# SOLID MECHANICS
# STATICS
# STRESS PART 1
# DIRECT STRESS AND STRAIN

**This is set at the British Edexcel National level NQF 3.**

Learning Outcome 4 Part 1

On completion of this tutorial you should be able to explain and define basic direct stress and strain and apply it to loaded structures. You should be able to define the elastic properties of materials and use it to solve some complex stress problems. In part 2 you will extend this to shear stress and strain.

**Contents**



## *1. Introduction*

This tutorial is about stress and strain in loaded structures and components made from elastic materials. The tutorial starts with the basics and progresses to more advanced work. The solutions to the self assessment exercises are given separately but require a small donation to access them.

Typical values for the various material properties mentioned in this tutorial may be found in tables on the home page for www.freestudy.co.uk.

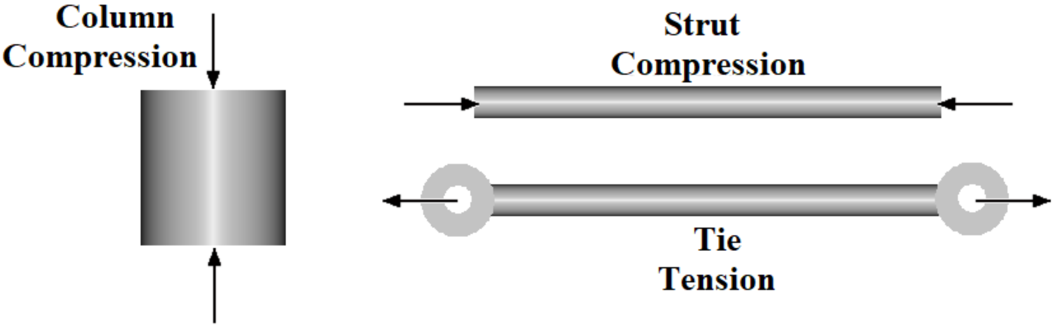
2. ***Static Forces and Structures***

***Static Forces*** will deform a body in one or more of the following ways.

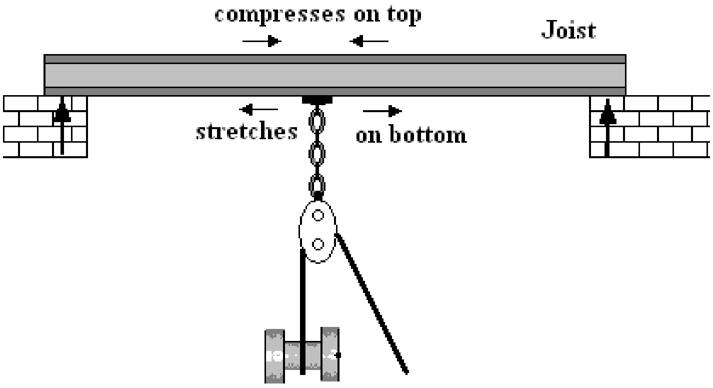
- The force may stretch the body, in which case it is called a ***Tensile Force***.

- The force may squeeze the body in which case it is called a ***Compressive Force***.

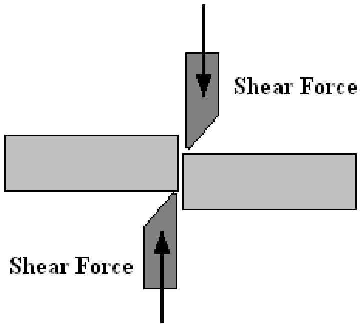
A rod or rope used in a frame to take a tensile load is called a ***Tie***. If it takes a compressive force it is called a ***Strut***. A strut that is thick compared to its length is called a ***Column***.



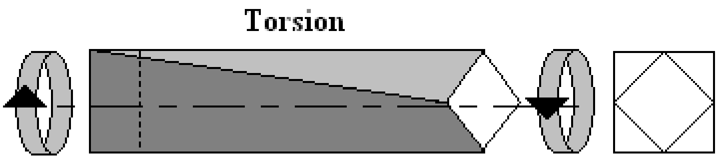

- The force may bend the body in which case both tensile and compressive forces may occur. A structure used to support a bending load is called a ***Beam*** or ***Joist***.




- The force may try to shear the body in which case the force is called a ***SHEAR FORCE.*** A scissors or guillotine produces shear forces.

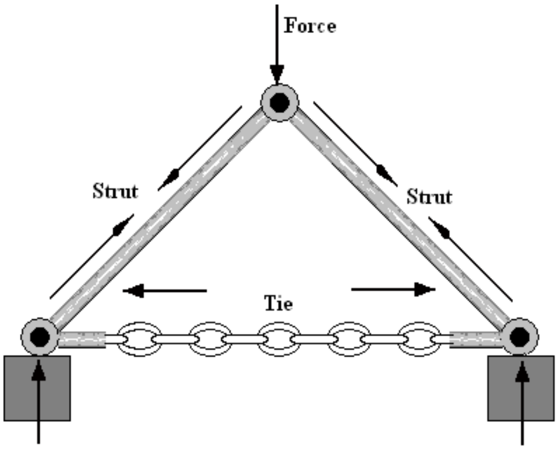



- The force may twist the body in which case ***Shear Forces*** occur. A structure that transmits rotation is called a ***Shaft*** and it experiences ***Torsion***.



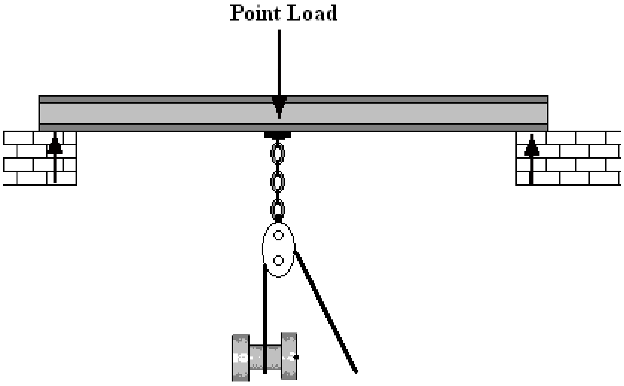
3. ***Frames***

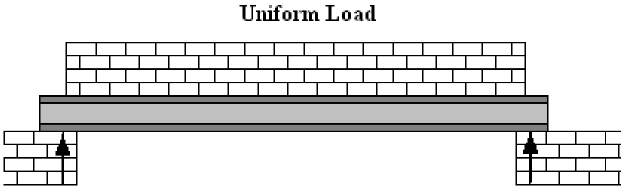
Struts and ties make up the members of lattice frames such as the simple one shown. The two side members are compressed and so are struts but the bottom one is stretched and is a tie. This could be a chain.



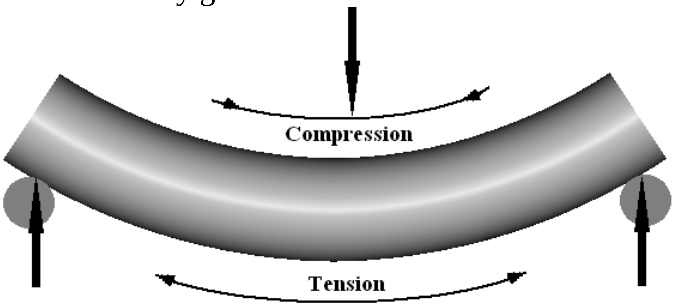

***4. Beams***

A beam is a structure, which is loaded transversely (sideways). The loads may be point loads or uniformly distributed loads (udl). The diagrams show the way that point loads and uniform loads are illustrated.




Transverse loading causes bending and bending is a very severe form of stressing a structure. The bent beam goes into tension (stretched) on one side and compression on the other. The transverse loads also produce shear stress on any given section.



## 5. *Safety Factors*

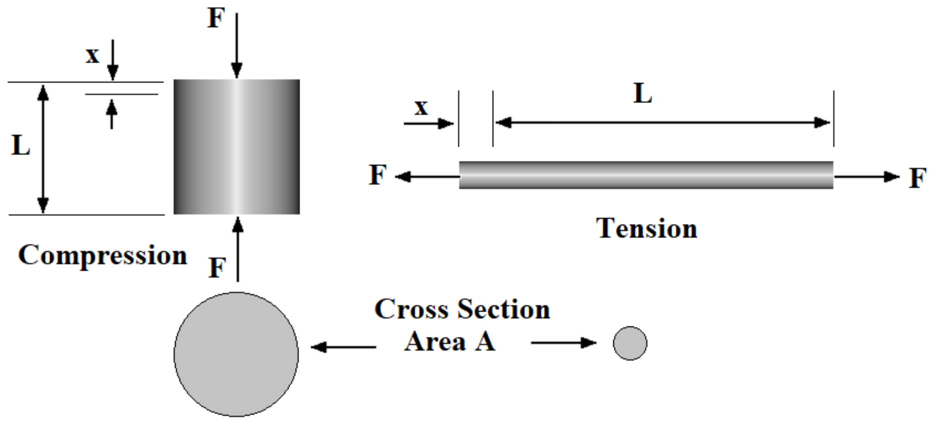
In the following sections you will learn how to calculate the stress in some basic engineering structures and components. If we want to be confident that the structure or component does not fail by being overstressed, we design it so that working stress is smaller than the stress at which it fails. If we want to define what failure means, we would need to study stress and strain and material science to a much greater depth than covered by this unit. Failure might mean that the material has yielded or that it has broken. Also calculation of maximum stress in a component is much more complicated than the cases studies here. If a material has direct stress and shear stress at the same time, the maximum stress may well exceed either of them on their own. Also the stress may be raised by things like sharp corners and undercuts. Other factors such as fatigue and creep also affect the stress. Don't think that this tutorial is anything but the start of your studies in that area.

If we decide what the fail stress is for a given material, then we design the component so that the working stress is less. We define the safety factor as the ratio such that:

$$\text{Safety Factor} = \frac{\text{Stress at Failure}}{\text{Maximum Working Stress}}$$

## 6. *Direct Stress σ*

When a force is applied to an elastic body, the body deforms. The way in which the body deforms depends upon the type of force applied to it. A compression force makes the body shorter. A tensile force makes the body longer.




Tensile and compressive forces are called ***Direct Forces.***

Stress is the force per unit area upon which it acts. **Stress = σ = Force/Area** N/m$^2$ or Pascals.

The symbol **σ** is called *Sigma*

***Note On Units*** **- The fundamental unit of stress is 1 N/m$^2$ and this is called a Pascal. This is a small quantity in most fields of engineering so we use the multiples kPa, MPa and GPa.**

**Areas may be calculated in mm$^2$ and units of stress in N/mm$^2$ are quite acceptable.**
**Since 1 N/mm$^2$ converts to 1 000 000 N/m$^2$ then it follows that the N/mm$^2$ is the same as a MPa**

## 7. *Direct Strain ε*

In each case, a force F produces a deformation x. In engineering we usually change this force into stress and the deformation into strain and we define these as follows.

Strain is the deformation per unit of the original length

$$\textbf{Strain} = \varepsilon = \frac{\mathbf{x}}{\mathbf{L}}$$

The symbol **ε** is called *Epsilon*

Strain has no units since it is a ratio of length to length. Most engineering materials do not stretch very much before they become damaged so strain values are very small figures. It is quite normal to change small numbers in to the exponent for of $10^{-6}$. Engineers use the abbreviation με (micro strain) to denote this multiple.

For example a strain of 0.000068 could be written as 68 x $10^{-6}$ but engineers would write 68 με.

**Note that when conducting a British Standard tensile test the symbols for original area are $S_o$ and for Length is $L_o$.**

**WORKED EXAMPLE No. 1**

A metal wire is 2.5 mm diameter and 2 m long. A force of 1200 N is applied to it and it stretches 0.3 mm. Assume the material is elastic. Determine the following.

i. The stress in the wire (σ).
ii. The strain in the wire (ε).

If the stress that produces failure is 300 MPa, calculate the safety factor.

**SOLUTION**

$$A = \frac{\pi d^2}{4} = \frac{\pi \text{ x } 2.5^2}{4} = 4.909 \text{ mm}^2 \quad \sigma = \frac{F}{A} = \frac{1\,200}{4.909} = 244 \text{ N/mm}^2$$

Answer (i) is hence 244 MPa

$$\varepsilon = \frac{x}{L} = \frac{0.3 \text{ mm}}{2\,000 \text{ mm}} = 0.00015 \text{ or } 150\ \mu\varepsilon$$

Safety Factor = 300/244 = 1.23

**SELF ASSESSMENT EXERCISE No. 1**

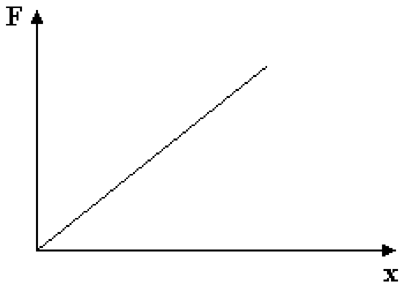
1. A steel bar is 10 mm diameter and 2 m long. It is stretched with a force of 20 kN and extends by 0.2 mm. Calculate the stress and strain. If the stress at failure is 300 MPa calculate the safety factor.
   (Answers 254.6 MPa, 100 $\mu\varepsilon$ and 1.18)

2. A rod is 0.5 m long and 5 mm diameter. It is stretched 0.06 mm by a force of 3 kN. Calculate the stress and strain. If the stress at failure is 250 MPa calculate the safety factor.
   (Answers 152.8 MPa, 120$\mu\varepsilon$ and 1.64)

## 8. *Modulus of Elasticity E*

Elastic materials always spring back into shape when released. They also obey ***Hooke's Law***. This is the law of a spring which states that deformation is directly proportional to the force.

$$F/x = \text{stiffness} = k \text{ N/m}$$




The stiffness is different for different materials and different sizes of the material. We may eliminate the size by using stress and strain instead of force and deformation as follows.

If F and x refer to direct stress and strain then

$F = \sigma A \quad x = \varepsilon L$ hence

$$\frac{F}{x} = \frac{\sigma A}{\varepsilon L} \text{ and } \frac{FL}{Ax} = \frac{\sigma}{\varepsilon}$$

The stiffness is now in terms of stress and strain only and this constant is called the ***Modulus of Elasticity*** **and it has a symbol E.**

$$E = \frac{FL}{Ax} = \frac{\sigma}{\varepsilon}$$

A graph of stress against strain will be a straight line with a gradient of E. The units of E are the same as the units of stress.

9. ***Ultimate Tensile Stress***

If a material is stretched until it breaks, the tensile stress has reached the absolute limit and this stress level is called the ultimate tensile stress.

**WORKED EXAMPLE No. 2**

A steel tensile test specimen has a cross sectional area of 100 $mm^2$ and a gauge length of 50 mm. In a tensile test the graph of force - extension produces a gradient in the elastic section of 410 x $10^3$ N/mm. Determine the modulus of elasticity.

The specimen breaks when the force is 35 kN. What is the ultimate tensile stress based on the original area?

**SOLUTION**

The gradient gives the ratio F/A = and this may be used to find E.

$$E = \frac{\sigma}{\varepsilon} = \frac{FL}{Ax} = 410 \text{ x} 10^3 \text{x} \frac{50}{100} = 205\ 000 \frac{N}{mm^2} \text{ or } 205\ 000 \text{ MPa or } 205 \text{ GPa}$$

$$\sigma_u = \frac{\text{Breaking Force}}{\text{Area}} = \frac{35\ 000}{100} = 350 \frac{N}{mm^2} \text{ or } 350 \text{ MPa}$$

**WORKED EXAMPLE No. 3**

A Steel column is 3 m long and 0.4 m diameter. It carries a load of 50 MN. Given that the modulus of elasticity is 200 GPa, calculate the compressive stress and strain and determine how much the column is compressed.

**SOLUTION**

$$A = \frac{\pi d^2}{4} = \frac{\pi \text{ x } 0.4^2}{4} = 0.126 \text{ m}^2 \quad \sigma = \frac{F}{A} = \frac{50 \text{ x } 10^6}{0.126} = 397.9 \text{ x } 10^6 \text{ Pa}$$

$$E = \frac{\sigma}{\varepsilon} \text{ so } \varepsilon = \frac{\sigma}{E} = \frac{397.9 \text{ x } 10^6}{200 \text{ x } 10^9} = 0.001989$$

$$\varepsilon = \frac{x}{L} \text{ so } x = \varepsilon L = 0.001989 \text{ x } 3\ 000 = 5.97 \text{ mm}$$

**SELF ASSESSMENT EXERCISE No. 2**

1. A bar is 500 mm long and is stretched to 500.45 mm with a force of 15 kN. The bar is 10 mm diameter.

   Calculate the stress and strain.

   Assuming that the material has remained within the elastic limit, determine the modulus of elasticity.

   If the fail stress is 250 MPa calculate the safety factor.

   (Answers 191 MPa, 900με , 212.2 GPa and 1.31.)

2. The maximum safe stress in a steel bar is 300 MPa and the modulus of elasticity is 205 GPa. The bar is 80 mm diameter and 240 mm long. If a factor of safety of 2 is to be used, determine the following.
   i. The maximum allowable stress. (150 MPa)
   ii. The maximum tensile force allowable. (754 kN)
   iii. The corresponding strain at this force. (731.7 με)
   iv. The change in length. (0.176 mm)

3. A circular metal column is to support a load of 500 Tonne and it must not compress more than 0.1 mm. The modulus of elasticity is 210 GPa. The column is 2 m long. Calculate the following.
   i. The cross sectional area(0.467 $m^2$)
   ii. The diameter. (0.771 m)

   Note 1 Tonne is 1000 kg.

## 2. *Temperature Stresses*

Metals expand when heated. This can be put to good use. For example a ring may be expanded by warming it and then fitted onto a shaft and on cooling grips the shaft very tightly.

Thermal expansion can also produce unwanted stresses in structures. For example, suddenly allowing hot fluid into a badly designed pipe could cause it to fracture as it tries to get longer but is prevented from doing so.

### *Coefficient of Linear Expansion*

All engineering materials expand when heated and this expansion is usually equal in all directions. If a bar of material of length L has its temperature increased by $\Delta\theta$ degrees, the increase of length $\Delta L$ is directly proportional to the original length L and to the temperature change $\Delta\theta$. Hence

$$\Delta L = \text{constant} \times L\,\Delta\theta$$

The constant of proportionality is called the coefficient of linear expansion ($\alpha$).

$$\Delta L = \alpha\, L\, \Delta\theta$$

### *Induced Stress in a Constrained Bar*

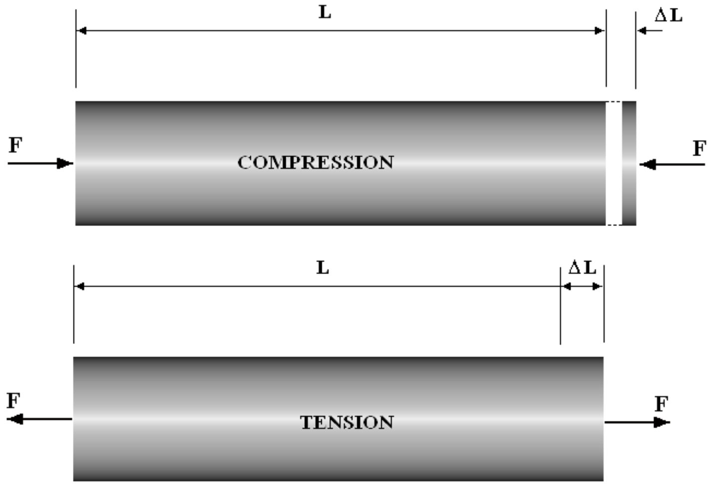
When a material is heated and not allowed to expand freely, stresses are induced which are known as "temperature stresses." Suppose the bar was allowed to expand freely by distance $\Delta L$ and then changed back to its original length. The strain is then

$$\varepsilon = \frac{\Delta L}{L} = \frac{\alpha L\,\Delta\theta}{L} = \alpha\,\Delta\theta$$

Since stress/strain = modulus of elasticity (E) then the induced stress is

$$\sigma = E\,\varepsilon = E\,\alpha\,\Delta\theta$$

This may be a tensile stress or a compressive stress depending whether the bar was pulled back to its original length or pushed back.



**WORKED EXAMPLE No. 4**

A thin steel band 850 mm diameter must be expanded to fit around a disc 851 mm diameter. Calculate the temperature change needed and the stress produced in the ring. The coefficient of linear expansion is 15 x $10^{-6}$ per $^oC$ and the modulus of elasticity E is 200 GPa.

**SOLUTION**

Initial circumference of ring = $\pi D = \pi \times 850$ = 2 670.35 mm
Required circumference = $\pi \times 851$ = 2 673.50 mm

$$\Delta L = 2\ 673.5 - 2\ 670.35 = 3.15 \text{ mm}$$

$$3.15 = 15 \times 10^{-6} \times 2\ 670.35 \times \Delta\theta$$

$$\Delta\theta = \frac{3.15}{15 \times 10^{-6} \times 2\ 670.35} = 78.6 \text{ K}$$

$$\sigma = E\alpha\,\Delta\theta = 200 \times 10^9 \times 15 \times 10^{-6} \times 78.6 = 235.8 \text{ MPa}$$

Alternatively

$$\varepsilon = \frac{\Delta L}{L} = \frac{3.15}{2\ 670.35} = 0.011796$$

$$\sigma = E\varepsilon = 200 \times 10^9 \times 0.011796 = 235.8 \text{ MPa}$$

**WORKED EXAMPLE No. 5**

A brass bar is 600 mm long and it is turned on a centre lathe to 100 mm diameter. It is held between the chuck jaws and a running tail stock so that it is not free to expand. During the turning process it has become heated from 20 $^oC$ to 95$^oC$. Calculate the thermal stress induced in the bar and the resulting thrust on the chuck and tail stock.
E for brass is 90 GPa and $\alpha$ is $18 \times 10^{-6}$ per $^oC$.

**SOLUTION**

Stress induced = $\sigma = E\alpha\,\Delta\theta = 90 \times 10^9 \times 18 \times 10^{-6} \times (95 - 20)$ = 121.5 MPa
Force = stress $\times$ cross sectional area = $(121.5 \times 10^6 \times \pi \times 0.1^2) \div 4$ = 954.3 kN

**WORKED EXAMPLE No. 6**
Determine the induced stress and thrust if the centre lathe flexed so that the bar changed length by 0.6 mm.

**SOLUTION**

Check your solution here. The free expansion of the bar is
$\Delta L = \alpha\, L\, \Delta\theta = 18 \times 10^{-6} \times 600 \times (95 - 20)$ = 0.81 mm

Actual change in length is 0.6 mm.
Strain induced = change in length/original length = (0.81 – 0.6)/600 = 0.00035
Stress induced $\sigma = E\,\varepsilon = 90 \text{ x } 10^9 \text{ x } 0.00035$ = 31.5 MPa

Force = stress $\times$ cross sectional area = $(31.5 \times 10^6 \times \pi \times 0.01^2) \div 4$ = 247.4 kN

**SELF ASSESSMENT EXERCISE No. 3**

1. A steel ring is 50 mm inside diameter. It must be fitted onto a shaft 50.1 mm diameter. Calculate the temperature to which it must be heated in order to fit on the shaft. The initial temperature is 20 $^oC$ and the coefficient of linear expansion is $15 \times 10^{-6}$ per $^oC$.

   (Answer 133.3 K)

2. A stub shaft 85.2 mm diameter must be shrunk to 85 mm diameter in order to insert it into housing. By how much must the temperature be reduced? Take the coefficient of linear expansion is $12 \times 10^{-6}$ per $^oC$.

   (Answer -195.6 K)

3. A steam pipe is 120 mm outer diameter and 100 mm inner diameter. It has a length of 30 m and passes through a wall at both ends where it is rigidly constrained. Steam at 200$^oC$ is suddenly released into the pipe. The initial temperature of the pipe is 15 $^oC$ and the coefficient of linear expansion is $15 \times 10^{-6}$ per $^oC$. E is 200 GPa.

   Calculate

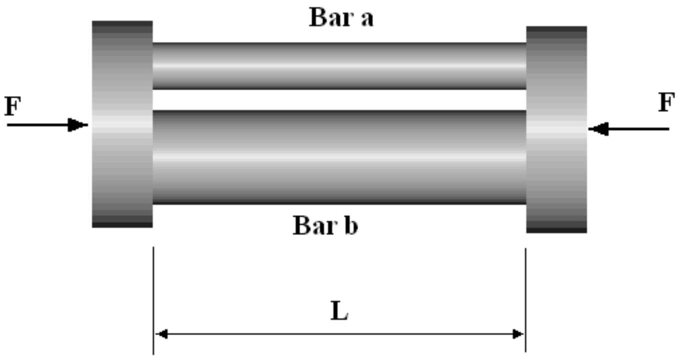
   i. the thermal stress produced. (555 MPa)

   ii. the force exerted by the pipe against the walls. (1.918 MN)

11. ***Composite Bars***

***Parallel***

A parallel compound bar may take many forms but essentially consists of two or more different materials that are arranged side by side so that they are strained by the same amount. A common example is reinforced concrete which has concrete and steel of the same length. When stressed, they change length by the same amount. Other examples occur in engineering structures such as rods with a tube around them. Consider two bars 'a' and 'b' made from two materials and squashed by a force F as shown. The Flanges at the ends are to spread the force between the two.




The two parts are the same length throughout and both parts are hence strained the same amount. Let the original length be L and the change in length be ΔL.

Strain = ε

$$\varepsilon_a = \frac{\Delta L}{L} \text{ for material a} \qquad \varepsilon_b = \frac{\Delta L}{L} \text{ for material b}$$

The cross areas are $A_a$ and $A_b$

$$\text{Stress in a} = \sigma_a = \frac{F}{A_a} \qquad F_1 = \sigma_a A_a$$

$$\text{Stress in b} = \sigma_b = \frac{F}{A_b} \qquad F_2 = \sigma_b A_b$$

$F = F_1 + F_2$

$$F = \sigma_a A_a + \sigma_b A_b \ \text{.........(1)}$$

The strain in a is

$$\varepsilon_a = \frac{\Delta L}{L}$$

The strain in b is

$$\varepsilon_b = \frac{\Delta L}{L}$$

The modulus of elasticity for a is

$$E_a = \frac{\sigma_a}{\varepsilon_a} \qquad \varepsilon_a = \frac{\sigma_a}{E_a}$$

The modulus of elasticity for b is

$$E_b = \frac{\sigma_b}{\varepsilon_b} \qquad \varepsilon_b = \frac{\sigma_b}{E_b}$$

Since the strains are equal

$$\frac{\sigma_a}{E_a} = \frac{\sigma_b}{E_b} \qquad \sigma_a = \frac{E_a}{E_b}\sigma_b \ldots\ldots (2)$$

Combining equations (1) and (2)

$$F = \sigma_a\left[\left(\frac{E_b}{E_a}\right)A_b + A_a\right] \text{ and } F = \sigma_b\left[\left(\frac{E_a}{E_b}\right)A_a + A_b\right]$$

**WORKED EXAMPLE No. 7**

A steel cylinder is 0.5 m outer diameter and 0.4 m inner diameter and 1.5 m long. It is filled with concrete and used as a vertical column to support a weight of 30 kN. Determine the compression and the stresses in both the steel and concrete.

E is 205 GPa for the steel and 10 GPa for concrete.

**SOLUTION**

$$\frac{E_a}{E_b} = \frac{205 \text{ x } 10^9}{10 \text{ x } 10^9} = 20.5$$

$$A_a = \frac{\pi(0.5^2 - 0.45^2)}{4} = 0.0373 \text{ m}^2 (\text{steel})$$

$$A_b = \frac{\pi(0.45^2)}{4} = 0.159 \text{ m}^2 (\text{concrete})$$

$$\frac{\sigma_a}{E_a} = \frac{\sigma_b}{E_b} \quad \sigma_a = \frac{E_a \sigma_b}{E_b} = 20.5\sigma_b$$

$$F = 30\,000 = \sigma_a A_a + \sigma_b A_b = \frac{E_a \sigma_b A_a}{E_b} + \sigma_b A_b$$

$$30\,000 = \quad 20.5 \text{ x } 0.0373\sigma_b + 0.159 \text{ x } \sigma_b$$

$$30\,000 = 0.7466\sigma_b + 0.159\,\sigma_b = 0.923\sigma_b$$

$$\sigma_b = \frac{30\,000}{0.923} = 32.48 \text{ kPa}$$

$$\sigma_a = \frac{E_a}{E_b}\sigma_b = 20.5 \text{ x } 32.48 = 665.8 \text{ kPa}$$

$$\varepsilon_b = \sigma_b/E_b$$

$$\varepsilon_b = 32480/10 \text{ x } 10^9 = 3.248 \text{ x } 10^{-6}$$

$$\Delta L = \varepsilon_b L = 3.248 \text{ x } 10^{-6} \text{ x } 1.5 = 4.872\ 10^{-6} \text{m}$$

Check for material a.

$$\varepsilon_a = \sigma_a/E_a$$

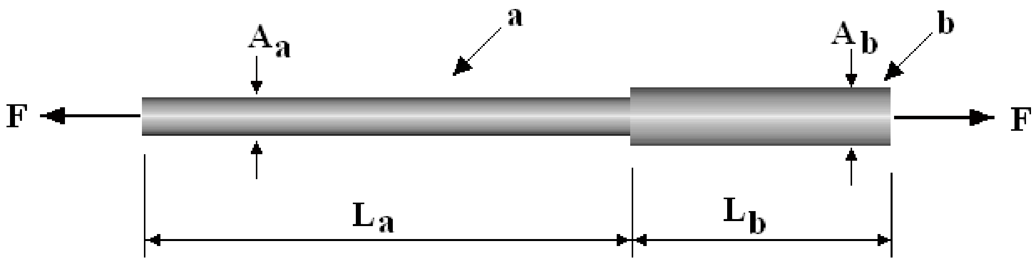
$$\varepsilon_a = 665\,800/205 \text{ x } 10^9 = 3.248 \text{ x } 10^{-6}$$

$$\Delta L = \varepsilon_a L = 3.248 \times 10^{-6} \text{ x } 1.5 = 4.872\ 10^{-6} \text{ m}$$

***Series***

A series compound bar consists of two or more different materials arranged end to end. In this case the common factor is the force which is the same in each series member.

Consider two bars 'a' and 'b' in series as shown.




$$F = \sigma_a A_a \qquad \sigma_a = \frac{F}{A_a} \qquad \varepsilon_a = \frac{\sigma_a}{E_a} \qquad \Delta L_b = \varepsilon_b L_a = \frac{FL_a}{E_a A_a}$$

$$F = \sigma_b A_b \qquad \sigma_b = \frac{F}{A_b} \qquad \varepsilon_b = \frac{\sigma_b}{E_b} \qquad \Delta L_b = \varepsilon_b L_b = \frac{FL_b}{E_b A_b}$$

Total change in length

$$\Delta L = \Delta L_a + \Delta L_b = \left[\frac{FL_a}{E_a A_a} + \frac{FL_b}{E_b A_b}\right]$$

$$\Delta L = F\left[\frac{L_a}{E_a A_a} + \frac{L_b}{E_b A_b}\right]$$

**WORKED EXAMPLE No. 8**

A series compound bar is formed by screwing steel rod 10 mm diameter and 300 mm long into the end of a brass rod 20 mm diameter and 100 mm long. Given E for steel is 200 GPa and E for brass is 100 GPa, calculate the change in length when a force of 5 kN is applied to stretch it.

**SOLUTION**

$$A_a = \frac{\pi \times 10^2}{4} = 78.54 \text{ mm}^2 \qquad A_b = \frac{\pi \times 20^2}{4} = 314.159 \text{ mm}^2$$

$$\Delta L = F\left[\frac{L_a}{E_a A_a} + \frac{L_b}{E_b A_b}\right]$$

$$\Delta L = 5\,000\left[\frac{0.3}{200 \times 10^9 \times 78.54 \times 10^{-6}} + \frac{0.1}{100 \times 10^9 \times 314.159 \times 10^{-6}}\right]$$

$$\Delta L = 111.4\text{x } 10^{-6} \text{ m or } 111.4 \text{ μm}$$

**SELF ASSESSMENT EXERCISE No. 4**

1. A reinforced concrete column is rectangular in section and measures 0.4 m × 0.6 m. The column contains 40 steel rods 10 mm diameter. Calculate the maximum load which can be supported given that the stress in the steel must not exceed 400 MPa.

   E is 200 GPa for the steel and 12 GPa for concrete.

   (Answer 6.94 MN)

2. A cast iron cylinder is filled with concrete and used as a pillar to support a weight of 40 kN. The cylinder is 0.5 m outer diameter and 0.49 m inner diameter. Determine the stress in the iron and concrete.

   E is 205 GPa for the steel and 10 GPa for concrete.

   (Answer 115 kPa 2.36MPa)

3. A compound bar is constructed by riveting a steel strip 5 mm × 20 mm to a brass strip 8 mm × 20 mm to form a cross section 13 mm × 20 mm. The bar is 300 mm long and is stretched until the stress in the brass is 5 MPa. Determine

   i. the force used to stretch the bar. (1.8 kN)
   ii. the extension of the bar. (0.015 mm)

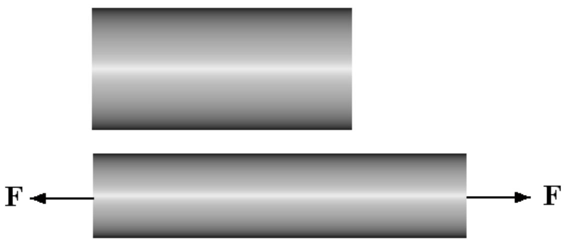
   Take E for steel as 200 GPa and for brass as 100 GPa.

4. An aluminium rod 25 mm diameter and 500 mm long is pin-jointed to a steel rod 12.5 mm diameter and 1 m long. The assembly is stretched with a force of 15 kN. What is the change in length? Take E for steel as 200 GPa and for aluminium as 71 GPa.

   (Answer 826 μm)

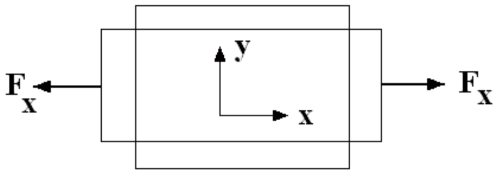
### *12. Poisson's Ratio*

Consider a bar or tie that is stretched with a force F as shown. The bar will not only get longer in the direction it is stretched but it will also get thinner as shown.




Let's reduce this to two dimensions x and y.

The stress in the x direction is $\sigma_x$ and there is no stress in the y direction. When it is stretched in the x direction, it causes the material to get thinner in all the other directions at right angles to it. This means that a negative strain is produced in the y direction. This is called the lateral strain. For elastic materials it is found that the lateral strain is always directly proportional to the applied such that




$$\frac{\varepsilon_y}{\varepsilon_x} = -\nu$$

ν (Nu) is an elastic constant called Poisson's' ratio.

The strain produced in the y direction is: $\varepsilon_y = -\nu\varepsilon_x$

If stress is applied in the y direction then the resulting strain in the x direction would similarly be $\varepsilon_x = -\nu\varepsilon_y$

**NOTE** that a force in the x direction acts on a plane in the y direction and many text books take $\sigma_x$ and $\varepsilon_x$ to mean the stress and strain on the x plane. Here we take it to mean the stress and strain in the x direction.

**NOTE** that we do not have to confine ourselves to the x and y directions and that the formula works for any two stresses at 90° to each other. In general we use $\sigma_1$ and $\sigma_2$ with corresponding strains $\varepsilon_1$ and $\varepsilon_2$. In the following examples we use $\sigma_L$ and $\sigma_D$ to mean in the direction of the length and any diameter.

**WORKED EXAMPLE No. 9**

A bar is 500 mm long and is stretched to 500.45 mm. The bar is 10 mm diameter. Given that ν = 0.23, calculate the new diameter.

**SOLUTION**

$$\varepsilon_L = \frac{\Delta L}{L} = \frac{0.45}{500} = 900\ \mu\varepsilon$$

$$\varepsilon_D = -\nu\sigma_L = -0.23 \times 900 = -207\ \mu\varepsilon$$

$$\Delta D = \varepsilon_D \times D = -207 \times 10^{-6} \times 10 = -2.07 \times 10^{-3}\ \text{mm}$$

The new diameter is 10 - 0.00207 = 9.99793 mm

**WORKED EXAMPLE No. 10**

A tie bar 2 m long and 10 mm diameter is stretched with a stress of 2 MPa. Given the elastic constants are E = 205 GPa and ν = 0.27, calculate the strains in both the longitudinal and diametral directions. Calculate the change in length and change in diameter.

**SOLUTION**

$$\varepsilon_L = \frac{\sigma_L}{E} = \frac{2 \times 10^6}{205 \times 10^9} = 9.756\ \mu\varepsilon$$

$$\varepsilon_D = -\nu\sigma_D = -0.27 \times 9.756 = -2.634\ \mu\varepsilon$$

$$\Delta D = \varepsilon_D \times D = -207 \times 10^{-6} \times 10 = -2.07 \times 10^{-3}\ \text{mm}$$

$$\Delta L = \varepsilon_L L = 9.756 \times 10^{-6} \times 2\,000 = 0.0196\ \text{mm}$$

$$\Delta D = \varepsilon_D D = -2.634 \times 10^{-6} \times 10 = -26.34 \times 10^{-6}\text{mm}$$

**WORKED EXAMPLE No. 11**

A metal bar 0.5 m long and 0.2 m diameter is compressed by an axial load of 800 kN. Given the elastic constants are E = 200 GPa and ν = 0.25, calculate the stresses and strains in both the longitudinal and diametral directions. Calculate the change in length and change in diameter.

**SOLUTION**

$A = \pi D^2/4 = 31.42 \times 10^{-3}\ m^2$

$\sigma_L = -F/A = 800 \times 10^3/31.42 \times 10^{-3} = -25.462$ MPa (Compressive)

$$\varepsilon_L = \frac{\sigma_L}{E} = \frac{-25.462 \times 10^6}{200 \times 10^9} = -127.3\ \mu\varepsilon$$

$$\varepsilon_D = -\nu\sigma_D = -0.25 \text{ x } (-127.3\ \mu\varepsilon) = 31.826\mu\varepsilon$$

$$\Delta L = \varepsilon_L L = -127.3 \times 10^{-6} \times 500 = -63.65\ \text{mm}$$

$$\Delta D = \varepsilon_D D = 31.826 \times 10^{-6} \times 200 = 6.365 \times 10^{-3}\text{mm}$$

**SELF ASSESSMENT EXERCISE No. 5**

1. A tie is 1 m long and 5 mm diameter is stretched by 0.3 mm. Given that ν = 0.25, calculate the new diameter. (4.99962 mm)

2. A metal bar 0.2 m long and 0.1 m diameter is compressed by an axial load of 1 MN. Given the elastic constants are E = 200 GPa and ν = 0.25, calculate the stresses and strains in both the longitudinal and diametral directions. Calculate the change in length and change in diameter. (-0.1273 mm and 0.015913 mm)

13. ***Stress in Two Mutually Perpendicular Directions.***

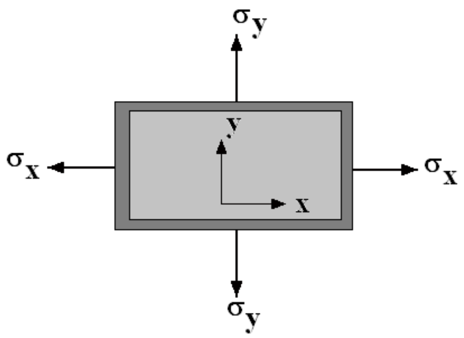
Now consider that the material has an applied stress in both the x and y directions. The resulting strain in any one direction is the sum of the direct strain and the lateral strain. Hence:

$$\varepsilon_x = \frac{\sigma_x}{E} = -\nu\sigma_y = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E}$$

$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\sigma_y\right) \ldots\ldots (A)$$

Similarly

$$\varepsilon_y = \frac{1}{E}\left(\sigma_y - \nu\sigma_x\right) \ldots\ldots (B)$$




The modulus E must be the same in both directions and such a material is not only elastic but ***Isotropic.***

**WORKED EXAMPLE No. 12**

A material has stresses of -2 MPa in the x direction and 3 MPa in the y direction. Given the elastic constants E = 205 GPa and ν= 0.27, calculate the strains in both direction.

**SOLUTION**

$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\sigma_y\right) = \frac{1}{205 \text{ x } 10^9}\left(-2 \text{ x } 10^6 - 0.27 \text{ x } 3 \text{ x } 10^6\right) = 13.7\ \mu\varepsilon$$

$$\varepsilon_y = \frac{1}{E}\left(\sigma_y - \nu\sigma_x\right) = \frac{1}{205 \text{ x } 10^9}\{3 \text{ x } 10^6 - 0.27(-2 \text{ x } 10^6)\} = 17.3\ \mu\varepsilon$$

**WORKED EXAMPLE No. 13**

A material has stresses of 2 MPa in the x direction and 3 MPa in the y direction. Given the elastic constants E = 205 GPa and ν = 0.27, calculate the strains in both direction.

**SOLUTION**

$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\sigma_y\right) = \frac{1}{205 \times 10^9}\left(2 \times 10^6 - 0.27 \times 3 \times 10^6\right) = 5.8\ \mu\varepsilon$$

$$\varepsilon_y = \frac{1}{E}\left(\sigma_y - \nu\sigma_x\right) = \frac{1}{205 \times 10^9}\{3 \times 10^6 - 0.27 \times 2 \times 10^6\} = 12\ \mu\varepsilon$$

**WORKED EXAMPLE No. 14**

A thin flat plate is 200 mm x 100 mm forms part of a structure and when in service a stress of 100 MPa is produced in the long direction and 150 MPa in the short direction. Given the elastic constants E = 205 GPa and ν = 0.25, calculate the strains in both direction. Calculate the change in dimensions ands area of the plate.

**SOLUTION**

$$\varepsilon_L = \frac{1}{E}(\sigma_L - \nu\sigma_S) = \frac{1}{205 \times 10^9}(100 \times 10^6 - 0.25 \times 150 \times 10^6) = 304.9\ \mu\varepsilon$$

$$\varepsilon_S = \frac{1}{E}(\sigma_S - \nu\sigma_L) = \frac{1}{205 \times 10^9}\{150 \times 10^6 - 0.25 \times 150 \times 10^6\} = 609.8\ \mu\varepsilon$$

Change in length = $\varepsilon_L \times 200 = 60.976 \times 10^{-3}$ mm

Change in side = $\varepsilon_S \times 100 = 60.976 \times 10^{-3}$ mm

Change in area = $60.976 \times 10^{-3} \times 60.976 \times 10^{-3} = 3.718 \times 10^{-3}$ mm

**SELF ASSESSMENT EXERCISE No. 6**

1. Solve the strains in both directions for the case below.

   E = 180 GPa ν =0.3 $\sigma_1$= - 3 MPa $\sigma_2$= 5 MPa

   (Answers 32.78 με and -25 με)

2. A large cylindrical pressure vessel is constructed from thin plate. When pressurised a tensile stress of 60 MPa is produced in the longitudinal direction and 120 MPa in the direction around the circumference. Calculate the strain in these directions.
   Part of the shell is a plate 100 mm x 100 mm and this may be treated as a flat plate. Calculate the change in dimensions and area.

   Take E = 205 GPa and ν = 0.25

   (146.3 με and 512.2 με )
   ($14.364 \times 10^{-3}$ mm and $51.22 \times 10^{-3}$ mm)
   (6.555 $mm^2$)

## *14. Three Dimensional Stress and Strain*

Equations A and B were derived for a 2 dimensional system. Suppose a material to be stressed in 3 mutually perpendicular directions x, y and z. The strain in any one of these directions is the direct strain plus the lateral strain from the other two directions. It follows that:

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y - \nu\sigma_z) = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] \;\ldots\ldots (C)$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x - \nu\sigma_z) = \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)] \;\ldots\ldots (D)$$

$$\varepsilon_Z = \frac{1}{E}(\sigma_z - \nu\sigma_x - \nu\sigma_y) = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)] \;\ldots\ldots (E)$$

If the stress is the same in all directions, then the strains are the same.

**WORKED EXAMPLE No. 15**

A solid cube of metal has sides of 200 mm. It is compressed by a pressure of 80MPa on all its faces. Determine the change in length of each side and the reduction of volume.
E is 71 GPa and Poisson's ratio $\nu$ is 0.34.

**SOLUTION**

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] = \frac{1}{71 \times 10^9}[80 \times 10^6 - 0.34(80 \times 10^6 + 80 \times 10^6)]$$

$$\varepsilon_x = 360.6 \ \mu\varepsilon$$

The strain will be the same in the y and z directions

Change in length = $\delta = \varepsilon_x \times L = 360.6 \times 10^{-6} \times 200 = 72.113 \times 10^{-3}$ mm

The change in volume in each direction is $\delta \times L^2 = 72.113 \times 10^{-3} \times 200^2 = 2\,885$ mm$^3$

The total change in volume is $3 \times 2\,885 = 8\,654$ mm$^3$

Note this is not quite true because we should also subtract the corner common to all three which is $(72.113 \times 10^{-3})^3 = 0.000\,375$ mm$^3$ and is negligible.

**SELF ASSESSMENT EXERCISE No. 7**

1. A cube is stressed in 3 mutually perpendicular direction x, y and z. The stresses in these directions are

   $\sigma_x$ = 50 MPa $\sigma_y$ = 80 MPa $\sigma_z$ = -100 MPa

   Determine the strain in each direction.
   ν is 0.34 and E is 71 GPa.

   (Answer $800 \times 10^{-6}$, $1.366 \times 10^{-9}$ and $-2.031 \times 10^{-3}$)

2. A cube of metal of side 30 mm is placed inside a pressure vessel and the pressure is raised to 20 MPa. Given that E = 205 GPa and ν = 0.25 determine the change in volume of the cube.
   (Answer 3.951 $mm^3$)

3. Repeat question 2 for a solid sphere 30 mm diameter. (-2.069 $mm^3$)