SOLID MECHANICS
STATICS
STRAIN ENERGY

You should judge your progress by completing the self assessment exercises.

You will find in this tutorial material to cover the definition and derivation of the various forms of strain energy. This is extended to the theory that allows you to solve the stress and deflection in various structures with single and multiple loads. It also covers the effect of suddenly applying a load to a structure.

Contents

It is assumed that students doing this tutorial are already familiar with the following.

- ***Basic stress and strain,***
- ***The elastic properties of materials***
- ***Basic bending theory.***
- ***Basic torsion theory.***
- ***Calculus including partial differentiation.***

1. Introduction

When an elastic body is deformed, work is done. The energy used up is stored in the body as strain energy and it may be regained by allowing the body to relax. The best example of this is a clockwork device which stores strain energy and then gives it up.

We will examine strain energy associated with the most common forms of stress encountered in structures and use it to calculate the deflection of structures. Strain energy is usually given the symbol U.

2. Strain Energy Due To Direct Stress.

Consider a bar of length L and cross sectional area A. If a tensile force is applied it stretches and the graph of force v extension is usually a straight line as shown. When the force reaches a value of F and corresponding extension x, the work done (W) is the area under the graph. Hence **W = Fx/2.** (The same as the average force x extension)

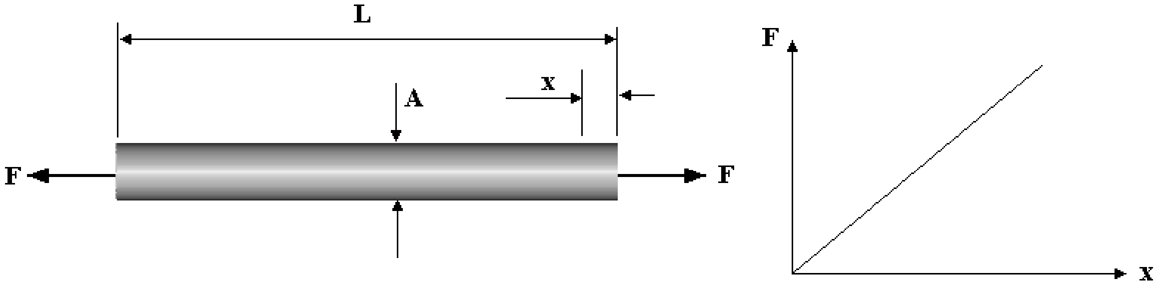


Figure 1

Since the work done is the energy used up, this is now stored in the material as strain energy hence

$$\mathbf{U = \frac{Fx}{2}}$$

The stress in the bar is $\sigma = F/A$ hence $F = \sigma A$

The strain in the bar is $\varepsilon = x/L$ hence $x = \varepsilon L$

For an elastic material up to the limit of proportionality, $\sigma/\varepsilon = E$ (The modulus of elasticity) hence

$$\varepsilon = \frac{\sigma}{E}$$

Substituting we find

$$U = \frac{\sigma A \varepsilon L}{2} = \frac{\sigma^2 AL}{2E}$$

The volume of the bar is A L so

$$\mathbf{U = \frac{\sigma^2}{2E} \times volume\ of\ the\ bar}$$

WORKED EXAMPLE No. 1

A steel rod has a square cross section 10 mm x 10 mm and a length of 2 m. Calculate the strain energy when a stress of 400 MPa is produced by stretching it. Take E = 200 GPa

SOLUTION

$A = 10 \times 10 = 100 \text{ mm}^2$ or $100 \times 10^{-6} \text{ m}^2$. $V = AL = 100 \times 10^{-6} \times 2 = 200 \times 10^{-6} \text{ m}^3$.

$\sigma = 400 \times 10^6 \text{ N/m}^2$ and $E = 200 \times 10^9 \text{ N/m}^2$

$$U = \frac{\sigma^2}{2E} \times \text{volume of the bar} = \frac{(400 \times 10^6)^2}{2 \times 200 \times 10^9} \times 200 \times 10^6 = 80 \text{ Joules}$$

3. Strain Energy Due To Pure Shear Stress

Consider a rectangular element subjected to pure shear so that it deforms as shown. The height is h and plan area A. It is distorted a distance x by a shear force F. The graph of Force plotted against x is normally a straight line so long as the material remains elastic. The work done is the area under the F - x graph so

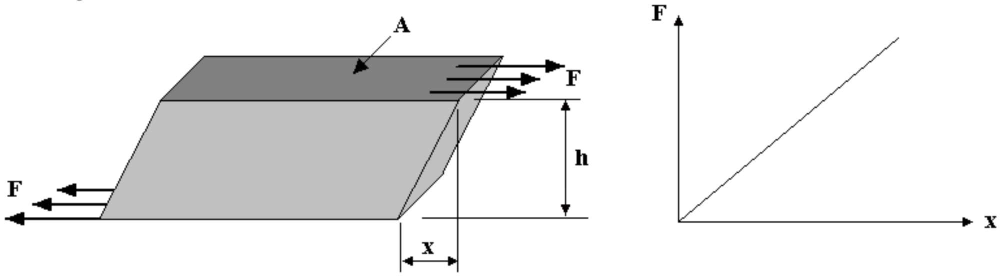


Figure 2

The work done is the strain energy stored hence

$$W = \frac{Fx}{2} = \mathbf{U} = \frac{\mathbf{Fx}}{\mathbf{2}}$$

The shear stress is $\tau = F/A$ hence $F = \tau A$ The shear strain is $\gamma = x/h$ hence $x = \gamma h$

Note that since x is very small it is the same length as an arc of radius h and angle γ. It follows that the shear strain is the angle through which the element is distorted.

For an elastic material $\tau/\gamma = G$ (The modulus of Rigidity) hence $\gamma = \tau/G$
Substituting we find

$$\mathbf{U} = \frac{\tau A\gamma h}{2} = \frac{\tau^2 Ah}{2G} = \frac{\boldsymbol{\tau}^{\mathbf{2}}}{\mathbf{2G}} \times \textbf{volume}$$

Pure shear does not often occur in structures and the numerical values are very small compared to that due to other forms of loading so it is often (but not always) ignored.

WORKED EXAMPLE No. 2

Calculate the strain energy due to the shear strain in the structure shown.
Take G = 90GPa

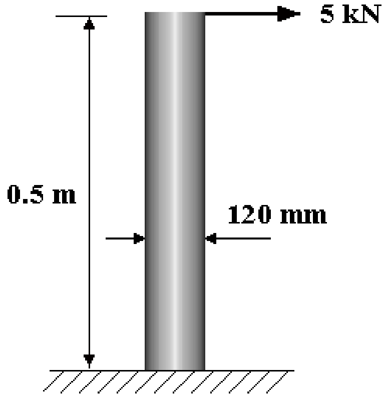


Figure 3

SOLUTION

$$A = \frac{\pi d^2}{4} = \frac{\pi \times 0.12^2}{4} = 11.31 \times 10^3 \text{ m}^2$$

$$\tau = \frac{F}{A} = \frac{5\ 000}{11.31 \times 10^3} = 56.55 \text{ Pa}$$

Volume = A h = $11.31 \times 10^{-3} \times 0.5$
Volume = 5.65×10^{-3} m^3

$$U = \frac{\tau^2}{2G} \times \text{volume} = \frac{56.55^2}{2 \times 90 \times 10^9} \times 5.65 \times 10^{-3}$$

$$U = 100.5 \times 10^{-12} \text{ Joules}$$

Note that the structure is also subject to bending.
The strain energy due to bending is covered later.

4. *Strain Energy Due To Torsion*

Consider a round bar being twisted by a torque T. A line along the length rotates through angle γ and the corresponding radial line on the face rotates angle θ. γ is the shear strain on the surface at radius R.

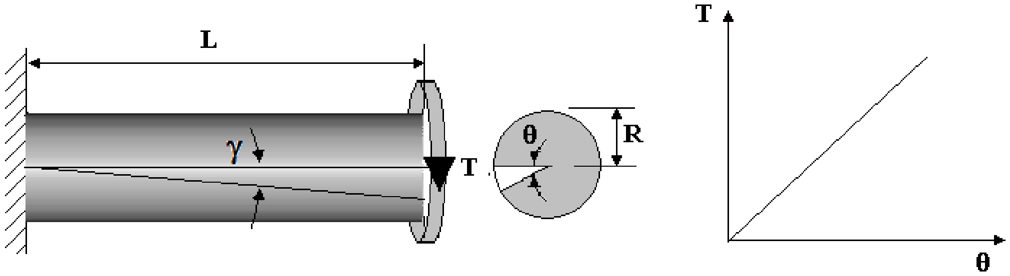


Figure 4

The relationship between torque T and angle of twist θ is normally a straight line. The work done is the area under the torque-angle graph.

For a given pair of values

$$W = \frac{T\theta}{2}$$

The strain energy stored is equal to the work done hence

$$\mathbf{U = \frac{T\theta}{2}}$$

From the theory of torsion (not covered here)

$$\theta = \frac{TL}{GJ}$$

G is the modulus of rigidity and J is the polar second moment of area.
Substitute for θ

$$U = \frac{T^2 L}{2GJ}$$

Also from torsion theory

$$T = \frac{\tau J}{R}$$

Substitute

$$U = \frac{\left(\frac{\tau J}{R}\right)^2 L}{2GJ} = \frac{\tau^2 JL}{2GR^2}$$

τ is maximum shear stress on the surface.

$$J = \frac{\pi R^4}{2}$$

Substitute

$$U = \frac{\tau^2 \pi R^4 L}{4GR^2} = \frac{\tau^2 \pi R^2 L}{4G}$$

The volume of the bar is $AL = \pi R^2 L$ so it follows that:

$$\mathbf{U = \frac{\tau^2}{4G} \times \text{volume of the bar}}$$

(τ is the maximum shear stress on the surface)

WORKED EXAMPLE No. 3

A solid bar is 20 mm diameter and 0.8 m long. It is subjected to a torque of 30 Nm. Calculate the maximum shear stress and the strain energy stored. Take G = 90GPa

SOLUTION

R = 10 mm = 0.01 m L = 0.8 m $\quad\quad$ $A = \pi R^2 = \pi \times 0.01^2 = 314.16 \times 10^{-6}\ m^2$

Volume of bar = AL = $314.16 \times 10^{-6} \times 0.8 = 251.3 \times 10^{-6}\ m^3$

$$J = \frac{\pi R^4}{2} = \frac{\pi \times 0.01^4}{2} = 15.7 \times 10^{-9}\ m^4$$

$$\tau = \frac{TR}{J} = \frac{30 \times 0.01}{15.7 \times 10^{-9}} = 19.1 \times 10^6\ Pa$$

$$U = \frac{\tau^2}{4G} \times V = \frac{(19.1 \times 10^6)^2}{4 \times 90 \times 10^9} \times 251.3 \times 10^{-6} = 0.255\ \text{Joules}$$

WORKED EXAMPLE No. 4

A helical spring is constructed by taking a wire of diameter d and length L and coiling it into a helix of mean diameter D with n coils. Show that the stiffness of the helical spring shown below is given by the formula

$$\frac{F}{y} = \frac{Gd}{8nD^3}$$

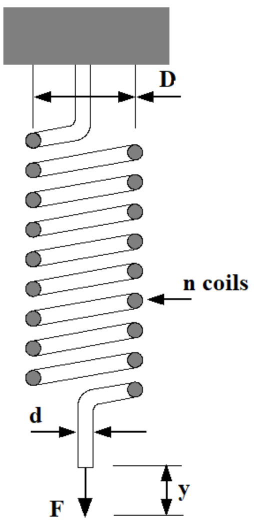


Figure 5

SOLUTION

When a force F is applied to the end it deflects down by a distance y. Looking at the bottom coil, it can be seen that a torque T = FD/2 is twisting the cross section of the wire. This torsion is transmitted throughout the entire length of the wire.

Starting with the strain energy due to torsion we have:

$$U = \frac{\tau^2}{4G} \times \text{volume}$$

We will substitute the following as we progress

$$V = AL \qquad A = \frac{\pi d^2}{4} \qquad J = \frac{\pi d^2}{32} \text{ and } \tau = \frac{TD}{2J}$$

$$U = \frac{\left(\frac{TD}{2J}\right)^2}{4G} \times AL = \frac{T^2 d^2}{16GJ^2} \times AL$$

$$U = \frac{T^2 \pi d^4}{16GJ^2} \times L = \frac{T^2}{2GJ} \times L$$

The work done by thea force F stretching the spring by distance y is

$$W = \frac{Fy}{2}$$

Equate to U

$$\frac{Fy}{2} = \frac{T^2}{2GJ} \times L = \frac{\left(\frac{FD}{2}\right)^2}{2GJ} \times L = \frac{F^2 D^2}{8GJ} \times L$$

$$\frac{4GJ}{LD^2} = \frac{F}{y} = \frac{4G\left(\frac{\pi d^2}{32}\right)}{LD^2} = \frac{G\pi d^4}{8LD^2}$$

$$L = n\pi D$$

$$\frac{F}{y} = \frac{Gd^4}{8nD^3}$$

This is the well known equation for the stiffness of a helical spring and the same formula may be derived by other methods.

5. Strain Energy Due To Bending.

The strain energy produced by bending is usually large in comparison to the other forms. When a beam bends, layers on one side of the neutral axis are stretched and on the other side they are compressed. In both cases, this represents stored strain energy. Consider a point on a beam where the bending moment is M.

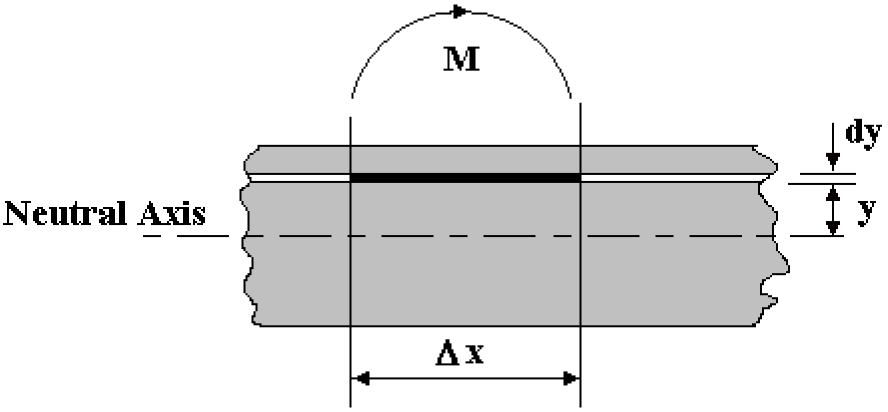


Figure 6

Now consider an elementary layer within the material of length Δx and thickness dy at distance y from the neutral axis. The cross sectional area of the strip is dA.

The bending stress is zero on the neutral axis and increases with distance y. This is tensile on one side and compressive on the other. If the beam has a uniform section the stress distribution is as shown.

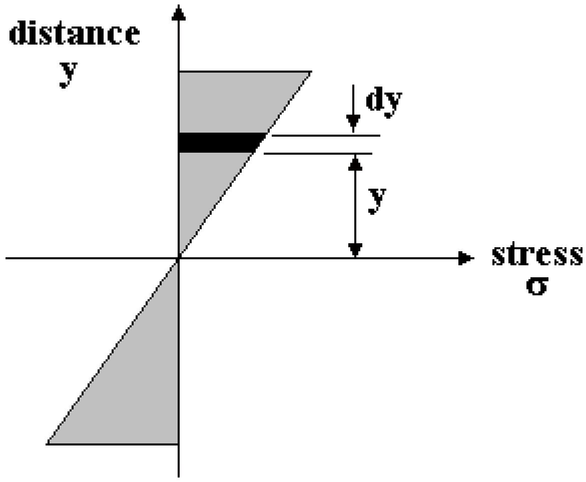


Figure 7

Each elementary layer has a direct stress (σ) on it and the strain energy stored (section 2) has been shown to be

$$U = \frac{\sigma^2}{2E} \times \text{volume}$$

The volume of the strip is Δx dA

The strain energy in the strip is part of the total so let u be the strain energy in this small part

$$du = \frac{\sigma^2}{2E} \times \Delta x \, dA$$

From bending theory (not covered here) we have

$$\sigma = \frac{My}{I} \quad \text{(I is the second moment of area)}$$

Substituting for σ we get

$$du = \frac{\left(\frac{My}{I}\right)^2}{2E} \times \Delta x\, dA$$

In the limit as $\Delta x \to dx$

$$du = \frac{\left(\frac{My}{I}\right)^2}{2E} \times dx\, dA = \frac{M^2}{2EI^2} dx\, y^2 dA$$

The strain energy stored in an element of length dx is then

$$u = \frac{M^2}{2EI^2} dx \int y^2 dA$$

By definition

$$I = \int y^2 dA$$

Substitute

$$u = \frac{M^2}{2EI} dx$$

In order to solve the strain energy stored in a finite length, we must integrate with respect to x.
For a length of beam the total strain energy is

$$U = \frac{1}{2EI} \int M^2 dx$$

The problem however, is that M varies with x and M as a function of x has to be substituted.

WORKED EXAMPLE No. 5

Determine the strain energy in the cantilever beam shown. The flexural stiffness EI is 200 kNm^2.

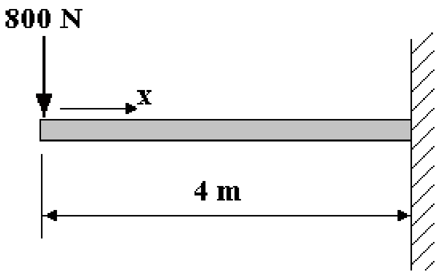


Figure 8

SOLUTION

This is a bending problem so

$$U = \frac{1}{2EI} \int M^2 dx$$

The beam is a simple cantilever so the bending moment at any distance x from the end is simply

$$M = -800\, x$$

(The minus sign for hogging makes no difference since it will be squared)

$$U = \frac{1}{2EI}\int_0^4 M^2 dx = \frac{1}{2EI}\int_0^4 (-800x)^2 dx = \frac{1}{2EI}\int_0^4 640\ 000\ x^2 dx$$

$$U = \frac{640\ 000}{2EI}\int_0^4 x^2 dx = \frac{640\ 000}{2 \times 2 \times 10^5}\left[\frac{x^3}{3}\right]_{04}^4$$

$$U = \frac{640\ 000}{2 \times 2 \times 10^5}\left[\frac{4^3}{3} - 0\right] = 34.13$$

The example was a simple one because the bending moment was easily integrated.

6. Deflection

The deflection of simple structures may be found by equating the strain energy to the work done. This is covered in detail later but for the simple cantilever beam it can be demonstrated easily as follows.

WORKED EXAMPLE No. 6

Calculate the deflection for the cantilever beam in W. E. No. 4.

SOLUTION

The deflection of the beam y is directly proportional to the force F so the work done by the force is

$$W = \frac{Fy}{2} \text{ (the area under the F} - \text{y graph)}$$

Equate the strain energy to the work done and

$$\frac{Fy}{2} = U$$

$$y = \frac{2U}{F} = \frac{2 \times 34.13}{800} = 0.085 \text{ m}$$

We can check the answer with the standard formula for the deflection of a cantilever (covered in the beams tutorials).

$$y = \frac{FL^3}{3EI} = \frac{800 \times 4^3}{3 \times 200 \times 10^5} = 0.085 \text{ m}$$

The most severe forms of stress are bending and torsion. If bending or torsion occurs in a structure, they will normally be much larger than that due to direct stress or shear and these are usually neglected

SELF ASSESSMENT EXERCISE No. 1

1. A metal rod is 15 mm diameter and 1.5 m long. It is stretched with a force of 3 000 N. Calculate the stress level and the strain energy stored in it assuming it has not reached the limit of proportionality. Take E = 180 GPa

   (Answers 16.98 MPa and 0.212 Joules

2. A bar of metal is 200 mm diameter and 0.5 m long. It has a shear stress of 4 MPa throughout its entire length. Take G = 90 GPa. Calculate the strain energy stored.

   (Answer 1.4 J)

3. A hollow bar is 60 mm outer diameter, 40 mm inner diameter and 0.6 m long. It is subjected to a torque of 500 Nm. Calculate the maximum shear stress and the strain energy stored. Take G = 90GPa

   (Answers 14.7 MPa and 0.565 J)

4. A cantilever beam is 4 m long and has a point load of 100 kN at the free end. The flexural stiffness is 300 MN m^2. Determine the strain energy stored and the deflection.

   (Answers 355.5 J and 7.11 mm)

7. *Harder Beam Problems*

When the bending moment function is more complex, integrating becomes more difficult and a maths package is advisable for solving them outside of an examination. In an examination you will need to do it the hard way. For example, the bending moment function changes at every load on a simply supported beam so it should be divided up into sections and the strain energy solved for each section. The next example is typical of a solvable problem.

WORKED EXAMPLE No. 7

Calculate the strain energy in the beam shown and determine the deflection under the load. The flexural stiffness is 25 MNm^2.

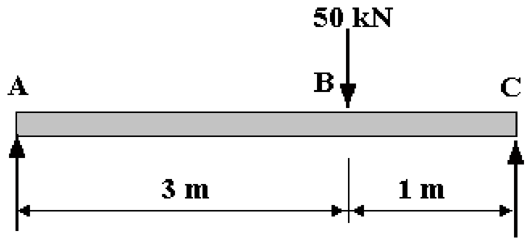


Figure 9

SOLUTION

First calculate the reactions by taking moments about the ends.

$R_C \times 4 = 50 \times 3 \quad R_C = 37.5$ kN
$R_A \times 4 = 50 \times 1 \quad R_A = 12.5$ kN
Check that they add up to 50 kN.

The bending moment equation is different for section AB and section BC so the solution must be done in 2 parts. The origin for x is the left end. First section AB

$M = R_A\, x = 12\,500\, x$

$$U = \frac{1}{2EI}\int_0^3 M^2 dx = \frac{1}{2EI}\int_0^3 (12\,500x)^2 dx$$

$$U = \frac{12\,500^2}{2 \times 25 \times 10^6}\int_0^4 x^2 dx = \frac{12\,500^2}{2 \times 25 \times 10^6}\left[\frac{3^3}{3} - 0\right] = 28.125 \text{ J}$$

Next solve for section BC. To make this easier, let the origin for x be the right hand end.

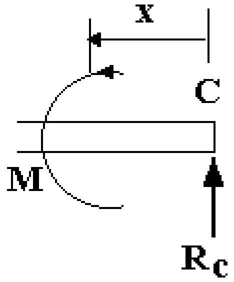


Figure 10

$$M = R_C\, x = 37\,500\, x$$

$$U = \frac{1}{2EI}\int_0^1 M^2 dx = \frac{1}{2EI}\int_0^1 (37\,500x)^2 dx$$

$$U = \frac{37500^2}{2 \times 25 \times 10^6}\int_0^1 x^2 dx = \frac{37\,500^2}{2 \times 25 \times 10^6}\left[\frac{1^3}{3} - 0\right] = 9.375 \text{ J}$$

The total strain energy is U =28.125+9.375 = 37.5 J

Equate the strain energy to the work done and

$$\frac{Fy}{2} = U$$

$$y = \frac{2U}{F} = \frac{2 \times 37.5}{50\,000} = 0.0015 \text{ m or } 1.5 \text{ mm m}$$

SELF ASSESSMENT EXERCISE No. 2

Determine the strain energy and for the beam shown and determine the deflection under the load. The flexural stiffness is 200 kNm^2.

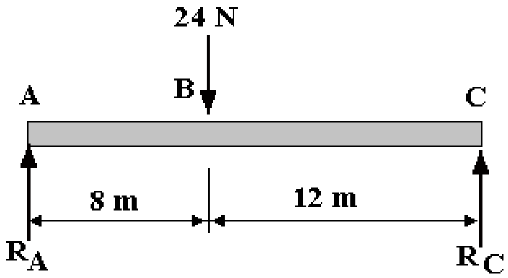


Figure 11

(Answers 0.221 J and 18.4 mm)

Now consider a problem involving torsion and bending. The force F will produce a torque on the bar and so F will deflect due to Torsion. In addition, F will deflect because both the bar and lever bends. Remember that distances are usually best measured from the free ends and that this can be changed at will in order that we may find the deflection at the force, we must evaluate all strain energies in terms of F.

WORKED EXAMPLE No. 8

The diagram shows a torsion bar held rigidly at one end and with a lever arm on the other end. Solve the strain energy in the system and determine the deflection at the end of the lever arm.
The force is 5 kN applied vertically. The following are the relevant stiffnesses.

Lever EI = 5 kNm^2.
Bar EI = 60 kNm^2.
Bar GJ = 50 kNm^2.

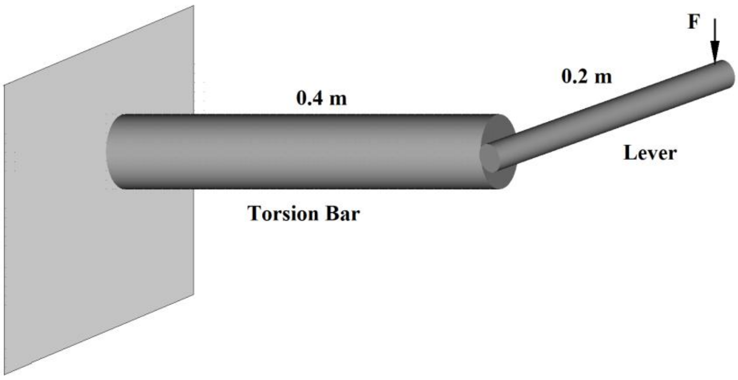


Figure 12

SOLUTION

The stresses to be considered are Bending in the lever
Bending in the bar
Torsion in the bar

LEVER
Make the origin for x the end as shown.
The bending moment is M = Fx

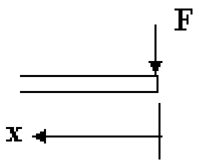


Figure 13

$$U = \frac{1}{2EI}\int_0^{0.2} M^2 dx = \frac{1}{2EI}\int_0^{0.2} (Fx)^2 dx$$

$$U = \frac{F^2}{2 \times 5 \times 10^3}\int_0^{0.2} x^2 dx = \frac{F^2}{2 \times 5 \times 10^3}\left[\frac{0.2^3}{3} - 0\right] = 266.7 \times 10^{-9}\, F^2$$

The numeric value is 6.67 J if evaluated

BAR

Viewed as shown we can see that the force F acts at the end of the bar as it is transmitted all along the length of the lever to the bar.

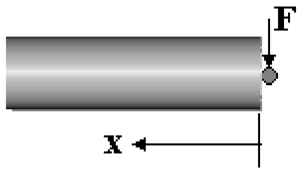


Figure 14

$$U = \frac{1}{2EI}\int_0^{0.4} M^2 dx = \frac{1}{2EI}\int_0^{0.4} (Fx)^2 dx$$

$$U = \frac{F^2}{2 \times 60 \times 10^3}\int_0^{0.4} x^2 dx = \frac{F^2}{2 \times 60 \times 10^3}\left[\frac{0.4^3}{3} - 0\right] = 177.7 \times 10^{-9} F^2$$

The numeric value is 4.444 J when evaluated

TORSION OF BAR

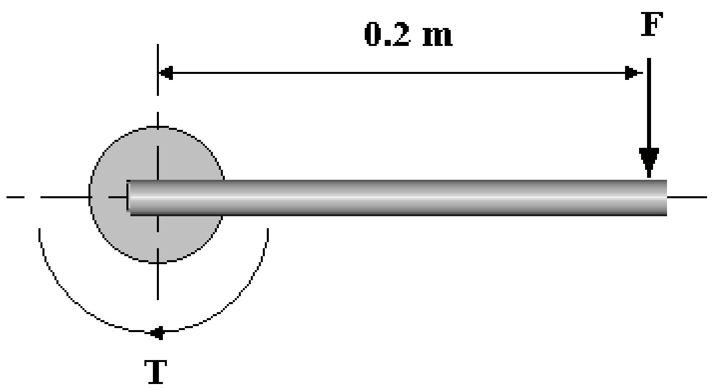


Figure 15

The torque in the bar is T = F × 0.2

For torsion

$$U = \frac{T^2}{2GJ} = \frac{0.04F^2}{2 \times 50\,000} = 160 \times 10^{-9} F^2$$

The numeric value is 4 J when evaluated

The total strain energy is then $(266.7F^2 + 177.7\,F^2 + 160\,F^2) \times 10^{-9}$

$$U = 605 \times 10^{-9}\,F^2$$

The work done is Fy/2 so equating

$$y = 2 \times 6.05\,F \times 10^{-7}$$

$$y = 12.1 \times 5000 \times 10^{-7} = 0.00605 \text{ m or } 6.05 \text{ mm}$$

WORKED EXAMPLE No. 9

Solve the deflection of the curved structure shown. The radius is 0.2 m and the force is 30 N. Take EI = 500 Nm^2.

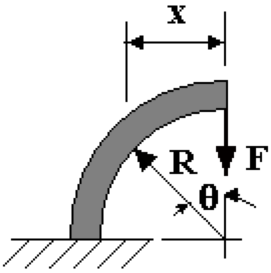


Figure 16

SOLUTION

Consider the point shown. The horizontal distance is x. the bending moment is Fx.

$$U = \frac{1}{2EI}\int M^2 dx = \frac{1}{2EI}\int (Fx)^2 dx = \frac{F^2}{2EI}\int x^2 dx$$

In order to solve the problem we must work in terms of angle rather than x

The curved length is Rθ

$$x = R\sin\theta$$

Differentiate

$$dx = R\cos\theta$$

Substitute these expressions and integrate between θ and ½ π (a quarter of a circle)

$$U = \frac{F^2}{2EI}\int_0^{\pi/2} (R\sin\theta)^2 R\cos\theta = \frac{R^3F^2}{2EI}\int_0^{\pi/2} (\sin\theta)^2\ \cos\theta$$

The integration yields the following (you should solve it, look it up or use an online integrator tool)

$$U = \frac{R^3F^2}{2EI}\left[\frac{\sin^3\theta}{3}\right]_0^{\pi/2} = \frac{0.2^3 \times F^2}{6 \times 500}\left[\sin^3\frac{\pi}{2} - \sin^3 0\right]$$

$$U = \frac{R^3F^2}{2E}[\sin^3\theta]_0^{\pi/2} = 2.67 \times 10^{-6}F^2[1 - 0]$$

$$U = 2.67 \times 10^{-6}F^2$$

Equate to the work done

$$\frac{Fy}{2} = 2.67 \times 10^{-6}F^2$$

$$y = \frac{2 \times 2.67 \times 10^{-6}F^2}{F} = 2 \times 2.67 \times 10^{-6}F$$

$$y = 2 \times 2.67 \times 10^{-6} \times 30 = 160 \times 10^{6}\ \text{m}$$

y = 0.16 mm

SELF ASSESSMENT EXERCISE No. 3

1. The diagram shows a 20 mm diameter bar formed into a 90^o crank attached to a fixed object. Solve the deflection at the end where the force is applied. Treat the bend as a sharp corner.
   Take E = 210 GPa and G = 81 GPa.

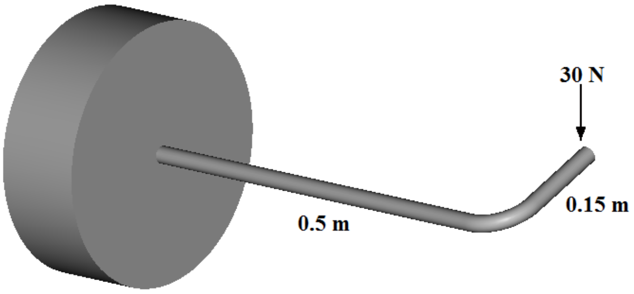


Figure 17

(Answer 1.04 mm)

2. Calculate the horizontal deflection of the curved member shown. EI = 600 N m^2.

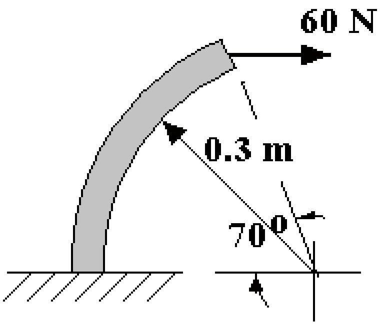


Figure 18

(Answer 0.747 mm)

8. Application to Impact Loads

When a mass M rests on a collar at the end of a suspended rod, the static deflection (x_s) is the extension due to the weight W = Mg. The static stress and strain due to the weight W are respectively

$$\sigma_s = \frac{W}{A} = \frac{Mg}{A} \qquad \varepsilon_s = \frac{x_s}{L}$$

If the elastic limit has not been exceeded then

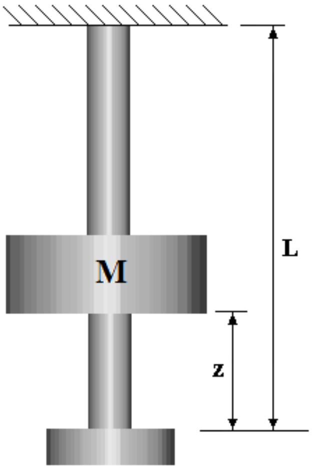


Figure 19

$$\frac{\sigma_s}{E} = \varepsilon_s = \frac{x_s}{L} \quad \text{so} \quad \sigma_s = \frac{E}{L}x_s \quad \text{and} \quad x_s = \frac{L}{E}\sigma_s$$

When the weight impacts on the collar by dropping down on to it, the stress and deflection resulting is larger than the static stress and deflection. Consider the mass M in the diagram is dropped from a height z. The bar has a length L and a cross sectional area A.

At the moment the bar is stretched to its maximum the force in the bar is F and the extension is x. The corresponding dynamic stress and strain respectively are

$$\sigma = \frac{F}{A} \quad \text{and} \quad \varepsilon = \frac{x}{L}$$

If the elastic limit is not exceeded

$$E = \frac{\sigma}{\varepsilon} \quad \text{hence } x = \frac{\sigma L}{E}$$

The strain energy in the bar is

$$U = \frac{\sigma^2 AL}{2E}$$

The potential energy given up by the falling mass is P. E. = Mg(z + x)

Equating the energy lost to the strain energy gained we have

$$Mg(z + x) = \frac{\sigma^2 AL}{2E}$$

This leads to the solution for the dynamic stress σ and dynamic deflection x

8.1 Simplified Solution

If the extension x is small compared to the distance z then

$$Mgz = \frac{\sigma^2 AL}{2E}$$

Hence

$$\sigma = \sqrt{\frac{2MgzE}{AL}}$$

With substitutions from above

$$\sigma = \sigma_s\sqrt{\frac{2z}{x_s}}$$

8.2 Exact Solution

***In terms of stress* -** Equating P. E and Strain energy we have

$$\mathrm{Mg}(z+x)=\frac{\sigma^2 AL}{2E}$$

$$(z+x)=\frac{\sigma^2 AL}{2MgE}$$

Make the following substitutions

$$x=\sigma\frac{L}{E}\qquad \frac{x_s}{\sigma_s}=\frac{L}{E}\ \text{ and }\ x=\frac{\sigma}{\sigma_s}x_s$$

$$\left(z+\frac{\sigma}{\sigma_s}x_s\right)=\frac{\sigma^2 AL}{2MgE}=\frac{\sigma^2 L}{2\sigma_s E}=\frac{\sigma^2 x_s}{2{\sigma_s}^2}$$

$$\left(z+\frac{\sigma}{\sigma_s}x_s\right)=\frac{\sigma^2 x_s}{2{\sigma_s}^2}$$

Rearrange into a quadratic equation and solve

$$\frac{x_s}{2{\sigma_s}^2}\sigma^2-\frac{x_s}{\sigma_s}\sigma-z=0$$

$$\sigma=\frac{\frac{x_s}{\sigma_s}\pm\sqrt{\left(\frac{x_s}{\sigma_s}\right)^2+\left(\frac{2x_s z}{{\sigma_s}^2}\right)}}{\frac{x_s}{{\sigma_s}^2}}=\sigma_s\pm\sqrt{{\sigma_s}^2+\frac{2z}{x_s}{\sigma_s}^2}=\sigma_s\left\{1\pm\sqrt{1+\frac{2z}{x_s}}\right\}$$

In terms of deflection- Equating strain and potential energy again we have

$$\mathrm{Mg}(z+x)=\frac{\sigma^2 AL}{2E}\quad \text{substitute } \sigma=\frac{Ex}{L}$$

$$\mathrm{Mg}(z+x)=\left(\frac{Ex}{L}\right)^2\left(\frac{AL}{2E}\right)=\frac{AEx^2}{2L}$$

$$x^2=\left(\frac{2mgL}{AE}\right)(z+x)\ \text{ substitute } x_s=\frac{mgL}{AE}$$

$$x^2=2x_s(z+x)=2x_s z+2x_s x$$

Rearrange into a quadratic equation

$$x^2-2x_s x-2x_s z=0$$

Applying the quadratic formula

$$x=\frac{2x_s\pm\sqrt{(2x_s)^2+4(2x_s z)}}{2}=x_s\pm\sqrt{(x_s)^2+2x_s z}$$

$$x=x_s\left\{1\pm\sqrt{1+\frac{2z}{x_s}}\right\}$$

The minus option does not give an answer

8.3 Suddenly Applied Loads

A suddenly applied load occur when z = 0. This is not the same as a static load. Putting z = 0 yields the result

$$x = 2x_s \text{ and } \sigma = 2\sigma_s$$

The instantaneous stress is double the static stress.

This theory also applies to loads dropped on beams where the appropriate solution for the static deflection must be used.

WORKED EXAMPLE No. 10

A mass of 5 kg is dropped from a height of 0.3 m onto a collar at the end of a bar 20 mm diameter and 1.5 m long. Determine the extension and the maximum stress induced.
E = 205 GPa.

SOLUTION

$$A = \frac{\pi D^2}{4} = \frac{\pi \times 0.02^2}{4} = 314.159 \times 10^{-6}\ \text{m}^2$$

$$x_s = \frac{MgL}{AE} = \frac{5 \times 9.81 \times 1.5}{314.159 \times 10^{-6} \times 205 \times 10^9} = 1.142 \times 10^{-6}\ \text{m}$$

$$x = x_s\left\{1 + \sqrt{1 + \frac{2z}{x_s}}\right\} = 1.142 \times 10^{-6}\left\{1 + \sqrt{1 + \frac{2 \times 0.3}{1.142 \times 10^{-6}}}\right\}$$

$$x = 828.8 \times 10^{-6}\ \text{m}$$

$$\sigma = \frac{xE}{L} = \frac{828.8 \times 10^{-6} \times 205 \times 10^9}{1.5} = 113.28 \times 10^6\ \text{Pa or } 113.28\ \text{MPa}$$

SELF ASSESSMENT EXERCISE No. 4

1. Calculate the stress and extension produced in a wire 2 mm diameter and 5 m long if a load of 50 g is suddenly applied. E = 200 GPa..

   (Answers x_s = 0.0039 σ_s = 156.1 kPa, x = 0.0078 mm and σ = 312.3 kPa)

2. A rod is 25 mm diameter and 2.6 m long and is suspended vertically from a rigid support. A mass of 10 kg falls vertically 3 mm onto a collar at the end.

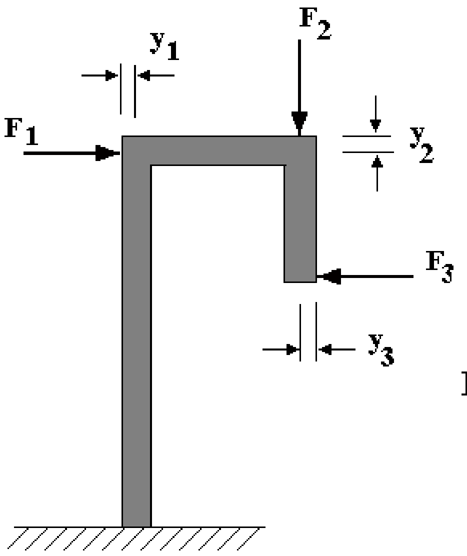
   Calculate the following.

   i. The static deflection (0.002598 mm)
   ii. The static stress (199.8 kPa)
   iii. The maximum deflection (0.1248 mm)
   iv. The maximum stress. (9.806 MPa)

   Take E = 200 GPa.

9. Castigliano's Theorem

Castigliano takes the work so far covered and extends it to more complex structures. This enables us to solve the deflection of structures which are subjected to several loads. Consider the structure shown.



The structure has three loads applied to it.

Consider the first point load. If the force was gradually increased from zero to F_1, the deflection would increase from zero to y_1 and the relationship would be linear as shown. The same would be true for the other two points as well.

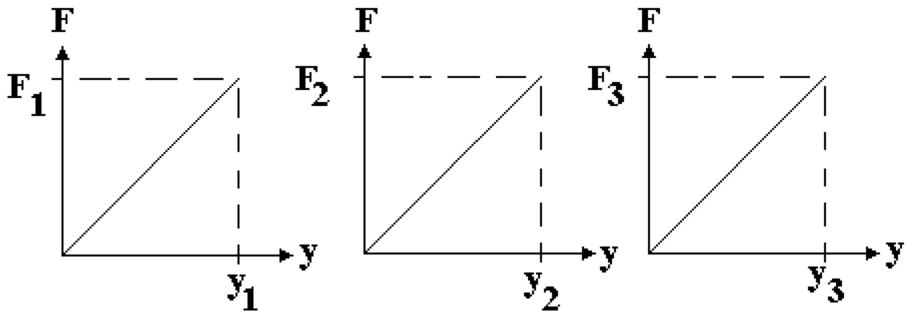


Figure 20

Figure 21

The work done by each load is the area under the graph. The total work is the sum of the three and this is equal to the strain energy hence:

$$W = U = \frac{F_1 y_1}{2} + \frac{F_2 y_2}{2} + \frac{F_3 y_3}{2} \dots\dots (A)$$

Next consider that F_1 is further increased by δF_1 but F_2 and F_3 remain unchanged. The deflection at all three points will change and for simplicity let us suppose that they increase as shown by δy_1, δy_2 and δy_3 respectively.

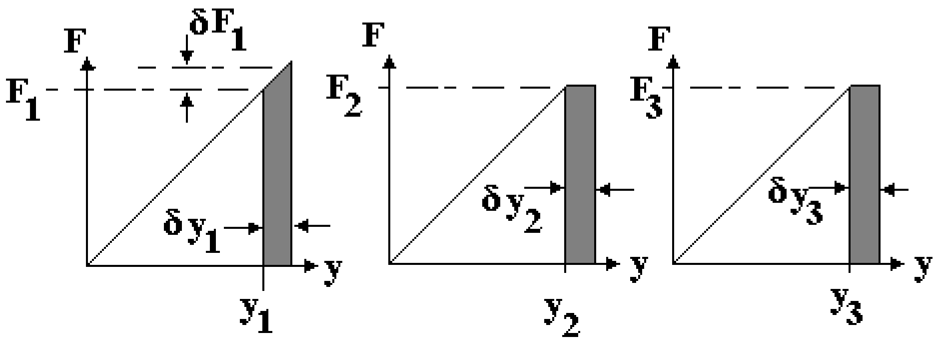


Figure 22

he increase in the work done and hence the strain energy δU is represented by the shaded areas (the increase in the areas) under the graphs. Note the first one is a tall rectangle with a small triangle on top and the other two are just tall rectangles.

$$\delta U = F_1 \delta y_1 + \frac{\delta F_1 \delta y_1}{2} + F_2 \delta y_2 + F_3 \delta y_3$$

The second term (the area of the small triangle) is very small and is ignored.

$$\delta U = F_1 \delta y_1 + F_2 \delta y_2 + F_3 \delta y_3 \dots\dots (B)$$

Now suppose that the same final points were arrived at by the gradual application of all three loads as shown.

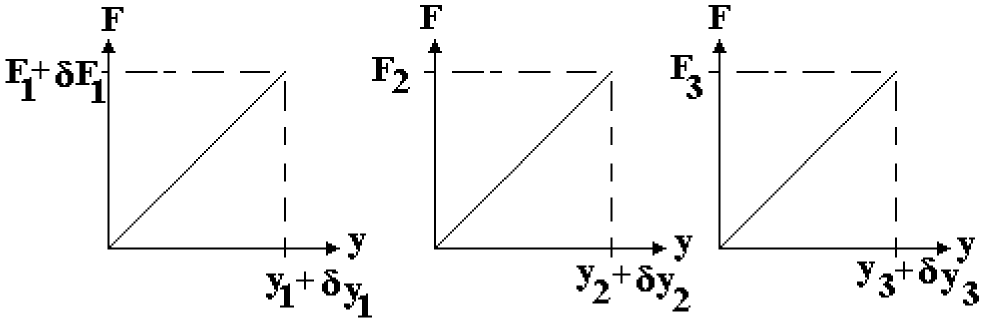


Figure 23

The work done and hence the strain energy is the area under the graphs.

$$U = \frac{(F_1 + \delta F_1)(y_1 + \delta y_1)}{2} + \frac{(F_2 + \delta F_2)(y_2 + \varepsilon y_2)}{2} + \frac{(F_3 + \delta F_3)(y_3 + \delta y_3)}{2} \ldots\ldots (C)$$

The change in strain energy is found this time by subtracting (A) from (C). This may be equated to (B). This is a major piece of algebra that you might attempt yourself. Neglecting small terms and simplifying we get the simple result

$$y_1 = \frac{\delta U}{\delta F_1}$$

Since this was found by keeping the other forces constant, we may express the equation in the form of partial differentiation since this is the definition of partial differentiation.

$$y_1 = \frac{\partial U}{\partial F_1}$$

If we repeated the process making F_2 change and keeping F_1 and F_3 constant we get:

$$y_2 = \frac{\partial U}{\partial F_2}$$

If we repeated the process making F_3 change and keeping F_1 and F_2 constant we get:

$$y_3 = \frac{\partial U}{\partial F_3}$$

This is Castigliano's theorem – the deflection at a point load is the partial differentiation of the rain energy with respect to that load.

Applying this is not so easy as you must determine the complete equation for the strain energy in the structure with all the forces left as unknowns until the end.

If the deflection is required at a point where there is no load, an imaginary force is placed there and then made zero at the last stage.

WORKED EXAMPLE No.11

The diagram shows a simple frame with two loads. Determine the deflection at both. The flexural stiffness of both sections is 2 MNm^2.

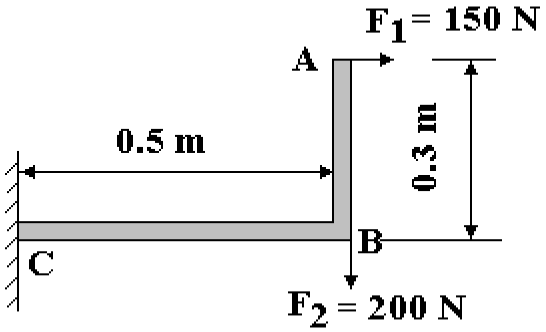


Figure 24

SOLUTION

It is important to note from the start that section AB bends and the bending moment at B turns the corner and section BC bends along its length due to both forces. Also, section BC is stretched but we will ignore this as the strain energy will be tiny compared to that produced by bending. Consider each section separately.

Section AB - Measure the moment arm x from the free end

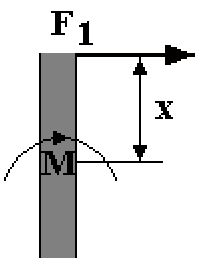


Figure 25

$M = F_1 x$ (x measured from the free end)

$$U = \frac{1}{2EI}\int_0^{0.3} M^2 dx = \frac{1}{2EI}\int_0^{0.3} (F_1 x)^2 dx = \frac{F_1^{\ 2}}{2EI}\int_0^{0.3} x^2 dx$$

$$U = \frac{F_1^{\ 2}}{2EI}\left[\frac{x^3}{3}\right]_0^{0.3} = \frac{F_1^{\ 2}}{2 \times 2 \times 10^6}\left[\frac{0.3^3}{3} - 0\right]$$

$$U = 2.25 \times 10^{-9} F_1^{\ 2}\text{Joules}$$

Section BC

The bending moment at point B is 0.3 F_1. This is carried along the section BC as a constant value. The moment arm x is measured from point B.
The second force produces additional bending moment of F_2 x. Both bending moments are in the same direction so they add. It is important to decide in these cases whether they add or subtract as deciding whether they are hogging (minus) or sagging (plus) is no longer relevant.

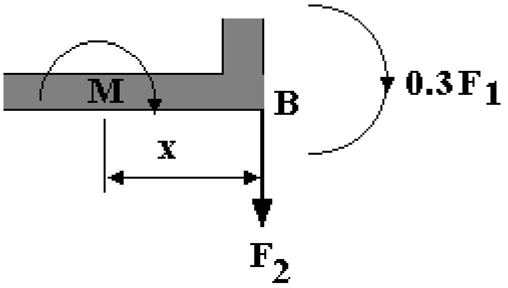


Figure 26

M = 0.3 F_1 + F_2 x

$$U = \frac{1}{2EI}\int_0^{0.5} M^2 dx = \frac{1}{2EI}\int_0^{0.5} (0.3F_1 + F_2 x)^2 dx = \frac{1}{2EI}\int_0^{0.5} \{(0.3F_1)^2 + (F_2 x)^2 + (0.6F_1F_2x)\}\, dx$$

$$U = \frac{1}{2EI}\int_0^{0.5} \{0.09{F_1}^2 + F_2^2x^2 + (0.6F_1F_2x)\}dx = \frac{1}{2EI}\left[0.09{F_1}^2 x + \frac{F_2^2x^3}{3} + \frac{0.6F_1F_2x^2}{2}\right]_0^{0.5}$$

$$U = \frac{1}{2 \times 2 \times 10^6}\left[0.09{F_1}^2 x + \frac{0.5^3F_2^2}{3} + \frac{0.6F_1F_2 \times 0.5}{2}\right]$$

$$U = 11.25{F_1}^2 \times 10^{-9} + 10.417{F_2}^2 \times 10^{-9} + 18.75F_1F_2 \times 10^{-9}$$

The total strain energy is

$$U = 11.25{F_1}^2 \times 10^{-9} + 10.417{F_2}^2 \times 10^{-9} + 18.75F_1F_2 \times 10^{-9} + 2.25 \times 10^{-9}{F_1}^2 \text{Joules}$$

$$U = 13.5{F_1}^2 \times 10^{-9} + 10.417{F_2}^2 \times 10^{-9} + 18.75F_1F_2 \times 10^{-9}\text{Joules}$$

To find y_1 carry out partial differentiation with respect to F_1.

$$y_1 = \frac{\partial U}{\partial F_1} = 27F_1 \times 10^{-9} + 0 + 18.75F_2 \times 10^{-9}$$

$$y_1 = 27 \times 150 \times 10^{-9} + 18.75 \times 200 \times 10^{-9} = 7.8 \times 10^{-6} \text{ m}$$

To find y_2 carry out partial differentiation with respect to F_2.

$$y_2 = \frac{\partial U}{\partial F_2} = 0 + 20.834F_2 \times 10^{-9} + 18.75F_1 \times 10^{-9}$$

$$y_2 = 20.834 \times 150 \times 10^{-9} + 18.75 \times 200 \times 10^{-9} = 7 \times 10^{-6} \text{ m}$$

SELF ASSESSMENT EXERCISE No. 5

Find the vertical deflection at the corner of the frame shown (point B).
EI = 1.8×10^6 N m^2 for both sections.

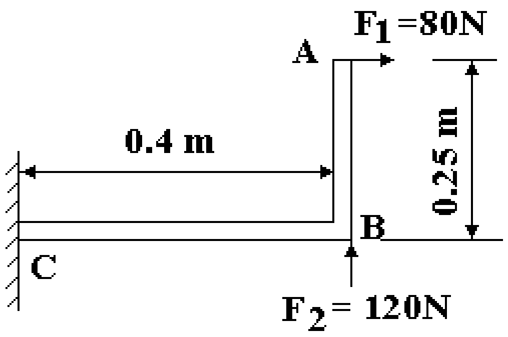


Figure 27

(Answer 533×10^{-9} m)