COMPLEX STRESS

TUTORIAL 2

STRESS AND STRAIN

This tutorial covers elements of the following syllabi.

- All of Edexcel HNC Mechanical Principles UNIT 21722P outcome 1.
- Part of the Statics section of the Eng. Council Certificate Exam C105
- Parts of the Engineering Council exam subject C103 Engineering Science. The relevant sections are coloured red.

You should judge your progress by completing the self assessment exercises. These may be sent for marking at a cost (see home page).

On completion of this tutorial you should be able to do the following.

- Define the elastic properties of materials.
- Define and use direct stress.
- Define and use direct strain.
- Define and use Modulus of Elasticity.
- Define and use Poisson’s Ratio.
- Define and calculate lateral strains.
- Solve problems involving two dimensional stress systems.
- Extend the work to 3 dimensional stress systems.
- Define and calculate volumetric strain.
- Define and use Bulk Modulus.
- Derive the relationships between the elastic constants.
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1. **MATERIAL TYPES**

An elastic material will spring back to its original shape and size when the forces causing it to deform are removed.

Elastic materials have several constants which you either already know or will learn soon. These are

- Modulus of Elasticity \( E \)
- Modulus of Rigidity \( G \)
- Bulk Modulus \( K \)
- Poissons' Ratio \( \nu \)

Materials may be further classified as follows.

**ISOTROPIC MATERIAL**

In this type of material the elastic constants are the same in all directions so if a specimen is cut from a bulk material, the direction in which it is cut has no affect on the values. This applies to most metals with no pronounced grain structure.

**ORTHOTROPIC MATERIAL**

In this type of material, the elastic constants have different values in the x, y and z directions so the results obtained in a test depend upon the direction in which the specimen was cut from the bulk material. This applies to materials with grain structures such as wood or rolled metals.

**NON-ISOTROPIC MATERIAL**

In this type of material, the elastic constants are unpredictable and the results from any two tests are never the same. This applies to materials such as glass and other ceramics.
2. **DIRECT STRESS $\sigma$**

When a force $F$ acts directly on an area $A$ as shown in figure 1, the resulting direct stress is the force per unit area and is given as

$$\sigma = \frac{F}{A}.$$  

$F$ is the force normal to the area in Newtons  
$A$ is the area in m$^2$  
$\sigma$ (sigma) is the direct stress in N/m$^2$ or Pascals.

Since 1 Pa is a small unit kPa, MPa and GPa are commonly used.

If the force pulls on the area so that the material is stretched then it is a tensile force and stress. This is regarded as positive.

If the force pushes on the surface so that the material is compressed, then the force and stress is compressive and negative.

![Diagram of stress](image)

Figure 1

3. **DIRECT STRAIN $\epsilon$**

Consider a piece of material of length $L$ as shown in figure 1. The direct stress produces a change in length $\Delta L$. The direct strain produced is $\epsilon$ (epsilon) defined as

$$\epsilon = \frac{\Delta L}{L}$$

The units of change in length and original length must be the same and the strain has no units. Strains are normally very small so often to indicate a strain of $10^{-6}$ we use the name micro strain and write it as $\mu \epsilon$.

For example we would write a strain of $7 \times 10^{-6}$ as $7 \mu \epsilon$.

Tensile strain is positive and compressive strain is negative.
4. **MODULUS OF ELASTICITY E**

Many materials are elastic up to a point. This means that if they are deformed in any way, they will spring back to their original shape and size when the force is released. It has been established that so long as the material remains elastic, the stress and strain are related by the simple formula

\[ E = \frac{\sigma}{\varepsilon} \]

E is called the MODULUS OF ELASTICITY. The units are the same as those of stress.

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**WORKED EXAMPLE No.1**

A metal bar which is part of a frame is 50 mm diameter and 300 mm long. It has a tensile force acting on it of 40 kN which tends to stretch it. The modulus of elasticity is 205 GPa. Calculate the stress and strain in the bar and the amount it stretches.

**SOLUTION**

\[ F = 40 \times 10^3 \text{ N.} \]
\[ A = \pi D^2/4 = \pi \times 50^2/4 = 1963 \text{ mm}^2 \]
\[ \sigma = \frac{F}{A} = \frac{(40 \times 10^3)}{(1963 \times 10^{-6})} = 20.37 \times 10^6 \text{ N/m}^2 = 20.37 \text{ MPa} \]
\[ E = \frac{\sigma}{\varepsilon} = 205 \times 10^9 \text{ N/m}^2 \]
\[ \varepsilon = \frac{\sigma}{E} = \frac{(20.37 \times 10^6)}{(205 \times 10^9)} = 99.4 \times 10^{-6} \text{ or } 99.4 \mu\varepsilon \]
\[ \varepsilon = \frac{\Delta L}{L} \]
\[ \Delta L = \varepsilon \times L = 99.4 \times 10^{-6} \times 300 \text{ mm} = 0.0298 \text{ mm} \]
5. **POISSON'S RATIO**

Consider a piece of material in 2 dimensions as shown in figure 2. The stress in the y direction is $\sigma_y$ and there is no stress in the x direction. When it is stretched in the y direction, it causes the material to get thinner in all the other directions at right angles to it. This means that a negative strain is produced in the x direction. For elastic materials it is found that the applied strain ($\varepsilon_y$) is always directly proportional to the induced strain ($\varepsilon_x$) such that

$$\frac{\varepsilon_x}{\varepsilon_y} = -\nu$$

$\nu$ (Nu) is an elastic constant called Poisson’s ratio.

The strain produced in the x direction is $\varepsilon_x = -\nu \varepsilon_y$

If stress is applied in x direction then the resulting strain in the y direction would similarly be $\varepsilon_y = -\nu \varepsilon_x$

Now consider that the material has an applied stress in both the x and y directions (figure 3).

![Figure 2](image2.png)

![Figure 3](image3.png)
The resulting strain in any one direction is the sum of the strains due to the direct force and the induced strain from the other direct force.

Hence

\[
\varepsilon_x = \frac{\sigma_x}{E} - \frac{\nu \sigma_y}{E} = \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E}
\]

\[
\varepsilon_x = \frac{1}{E} \left( \sigma_x - \nu \sigma_y \right), \quad \ldots (1A)
\]

Similarly \( \varepsilon_y = \frac{1}{E} \left( \sigma_y - \nu \sigma_x \right), \quad \ldots (1B) \)

The modulus E must be the same in both directions and such a material is not only elastic but ISOTROPIC.

**WORKED EXAMPLE No.2**

A material has stresses of 2 MPa in the x direction and 3 MPa in the y direction. Given the elastic constants \( E = 205 \text{ GPa} \) and \( \nu = 0.27 \), calculate the strains in both direction.

**SOLUTION**

\[
\varepsilon_x = \frac{1}{E} \left( \sigma_x - \nu \sigma_y \right) = \frac{1}{205 \times 10^9} \left( 2 \times 10^6 - 0.27 \times 3 \times 10^6 \right) = 5.8 \mu \varepsilon
\]

\[
\varepsilon_y = \frac{1}{E} \left( \sigma_y - \nu \sigma_x \right) = \frac{1}{205 \times 10^9} \left( 3 \times 10^6 - 0.27 \times 2 \times 10^6 \right) = 12 \mu \varepsilon
\]

**WORKED EXAMPLE No.3**

A material has stresses of -2 MPa in the x direction and 3 MPa in the y direction. Given the elastic constants \( E = 205 \text{ GPa} \) and \( \nu = 0.27 \), calculate the strains in both direction.

**SOLUTION**

\[
\varepsilon_x = \frac{1}{E} \left( \sigma_x - \nu \sigma_y \right) = \frac{1}{205 \times 10^9} \left( -2 \times 10^6 - 0.27 \times 3 \times 10^6 \right) = 13.7 \mu \varepsilon
\]

\[
\varepsilon_y = \frac{1}{E} \left( \sigma_y - \nu \sigma_x \right) = \frac{1}{205 \times 10^9} \left( 3 \times 10^6 - 0.27 \times (-2 \times 10^6) \right) = 17.3 \mu \varepsilon
\]

Note that we do not have to confine ourselves to the x and y directions and that the formula works for any two stresses at 90° to each other. In general we use \( \sigma_1 \) and \( \sigma_2 \) with corresponding strains \( \varepsilon_1 \) and \( \varepsilon_2 \).
5.1 CONverting Strains Into Stresses

We have already derived
\[
\varepsilon_1 = \frac{\sigma_1 - \nu \sigma_2}{E}
\]
\[
\varepsilon_2 = \frac{\sigma_2 - \nu \sigma_1}{E}
\]
Rearrange to make \( \sigma_2 \) the subject.
\[
\sigma_2 = \varepsilon_2 E + \nu \sigma_1
\]
Substitute this into the first formula
\[
\varepsilon_2 = \frac{\left( \varepsilon_2 E + \nu \sigma_1 \right) - \nu \sigma_1}{E}
\]
Rearrange
\[
\sigma_1 = \left( \frac{E}{1-\nu^2} \right) (\varepsilon_1 + \nu \varepsilon_2)
\]
If we do the same but make \( \sigma_2 \) the subject of the formula we get
\[
\sigma_2 = \left( \frac{E}{1-\nu^2} \right) (\varepsilon_2 + \nu \varepsilon_1)
\]

**Self Assessment Exercise No.1**

1. Solve the strains in both directions for the case below.

\( E = 180 \text{ GPa} \quad \nu = 0.3 \quad \sigma_1 = -3 \text{ MPa} \quad \sigma_2 = 5 \text{ MPa} \)

(Answers 32.78 \( \mu \varepsilon \) and -25 \( \mu \varepsilon \))

2. Solve the stresses in both directions for the cases below.

\( E = 200 \text{ GPa} \quad \nu = 0.26 \quad \varepsilon_1 = -4.9 \mu \varepsilon \quad \varepsilon_2 = -8.05 \mu \varepsilon \)

(Answers -1.5 MPa and -2 MPa)

3. \( E = 205 \text{ GPa} \quad \nu = 0.25 \quad \varepsilon_1 = 0.366 \mu \varepsilon \quad \varepsilon_2 = 2.195 \mu \varepsilon \)

(Answers 200 kPa and 500 kPa)
6. THREE DIMENSIONAL STRESS AND STRAIN

Equations A and B were derived for a 2 dimensional system. Suppose a material to be stressed in mutually perpendicular directions x, y, and z. The strain in any one of these directions is reduced by the effect of the strain in the other two directions and the three strains are

\[ \varepsilon_x = \frac{1}{E} \left( \sigma_x - \nu (\sigma_y + \sigma_z) \right) = \frac{1}{E} \left[ \sigma_x - \nu (\sigma_y + \sigma_z) \right] \] \hspace{1cm} (1C)

\[ \varepsilon_y = \frac{1}{E} \left( \sigma_y - \nu (\sigma_x + \sigma_z) \right) = \frac{1}{E} \left[ \sigma_y - \nu (\sigma_x + \sigma_z) \right] \] \hspace{1cm} (1D)

\[ \varepsilon_z = \frac{1}{E} \left( \sigma_z - \nu (\sigma_x + \sigma_y) \right) = \frac{1}{E} \left[ \sigma_z - \nu (\sigma_x + \sigma_y) \right] \] \hspace{1cm} (1E)

6.1 VOLUMETRIC STRAIN \( \varepsilon_V \)

Now consider a cube of material is stressed in the x direction by a compressive pressure as shown in figure 4. The change in volume is \( L^2 \Delta L \).

![Figure 4](image)

Now consider that the cube is strained by an equal amount in the y and z directions also. With very little error the total change in volume is \( 3L^2 \Delta L \). The original volume is \( L^3 \).

When a solid object is subjected to a pressure \( p \) such that the volume is reduced, the volumetric strain is \( \varepsilon_V = \frac{\text{change in volume}}{\text{original volume}} \).

In the case of a cube this becomes

\[ \varepsilon_V = \frac{3L^2 \Delta L}{L^3} = 3\Delta L / L \]

\[ \varepsilon_V = 3\varepsilon \] \hspace{1cm} (1F)

\( \varepsilon \) is the equal strain in all three directions.

It follows that when a material is compressed by a pressure which by definition must be equal in all directions, the volumetric strain is three times the linear strain in any direction.
6.2 **BULK MODULUS**

It has been established that the volumetric strain in an elastic material is directly proportional to the stress such that

\[
\frac{\sigma}{\varepsilon_v} = K
\]

K is called the Bulk Modulus. This is another of the elastic constants of a material.

6.3 **RELATIONSHIP BETWEEN THE ELASTIC CONSTANTS**

When the material is compressed by a pressure p the stress is equal to -p because it is compressive. The bulk modulus is then

\[
K = \frac{-p}{\varepsilon_v} \quad \text{(1G)}
\]

From equation F we have \( \varepsilon_v = 3\varepsilon \)

From equations C, D and E the strains in all direction being equal and the stresses being equal to -p, we have

\[
\varepsilon = \frac{1}{E}(-p + \nu 2p)
\]

The volumetric strain is

\[
\varepsilon_v = 3\varepsilon = \frac{3}{E}(-p + \nu 2p) \quad \text{.........(1H)}
\]

Combining equations G and H we have

\[
K = \frac{-p}{\left(\frac{3}{E}\right)(-p + \nu 2p)} = \frac{E}{3(1-2\nu)} \quad \text{.........(1I)}
\]

This shows the relationship between E, K and \( \nu \).

It may be shown that the relationship of the shear modulus G to the other elastic constants is given by

\[
G = \frac{E}{2(1+\nu)} \quad \text{.........(1J)}
\]

You would need to study three dimensional stress systems in depth in order to derive this equation.
WORKED EXAMPLE No.4

A solid piece of metal of volume 8000 mm$^3$ is compressed by a pressure of 80MPa. Determine the bulk modulus given that the elastic modulus E is 71 GPa and Poisson's ratio $\nu$ is 0.34. Determine the change in volume.

SOLUTION

$$K = \frac{E}{3(1-2\nu)} = \frac{71 \times 10^9}{3(1-2 \times 0.34)} = 73.96 \text{ GPa}$$

$$K = \frac{-P}{\epsilon_v} \quad \epsilon_v = \frac{-P}{K} = \frac{-80 \times 10^6}{73.96 \times 10^9} = 0.00108$$

$$\epsilon_v = \frac{\Delta V}{V} \quad \Delta V = V\epsilon_v = 8000 \times 0.00108 = 8.65 \text{ mm}^3$$

SELF ASSESSMENT EXERCISE No.2

1. Find the bulk modulus $K$ and shear modulus $G$ for aluminium given that the elastic modulus $E$ is 71 GPa and Poisson's ratio $\nu$ is 0.34.

(Answers 73.9 GPa and 26.49GPa).

2. A cube is stressed in 3 mutually perpendicular direction x, y and z. The stresses in these directions are

$\sigma_x = 50 \text{ kPa} \quad \sigma_y = 80 \text{ kPa} \quad \sigma_z = -100 \text{ kPa}$

Determine the strain in the x direction. $\nu$ is 0.34 and $E$ is 71 GPa.

(Answer $800 \times 10^{-9}$)