y)\sin 2\theta}{2} - \tau_{xy} \cos 2\theta \quad \dots\dots\dots(1.2)$$

If the value of σ_θ and τ_θ are plotted against θ for a given value of σ_x and σ_y the resulting graphs are as shown below.

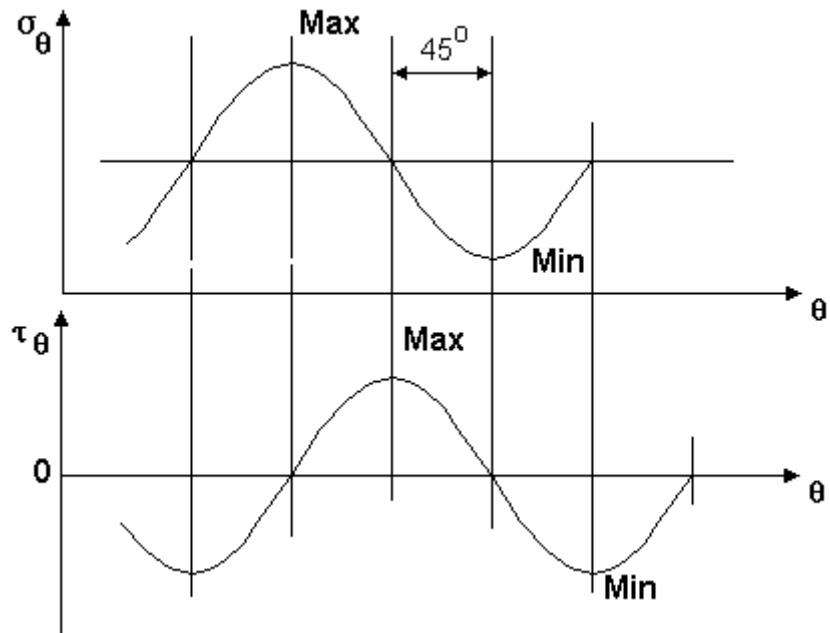


Figure 4

The graphs show that the direct stresses have a maximum and minimum value. These are called the **PRINCIPAL STRESSES**. The principal stresses occur on planes 90° apart. These planes are called the **PRINCIPAL PLANES**. Further, the shear stress has an equal maximum and minimum value on planes 90° apart and 45° from the principal planes. This is of interest because brittle materials fail on these planes. For example, if a brittle material is broken in a tensile test, the fracture occurs on a plane at 45° to the direction of pull indicating that they fail in shear. Further it can be seen that the principal planes have no shear stress so this is a definition of a principle plane.

1.2 DETERMINING THE PRINCIPAL STRESSES AND PLANES

The stresses are a maximum or minimum on the principal planes so using max. and min theory we have

$$\frac{d\sigma_\theta}{d\theta} = -(\sigma_x - \sigma_y)\sin 2\theta + 2\tau \cos 2\theta = 0$$

$$2\tau \cos 2\theta = (\sigma_x - \sigma_y)\sin 2\theta$$

$$\tan 2\theta = \frac{2\tau}{\sigma_x - \sigma_y} \dots\dots\dots(1.3)$$

There are two solutions to this equation giving answers less than 360° and they differ by 90°. From this the angle of the principal plane may be found. If this angle is substituted into equation A and algebraic manipulation conducted the stress values are then the principal stresses and are found to be given as

$$\sigma_{\max} = \sigma_1 = \frac{(\sigma_x + \sigma_y)}{2} + \frac{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}{2} \dots\dots\dots(1.4)$$

$$\sigma_{\min} = \sigma_2 = \frac{(\sigma_x + \sigma_y)}{2} - \frac{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}{2} \dots\dots\dots(1.5)$$

Repeating the process for equation B, the shear stress, we get

$$\tau_{\max} = \frac{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}{2} = \frac{\sigma_1 - \sigma_2}{2} \dots\dots\dots(1.6)$$

$$\tau_{\min} = \frac{-\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}{2} = -\left[\frac{\sigma_1 - \sigma_2}{2}\right] \dots\dots\dots(1.7)$$

WORKED EXAMPLE No. 1

Using the formulae, find the principal stresses for the case shown below and the position of the principal plane.

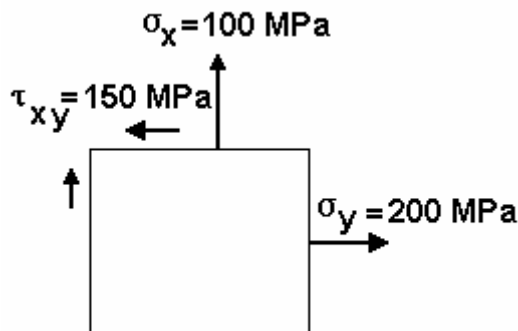


Figure 5

SOLUTION

$$\frac{2\tau}{\sigma_x - \sigma_y} = -\tan 2\theta = \frac{2 \times 150}{200 - 100} = -3$$

$$2\theta = -71.6^\circ \text{ or } 108.4^\circ$$

$$\theta = -35.8^\circ \text{ or } 54.2^\circ$$

Putting this into equations 1.1 and 1.2 we have

$$\sigma_\theta = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)\cos 2\theta}{2} + \tau_{xy} \sin 2\theta$$

$$\sigma_\theta = \frac{(200+100)}{2} + \frac{(200-100)\cos(-71.6^\circ)}{2} + 150\sin(-71.6^\circ)$$

$$\sigma_\theta = 308.1 \text{ MPa}$$

$$\sigma_1 = 308.1 \text{ MPa}$$

Using the other angle.

$$\sigma_\theta = \frac{(200+100)}{2} + \frac{(200-100)\cos(108.4^\circ)}{2} + 150\sin(108.4^\circ)$$

$$\sigma_\theta = -8.1 \text{ MPa}$$

$$\sigma_2 = -8.1 \text{ MPa}$$

$$\tau_\theta = \frac{(\sigma_x - \sigma_y)\sin 2\theta}{2} - \tau_{xy} \cos 2\theta$$

$$\tau_\theta = \frac{(100-200)\sin 71.6^\circ}{2} + 150\cos 71.6^\circ = 0 \text{ (This is as expected)}$$

From equations 1.6 and 1.7 we have

$$\tau_{\max} = (1/2)(\sigma_1 - \sigma_2) = 158.1 \text{ MPa}$$

$$\tau_{\min} = -(1/2)(\sigma_1 - \sigma_2) = -158.1 \text{ MPa}$$

It is easier to use equations 1.4 and 1.5 as follows.

$$\sigma_{\max} = \sigma_1 = \frac{(\sigma_x + \sigma_y)}{2} + \frac{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}{2}$$

$$\sigma_1 = \frac{(200+100)}{2} + \frac{\sqrt{(200-100)^2 + 4 \times 150^2}}{2} = 308.1 \text{ MPa}$$

$$\sigma_{\min} = \sigma_2 = \frac{(200+100)}{2} - \frac{\sqrt{(200-100)^2 + 4 \times 150^2}}{2} = -8.1 \text{ MPa}$$

1.3. MOHR'S CIRCLE OF STRESS

Mohr found a way to represent equations 1.1 and 1.2 graphically. The following rules should be used.

1. Draw point 'O' at a suitable position (which is possible to see with experience)
2. Measure σ_x and σ_y along the horizontal axis using a suitable scale.
3. Find the centre half way between the marked points.
4. Draw τ_{xy} up at x and down at y if positive or opposite if negative.
5. Draw the circle and diagonal.
6. Measure σ_1 , σ_2 , 2θ and τ_{max} .

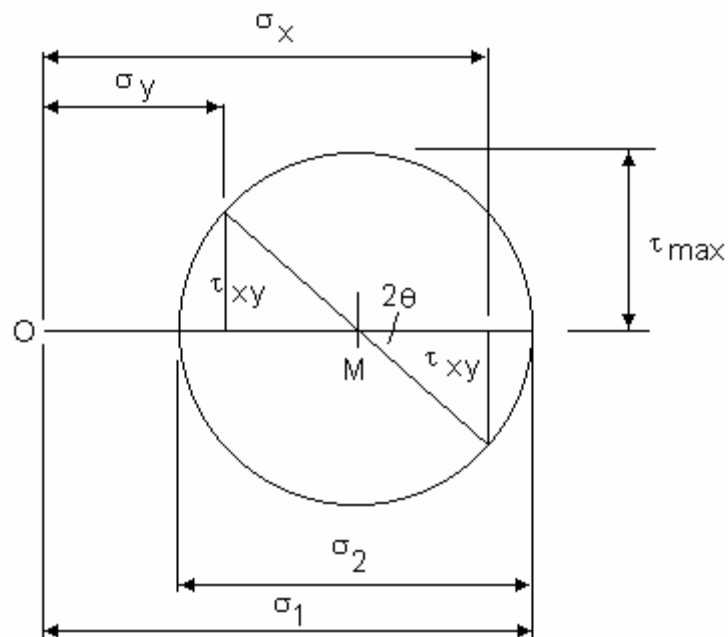


Figure 6

Consider the triangle. The sides are as shown.

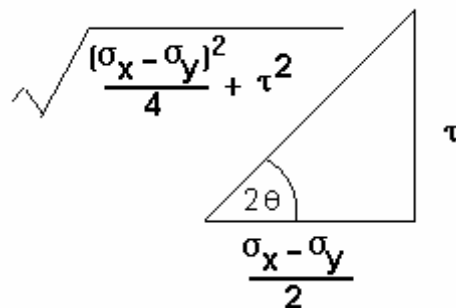


Figure 7

Applying trigonometry we find the same results as before so proving the geometry represents the equations.

$$\tan 2\theta = \frac{2\tau}{\sigma_x - \sigma_y} \dots\dots\dots(1.3)$$

$$\sigma_{\max} = \sigma_1 = \frac{(\sigma_x + \sigma_y)}{2} + \frac{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}{2} \dots\dots\dots(1.4)$$

$$\sigma_{\min} = \sigma_2 = \frac{(\sigma_x + \sigma_y)}{2} - \frac{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}{2} \dots\dots\dots(1.5)$$

WORKED EXAMPLE No.2

Show that the solution in worked example No.1 may be obtained by constructing the circle of stress.

SOLUTION

Following the rules for construction figure 8 is obtained.

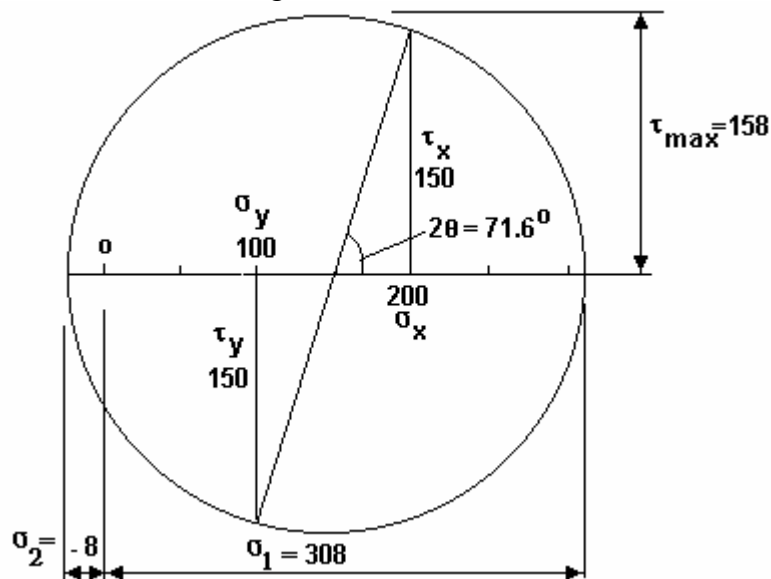


Figure 8

WORKED EXAMPLE No.3

A material has direct stresses of 120 MPa tensile and 80 MPa compressive acting on mutually perpendicular planes. There is no shear stress on these planes. Draw Mohr's circle of stress and determine the stresses on a plane 20° to the plane of the larger stress.

SOLUTION

Since there is no shear stress, σ_x and σ_y are the principal stresses and are at the edge of the circle. Select the origin '0' and plot +120 to the right and -80 to the left.

Draw the circle to encompass these points and mark the centre of the circle.

Using a protractor draw the required plane at 40° to the horizontal (assumed anti-clockwise) through the middle of the circle.

Draw the verticals. Scale the direct stress σ_θ and shear stress τ_θ . These may be checked with the formulae if confirmation is required.

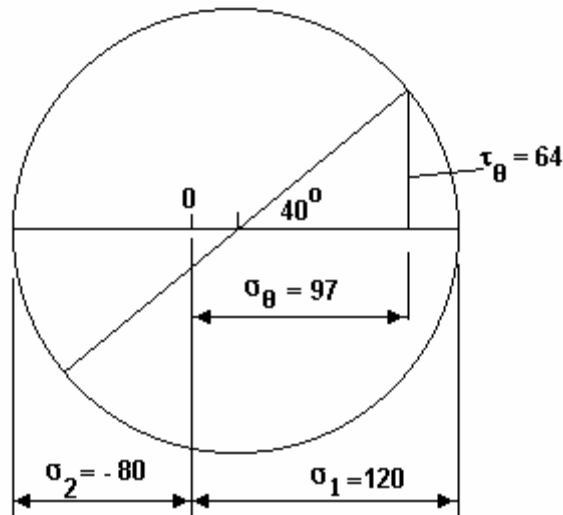


Figure 9

SELF ASSESSMENT EXERCISE No.1

- a) Figure 10 shows an element of material with direct stresses on the x and y planes with no shear stress on those planes. Show by balancing the forces on the triangular element that the direct and shear stress on the plane at angle θ anti-clockwise of the x plane is given by

$$\sigma_{\theta} = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)\cos 2\theta}{2}$$

$$\tau_{\theta} = \frac{(\sigma_x - \sigma_y)\sin 2\theta}{2}$$

Show how Mohr's circle of stress represents this equation.

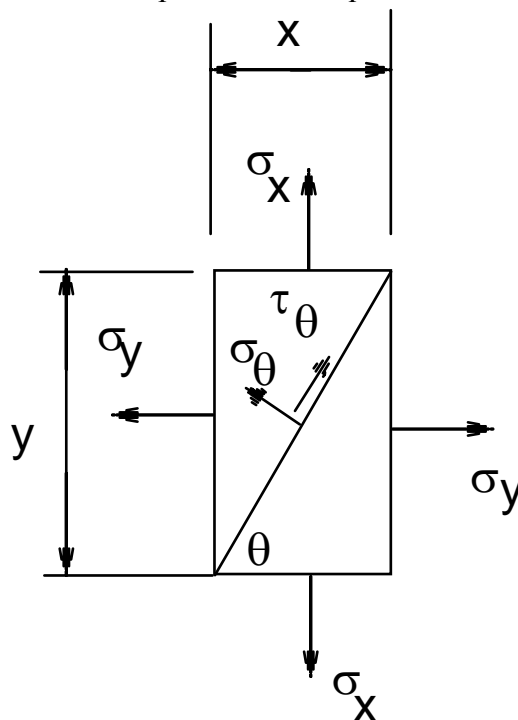


Figure 10

- b). An elastic material is subjected to two mutually perpendicular stresses 80MPa tensile and 40 MPa compressive. Determine the direct and shear stresses acting on a plane 30° to the plane on which the 80 MPa stress acts.

(Hint for solution) The derivation is the same as in the notes but with no shear stress, i.e. the stresses shown on fig.9 are the principal stresses.

(50 MPa and 52 MPa)

2. Define the terms principal stress and principal plane.

A piece of elastic material has direct stresses of 80 MPa tensile and 40 MPa compressive on two mutually perpendicular planes. A clockwise shear stress acts on the plane with the 80 MPa stress and an equal and opposite complementary shear stress acts on the other plane.

The maximum principal stress in the material is 100 MPa tensile. Construct Mohr's circle of stress and determine the following.

i. The shear stress on the planes. (52 MPa)

ii. The maximum shear stress. (80 MPa)

iii. The minimum principal stress. (-60 MPa)

iv. The position of the principal planes. (20° C.W. of 80 MPa direction)

(Hint for solution. Make the larger stress act on the x plane and the other on y plane. Mark off σ_x , σ_y and σ_1 . This is enough data to draw the circle.)

2. COMPLEX STRAIN

2.1 PRINCIPAL STRAINS

In any stress system there are 3 mutually perpendicular planes on which only direct stress acts and there is no shear stress. These are the principal planes and the stresses are the principal stresses. These are designates σ_1 , σ_2 and σ_3 . The corresponding strains are the principal strains ϵ_1 , ϵ_2 and ϵ_3 .

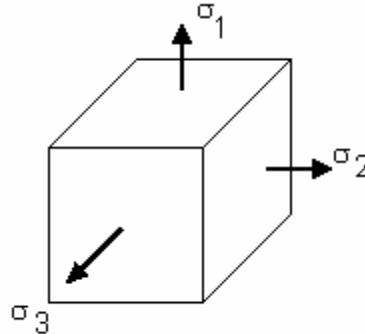


Figure 11

The strain in each direction is given by

$$\epsilon_1 = \frac{1}{E}(\sigma_1 - \nu\sigma_2 - \nu\sigma_3) = \frac{1}{E}[\sigma_1 - \nu(\sigma_2 + \sigma_3)]$$

$$\epsilon_2 = \frac{1}{E}(\sigma_2 - \nu\sigma_1 - \nu\sigma_3) = \frac{1}{E}[\sigma_2 - \nu(\sigma_1 + \sigma_3)]$$

$$\epsilon_3 = \frac{1}{E}(\sigma_3 - \nu\sigma_1 - \nu\sigma_2) = \frac{1}{E}[\sigma_3 - \nu(\sigma_1 + \sigma_2)]$$

Most often we only study 2 dimensional systems in which case $\sigma_3 = 0$ so

$$\epsilon_1 = \frac{1}{E}[\sigma_1 - \nu(\sigma_2)] \dots \dots \dots (2.1)$$

$$\epsilon_2 = \frac{1}{E}[\sigma_2 - \nu(\sigma_1)] \dots \dots \dots (2.2)$$

It is more practical to measure strains and convert them into stresses as follows. From equation 2.2 $E\epsilon_2 + \nu\sigma_1 = \sigma_2$

Substituting for σ_2 in equation 1 we have

$$\epsilon_1 = \frac{1}{E}[\sigma_1 - \nu(E\epsilon_2 + \nu\sigma_1)]$$

$$E\epsilon_1 = [\sigma_1 - \nu E\epsilon_2 + \nu^2\sigma_1]$$

$$E\epsilon_1 = \sigma_1(1 - \nu^2) - \nu E\epsilon_2$$

$$E\epsilon_1 + \nu E\epsilon_2 = \sigma_1(1 - \nu^2)$$

$$\sigma_1 = \frac{E}{1 - \nu^2}(\epsilon_1 + \nu\epsilon_2)$$

Similarly we can show

$$\sigma_2 = \frac{E}{1 - \nu^2}(\epsilon_2 + \nu\epsilon_1)$$

These formulas should be used for converting principal strains into principal stresses.

SELF ASSESSMENT EXERCISE No.2

1. The principal strains acting on a steel component are $12 \mu\epsilon$ and $6 \mu\epsilon$. Determine the principal stresses.

$$E = 205 \text{ GPa} \quad \nu = 0.32 \quad (\text{Ans. } 2.247 \text{ MPa and } 3.179 \text{ MPa})$$

2. The principal strains acting on a steel component are $-100 \mu\epsilon$ and $160 \mu\epsilon$. Determine the principal stresses.

$$E = 205 \text{ GPa} \quad \nu = 0.32 \quad (\text{Ans. } 29.2 \text{ MPa and } -11.14 \text{ MPa})$$

2.2 MOHR'S CIRCLE OF STRAIN

2 dimensional strain may be analysed in much the same way as 2 dimensional stress and the circle of strain is a graphic construction very similar to the circle of stress. First consider how the equations for the strain on any plane are derived.

Consider a rectangle A,B,C,D which is stretched to A',B',C',D' under the action of two principal stresses. The diagonal rotates an angle β from the original direction. The plane under study is this diagonal at angle θ to the horizontal.

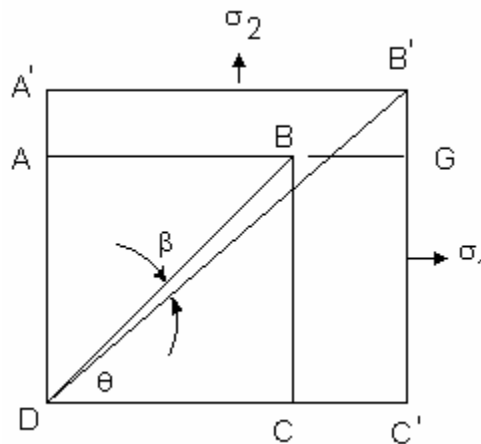


Figure 12

The principal strains are $\epsilon_1 = BE/AB$ so
and $\epsilon_2 = EB'/CB$ so

$$BE = \epsilon_1 AB$$

$$EB' = \epsilon_2 CB$$

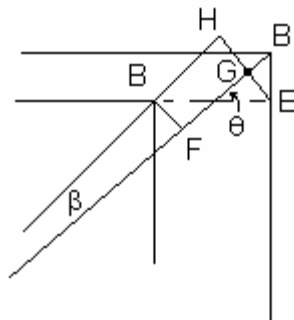


Figure 13

BH is nearly the same length as FG.

$$FB' = FG + GB' = BE \cos\theta + EB' \sin\theta = \epsilon_1 AB \cos\theta + \epsilon_2 CB \sin\theta$$

The strain acting on the plane at angle θ is $\epsilon_\theta = FB'/DB$

$$\epsilon_\theta = \epsilon_1 \cos^2\theta + \epsilon_2 \sin^2\theta$$

$$\epsilon_\theta = \frac{1}{2}(\epsilon_1 + \epsilon_2) + \frac{1}{2}(\epsilon_1 - \epsilon_2) \cos 2\theta \dots\dots\dots(2.3)$$

BF is nearly the same length as HG so

$$BF = HE - GE = BE \sin\theta - EB' \cos\theta$$

$$BF = \epsilon_1 AB \sin\theta + \epsilon_2 CB \cos\theta$$

$$\beta = BF/DB = \epsilon_1 \cos\theta \sin\theta - \epsilon_2 \cos\theta \sin\theta$$

$$\beta = \frac{1}{2}(\epsilon_1 - \epsilon_2) \sin 2\theta \dots\dots\dots(2.4)$$

Equations 2.3 and 2.4 may be compared to the equations for 2 dimensional stress which were

$$\sigma_\theta = \frac{1}{2}(\sigma_1 + \sigma_2) + \frac{1}{2}\{(\sigma_1 - \sigma_2)\} \cos 2\theta \dots\dots\dots(2.5)$$

$$\tau_\theta = \frac{1}{2}\{(\sigma_1 - \sigma_2)\} \sin 2\theta \dots\dots\dots(2.6)$$

It follows that a graphical construction may be made in the same way for strain as for stress. There is one complication. Comparing equations 4 and 6, τ_θ is the shear stress but β is not the shear strain. It can be shown that the rotation of the diagonal (in radians) is in fact half the shear strain on it (γ) and negative so equation 4 becomes

$$-\frac{1}{2} \gamma = \frac{1}{2}(\epsilon_1 - \epsilon_2) \sin 2\theta \dots\dots\dots(2.7)$$

These equations may be used to determine the strains on two mutually perpendicular planes x and y by using the appropriate angle. Alternatively, they may be solved by constructing the circle of strain.

2.3 CONSTRUCTION

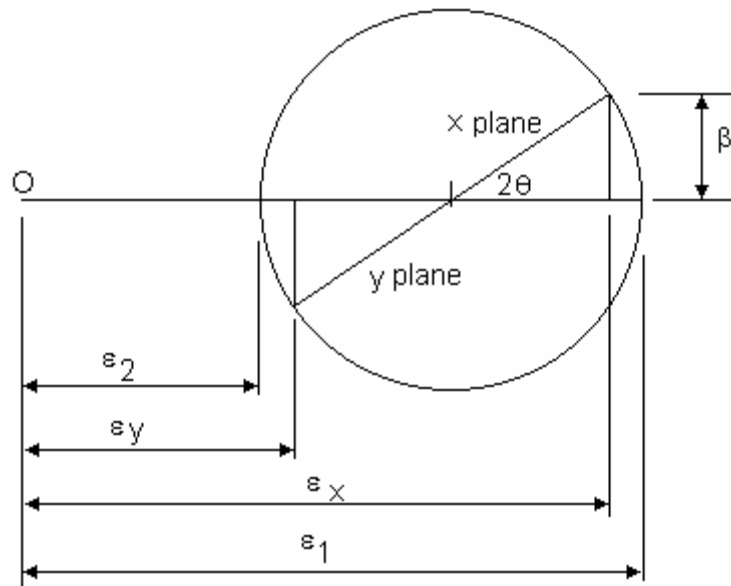


Figure 14

1. Draw point 'O' at a suitable position (which is possible to see with experience)
2. Measure ϵ_1 and ϵ_2 along the horizontal axis using a suitable scale.
3. Find the centre half way between the marked points.
4. Draw the circle.
5. Draw in the required plane at the double angle.
6. Measure ϵ_x and ϵ and β .

SELF ASSESSMENT EXERCISE No.3

1. The principal strains in a material are $500 \mu\epsilon$ and $300 \mu\epsilon$. Calculate the direct strain and shear strain on a plane 30° anti clockwise of the first principal strain.
(Answers $450 \mu\epsilon$ and $-173.2 \mu\epsilon$.)
2. Repeat the problem by constructing the circle of strain.
3. The principal strains in a material are $600 \mu\epsilon$ and $-200 \mu\epsilon$. Determine the direct strain and shear strain on a plane 22.5° clockwise of the first principal strain.
(Answers $480 \mu\epsilon$ and $-800 \mu\epsilon$.)

3. STRAIN GAUGE ROSETTES

It is possible to measure strain but not stress. Strain gauges are small surface mounting devices which, when connected to suitable electronic equipment, enable strain to be measured directly.

In order to construct a circle of strain without knowing the principal strains, we might expect to use the strains on two mutually perpendicular planes (x and y) and the accompanying shear strain. Unfortunately, it is not possible to measure shear strain so we need three measurements of direct strain in order to construct a circle. The three strain gauges are conveniently manufactured on one surface mounting strip and this is called a strain gauge rosette. There are two common forms. One has the gauges at 45° to each other and the other has them at 60° to each other. The method for drawing the circle of strain is different for each.

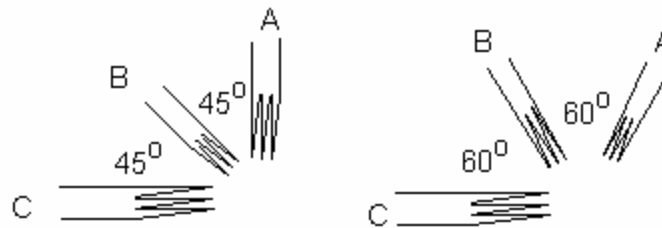


Figure 15

We shall label the gauges A, B and C in an anti clockwise direction as shown.

2.1 CONSTRUCTION OF THE 45° CIRCLE

We shall use a numerical example to explain the construction of the circle. Suppose the three strains are $\epsilon_A = 700 \mu\epsilon$ $\epsilon_B = 300 \mu\epsilon$ $\epsilon_C = 200 \mu\epsilon$

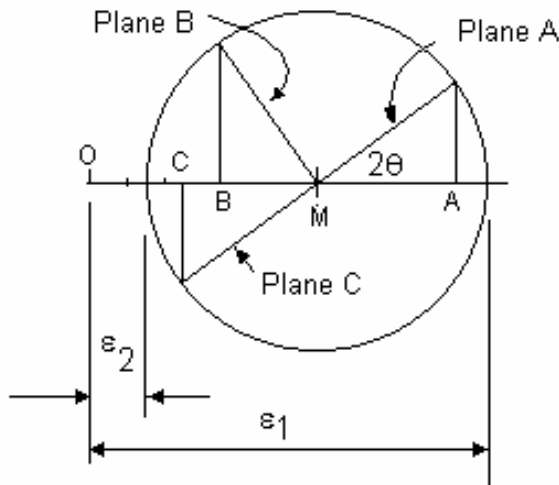


Figure 16

1. Choose a suitable origin O
2. Scale off horizontal distances for ϵ_A , ϵ_B and ϵ_C and mark them as A, B and C.
3. Mark the centre of the circle M half way between A and C.
4. Construct vertical lines through A, B and C
5. Measure distance BM
6. Draw lines A A' and C C' equal in length to BM
7. Draw circle centre M and radius M A' and M C'
8. Draw B B'

Scaling off the values we find $\epsilon_1 = 742 \mu\epsilon$, $\epsilon_2 = 158 \mu\epsilon$ and the angle $2\theta = 30^\circ$
The first principal plane is hence 15° clockwise of plane A.

SELF ASSESSMENT EXERCISE No.4

1. The results from a 45° strain gauge rosette are

$$\varepsilon_A = 200 \mu\varepsilon$$

$$\varepsilon_B = -100 \mu\varepsilon$$

$$\varepsilon_C = 300 \mu\varepsilon$$

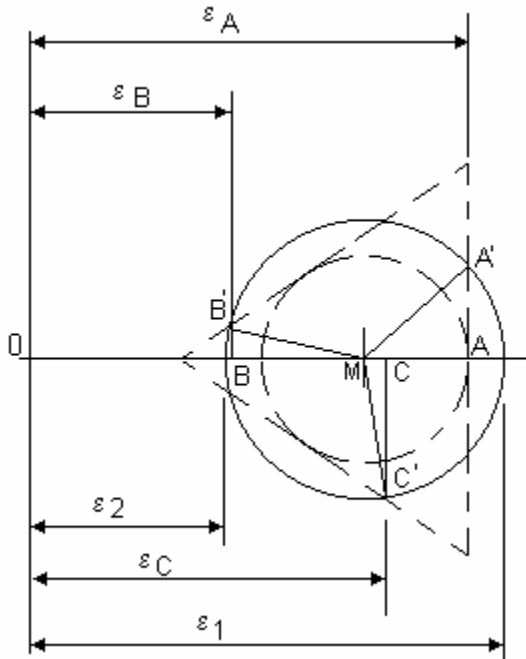
Draw the strain circle and deduce the principal strains. Determine the position of the first principal plane. Go on to convert these into principal stresses given $E = 205 \text{ GPa}$ and $\nu = 0.29$.

(Answers $\varepsilon_1 = 603 \mu\varepsilon$ $\varepsilon_2 = -103 \mu\varepsilon$ $\sigma_1 = 128.3 \text{ MPa}$ $\sigma_2 = 16 \text{ MPa}$

The first principal plane is 49° clockwise of plane A)

3.2

CONSTRUCTION OF A 60° ROSETTE



1. Select a suitable origin O
2. Scale of A, B and C to represent the three strains.
3. Calculate $OM = (A + B + C)/3$
4. Draw the inner circle radius MA
5. Draw the triangle (60° each corner)
6. Draw outer circle passing through B' and C'.
7. Make sure that the planes are A,B and C in an anti clockwise direction because you can obtain an upside down version for clockwise directions.
8. Scale off principal strains.

Figure 17

SELF ASSESSMENT EXERCISE No.5

1. The results from a 60° strain gauge rosette are

$$\epsilon_A = 700 \mu\epsilon$$

$$\epsilon_B = 200 \mu\epsilon$$

$$\epsilon_C = -100 \mu\epsilon$$

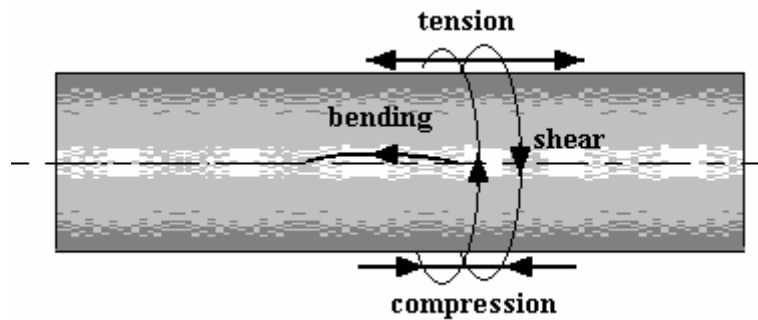
Draw the strain circle and deduce the principal strains. determine the position of the first principal plane. Go on to convert these into principal stresses given $E = 205 \text{ GPa}$ and $\nu = 0.25$.

(Answers $\epsilon_1 = 733 \mu\epsilon$ $\epsilon_2 = -198 \mu\epsilon$ $\sigma_1 = 149.4 \text{ MPa}$ $\sigma_2 = -3.6 \text{ MPa}$

The first principal plane is 11° anti clockwise of plane A)

4. COMBINED BENDING, TORSION AND AXIAL LOADING.

When a material is subjected to a combination of direct stress, bending and torsion at the same time, we have a complex stress situation. A good example of this is a propeller shaft in which torsion is produced. If in addition some misalignment of the bearings, the shaft will bend rotates. If a snap shot is one side of the shaft will be tension and one in compression. The shear direction depends upon the direction of the torque being transmitted.



there is
as it
taken,
in
stress

Figure 18

The bending and shear stresses on their own are a maximum on the surface but they will combine to produce even larger stresses. The maximum stress in the material is the principal stress and this may be found with the formulae or by constructing Mohr's circle. In earlier work it was shown that the maximum direct and shear stress was given by the following formula.

$$\sigma_{\max} = \sigma_1 = \frac{(\sigma_x + \sigma_y)}{2} + \frac{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}{2} \quad \tau_{\max} = \frac{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}{2}$$

If there is only one direct stress in the axial direction σ_x and an accompanying shear stress τ (assumed positive), then putting $\sigma_y = 0$ we have the following.

$$\sigma_{\max} = \frac{(\sigma)}{2} + \frac{\sqrt{(\sigma)^2 + 4\tau^2}}{2} \quad \tau_{\max} = \frac{\sqrt{(\sigma)^2 + 4\tau^2}}{2}$$

If the axial stress is only due to bending, then $\sigma = \sigma_B$. From the bending and torsion equations we have formula for σ_B and τ as follows.

$$\sigma_B = \frac{My}{I} = \frac{MD}{2I} = \frac{32M}{\pi D^3} \text{ and } \tau = \frac{TR}{J} = \frac{TD}{2J} = \frac{16T}{\pi D^3}$$

Substituting for σ and τ we get the following result.

$$\sigma_{\max} = \frac{16}{\pi D^3} \left[M + \sqrt{T^2 + M^2} \right] \text{ and } \tau_{\max} = \frac{16}{\pi D^3} \sqrt{T^2 + M^2}$$

It was shown earlier that the angle of the principal plane could be found from the following formula. $\tan 2\theta = \frac{2\tau}{\sigma_x - \sigma_y}$. Putting $\sigma_x = \sigma$ and $\sigma_y = 0$ this becomes $\tan 2\theta = \frac{2\tau}{\sigma}$. If there is

only bending stress and torsion we may substitute for σ_B and τ as before, we get the following formula.

$$\tan 2\theta = \frac{T}{M}$$

If the direct stress is due to bending and an additional axial load, (e.g. due to the propeller pushing or pulling), the direct stresses should be added together first to find σ as they are in the same direction. You could draw Mohr's circle to solve these problems or use the appropriate formulae.

WORKED EXAMPLE No.4

A propeller shaft has a bending stress of 7 MPa on the surface. Torsion produces a shear stress of 5 MPa on same point of the surface. The propeller pushes and puts a compressive stress of 2 MPa in the shaft.

Determine the following.

- The principal stresses on the surface.
- The position of the principal plane.

SOLUTION

Since we have stress values, the problem is best solved by drawing Mohr's circle. At the point considered we have two a direct stresses and a shear stress. The total direct stress is $7 - 2 = 5$ MPa. Let this be σ_x and let the shear stress be positive on this plane. σ_y will be zero and the shear stress will be negative on the y plane.

Construction of the circle yields principle stresses of 8.09 and -3.09 MPa. The principle plane is 31.7° clockwise of the x plane.

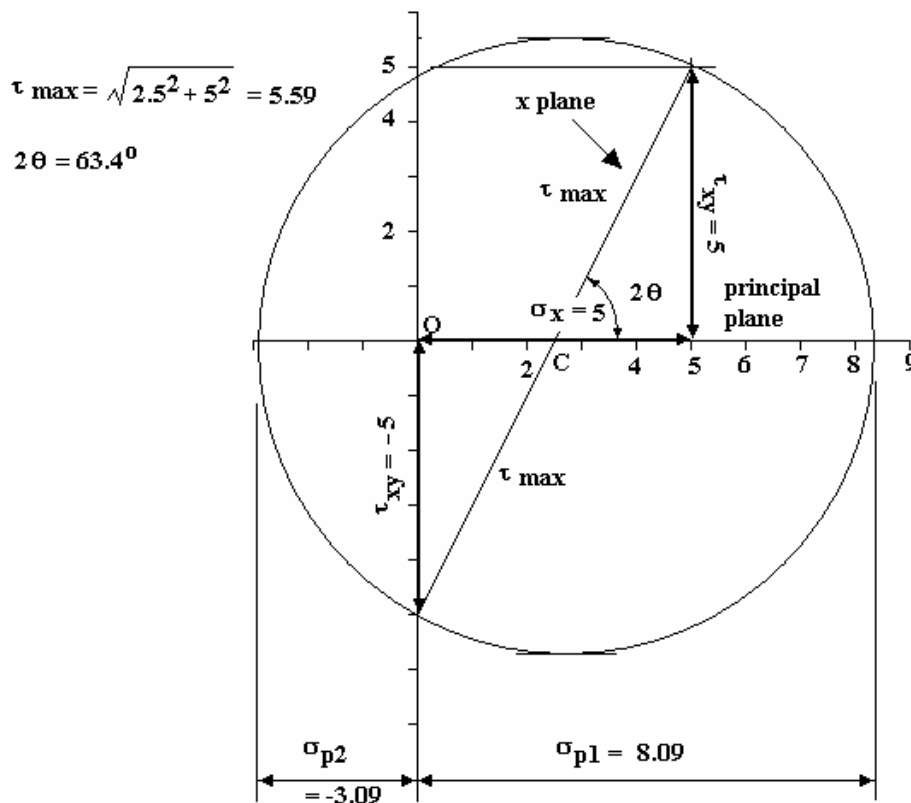


Figure 19

Check the answers by using the formulae.

$$\sigma_{\max} = \frac{(\sigma)}{2} + \frac{\sqrt{(\sigma)^2 + 4\tau^2}}{2} = \frac{5}{2} + \frac{\sqrt{5^2 + 4 \times 5^2}}{2} = 8.09 \text{ MPa}$$

$$\tau_{\max} = \frac{\sqrt{(\sigma)^2 + 4\tau^2}}{2} = \frac{\sqrt{5^2 + 4 \times 5^2}}{2} = 5.59 \text{ MPa}$$

$$\tan 2\theta = \frac{2\tau}{\sigma} = \frac{2 \times 5}{5} = 2 \quad 2\theta = 63.43^\circ \quad \theta = 31.7^\circ$$

WORKED EXAMPLE No.5

A solid circular shaft 100 mm diameter is subjected to a bending moment of 300 Nm and a Torque of 400 Nm. Calculate the maximum direct stress and shear stress in the shaft.

SOLUTION

As there is no other axial load we can use the simpler formulae.

$$\sigma_{\max} = \frac{16}{\pi D^3} \left[M + \sqrt{T^2 + M^2} \right] = \frac{16}{\pi (0.1)^3} \left[300 + \sqrt{400^2 + 300^2} \right]$$

$$\sigma_{\max} = 4.07 \text{ MPa}$$

$$\tau_{\max} = \frac{16}{\pi D^3} \sqrt{T^2 + M^2} = \tau_{\max} = \frac{16}{\pi (0.1)^3} \sqrt{400^2 + 300^2}$$

$$\tau_{\max} = 2.55 \text{ MPa}$$

$$\tan 2\theta = \frac{T}{M} = \frac{400}{300} = 1.333 \quad 2\theta = 53.13^\circ \quad \theta = 26.6^\circ$$

SELF ASSESSMENT EXERCISE No.5

1. A shaft is subjected to bending and torsion such that the bending stress on the surface is 80 MPa and the shear stress is 120 MPa. Determine the maximum stress and the direction of the plane on which it occurs relative to the axis of the shaft.

(166.5 MPa 35.8° clockwise of the axis)

2. A solid circular shaft 120 mm diameter is subjected to a torque of 2000 Nm and a bending moment of 8000 Nm. Calculate the maximum direct and shear stress and the angle of the principal plane.

(47.9 MPa, 24.3 MPa and 7°)

3. A solid circular shaft 60 mm diameter transmits a torque of 50 kNm and has a bending moment of 9 kNm. Calculate the maximum direct and shear stress and the angle of the principal plane.

(1.41 MPa, 1.2 MPa and 39.9°)

4. A solid circular shaft is 90 mm diameter. It transmits 80 kNm of torque and in addition there is a tensile axial force of 8 kN and a bending moment of 30 kNm. Calculate the following.

- i. The maximum stress due to bending alone. (41.9 MPa)
- ii. The shear stress due to torsion alone. (55.9 MPa)
- iii. The stress due to the axial force alone. (1.26 MPa)
- iv. The maximum direct stress. (81.5 MPa)
- v. The maximum shear stress. (59.9 MPa)
- vi. The position of the principal plane. (34.4° relative to the axis)