SOLID MECHANICS
STATICS
STRESS PART 1
DIRECT STRESS AND STRAIN

This material is set at the British NQF Levels 3 and 4.

On completion of this tutorial you should be able to do the following.

- Define the elastic properties of materials.
- Define and use direct stress.
- Define and use direct strain.
- Define and use Modulus of Elasticity.
- Define and use Poisson's Ratio.
- Define and calculate lateral strains.
- Solve problems involving two dimensional stress systems.
- Extend the work to 3 dimensional stress systems.
- Define and calculate volumetric strain.
- Define and use Bulk Modulus.
- Derive the relationships between the elastic constants.

Contents

1. Introduction

This tutorial is about stress and strain in loaded structures and components made from elastic materials. The tutorial contains the prerequisite material on stress and strain to complete the work on complex stress. **Click here for a more extensive tutorial**

The solutions to the self assessment exercises are given separately but require a small donation to access them.

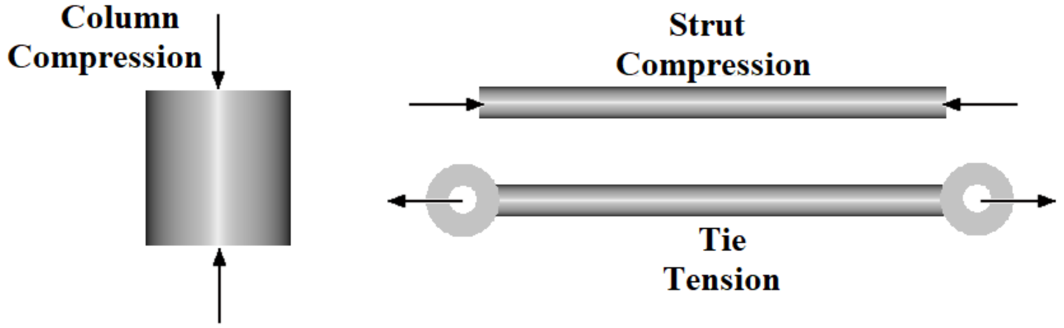
Typical values for the various material properties mentioned in this tutorial may be found in tables on the home page for www.freestudy.co.uk.

2. Static Forces and Structures

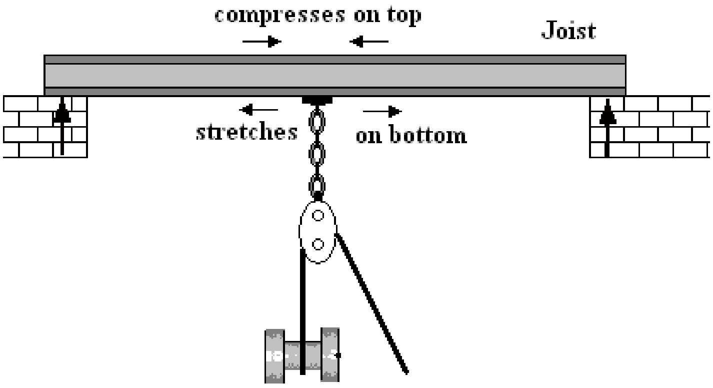
Static Forces will deform a body in one or more of the following ways.

- The force may stretch the body, in which case it is called a ***Tensile Force***.
- The force may squeeze the body in which case it is called a ***Compressive Force***.

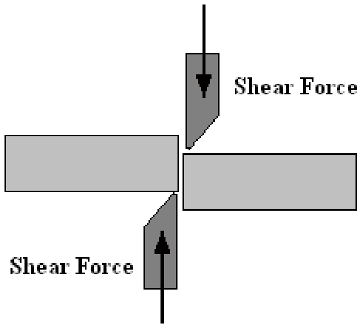
A rod or rope used in a frame to take a tensile load is called a ***Tie***. If it takes a compressive force it is called a ***Strut***. A strut that is thick compared to its length is called a ***Column***.



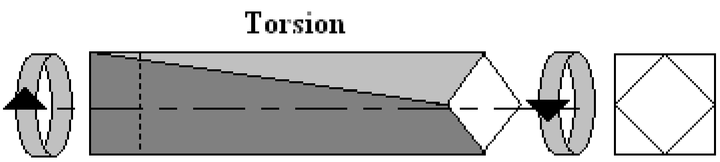
- The force may bend the body in which case both tensile and compressive forces may occur. A structure used to support a bending load is called a ***Beam*** or ***Joist***.



- The force may try to shear the body in which case the force is called a ***SHEAR FORCE.*** A scissors or guillotine produces shear forces.

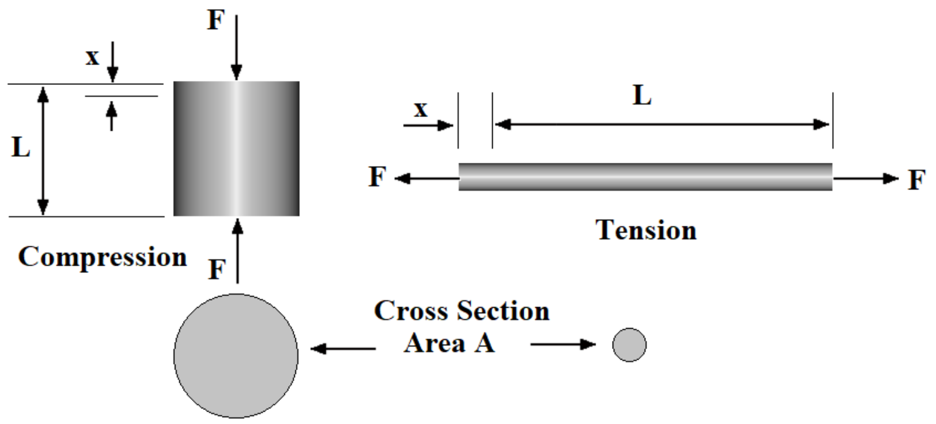


- The force may twist the body in which case ***Shear Forces*** occur. A structure that transmits rotation is called a ***Shaft*** and it experiences ***Torsion***.



3. *Direct Stress σ*

When a force is applied to an elastic body, the body deforms. The way in which the body deforms depends upon the type of force applied to it. A compression force makes the body shorter. A tensile force makes the body longer.



Tensile and compressive forces are called ***Direct Forces.***

Stress is the force per unit area upon which it acts. **Stress = σ = Force/Area** N/m^2 or Pascals.

The symbol **σ** is called *Sigma*

Note On Units **- The fundamental unit of stress is 1 N/m^2 and this is called a Pascal. This is a small quantity in most fields of engineering so we use the multiples kPa, MPa and GPa.**

Areas may be calculated in mm^2 and units of stress in N/mm^2 are quite acceptable.
Since 1 N/mm^2 converts to 1 000 000 N/m^2 then it follows that the N/mm^2 is the same as a MPa

4. *Direct Strain ε*

In each case, a force F produces a deformation x. In engineering we usually change this force into stress and the deformation into strain and we define these as follows.

Strain is the deformation per unit of the original length

$$\textbf{Strain} = \varepsilon = \frac{\mathbf{x}}{\mathbf{L}}$$

The symbol $\boldsymbol{\varepsilon}$ is called *Epsilon*

Strain has no units since it is a ratio of length to length. Most engineering materials do not stretch very much before they become damaged so strain values are very small figures. It is quite normal to change small numbers in to the exponent for of 10^{-6}. Engineers use the abbreviation $\mu\varepsilon$ (micro strain) to denote this multiple.

For example a strain of 0.000068 could be written as 68 x 10^{-6} but engineers would write 68 $\mu\varepsilon$.

Note that when conducting a British Standard tensile test the symbols for original area are S_o and for Length is L_o.

WORKED EXAMPLE No. 1

A metal wire is 2.5 mm diameter and 2 m long. A force of 1 200 N is applied to it and it stretches 0.3 mm. Assume the material is elastic. Determine the following.

i. The stress in the wire (σ).
ii. The strain in the wire (ε).

If the stress that produces failure is 300 MPa, calculate the safety factor.

SOLUTION

$$A = \frac{\pi d^2}{4} = \frac{\pi \text{ x } 2.5^2}{4} = 4.909 \text{ mm}^2 \quad \sigma = \frac{F}{A} = \frac{1\,200}{4.909} = 244 \text{ N/mm}^2$$

Answer (i) is hence 244 MPa

$$\varepsilon = \frac{x}{L} = \frac{0.3 \text{ mm}}{2\,000 \text{ mm}} = 0.00015 \text{ or } 150 \ \mu\varepsilon$$

Safety Factor = 300/244 = 1.23

SELF ASSESSMENT EXERCISE No. 1

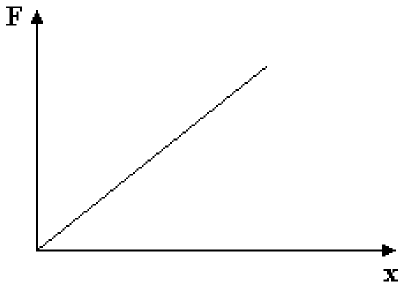
1. A steel bar is 10 mm diameter and 2 m long. It is stretched with a force of 20 kN and extends by 0.2 mm. Calculate the stress and strain. If the stress at failure is 300 MPa calculate the safety factor.
(Answers 254.6 MPa, 100 $\mu\varepsilon$ and 1.18)

2. A rod is 0.5 m long and 5 mm diameter. It is stretched 0.06 mm by a force of 3 kN. Calculate the stress and strain. If the stress at failure is 250 MPa calculate the safety factor.
(Answers 152.8 MPa, 120$\mu\varepsilon$ and 1.64)

5. *Modulus of Elasticity E*

Elastic materials always spring back into shape when released. They also obey ***Hooke's Law***. This is the law of a spring which states that deformation is directly proportional to the force.

$$F/x = \text{stiffness} = k \text{ N/m}$$



The stiffness is different for different materials and different sizes of the material. We may eliminate the size by using stress and strain instead of force and deformation as follows.

If F and x refer to direct stress and strain then

$F = \sigma A \quad x = \varepsilon L$ hence

$$\frac{F}{x} = \frac{\sigma A}{\varepsilon L} \text{ and } \frac{FL}{Ax} = \frac{\sigma}{\varepsilon}$$

The stiffness is now in terms of stress and strain only and this constant is called the ***Modulus of Elasticity* and it has a symbol E.**

$$E = \frac{FL}{Ax} = \frac{\sigma}{\varepsilon}$$

A graph of stress against strain will be a straight line with a gradient of E. The units of E are the same as the units of stress.

6. *Ultimate Tensile Stress*

If a material is stretched until it breaks, the tensile stress has reached the absolute limit and this stress level is called the ultimate tensile stress.

WORKED EXAMPLE No. 2

A steel tensile test specimen has a cross sectional area of 100 mm^2 and a gauge length of 50 mm. In a tensile test the graph of force - extension produces a gradient in the elastic section of 410 x 10^3 N/mm. Determine the modulus of elasticity.

The specimen breaks when the force is 35 kN. What is the ultimate tensile stress based on the original area?

SOLUTION

The gradient gives the ratio F/A = and this may be used to find E.

$$E = \frac{\sigma}{\varepsilon} = \frac{FL}{Ax} = 410 \text{ x}10^3 \text{x} \frac{50}{100} = 205\ 000 \frac{N}{mm^2} \text{ or } 205\ 000 \text{ MPa or } 205 \text{ GPa}$$

$$\sigma_u = \frac{\text{Breaking Force}}{\text{Area}} = \frac{35\ 000}{100} = 350 \frac{N}{mm^2} \text{ or } 350 \text{ MPa}$$

WORKED EXAMPLE No. 3

A Steel column is 3 m long and 0.4 m diameter. It carries a load of 50 MN. Given that the modulus of elasticity is 200 GPa, calculate the compressive stress and strain and determine how much the column is compressed.

SOLUTION

$$A = \frac{\pi d^2}{4} = \frac{\pi \text{ x } 0.4^2}{4} = 0.126 \text{ m}^2 \quad \sigma = \frac{F}{A} = \frac{50 \text{ x } 10^6}{0.126} = 397.9 \text{ x } 10^6 \text{ Pa}$$

$$E = \frac{\sigma}{\varepsilon} \text{ so } \varepsilon = \frac{\sigma}{E} = \frac{397.9 \text{ x } 10^6}{200 \text{ x } 10^9} = 0.001989$$

$$\varepsilon = \frac{x}{L} \text{ so } x = \varepsilon L = 0.001989 \text{ x } 3\ 000 = 5.97 \text{ mm}$$

SELF ASSESSMENT EXERCISE No. 2

1. A bar is 500 mm long and is stretched to 500.45 mm with a force of 15 kN. The bar is 10 mm diameter.

   Calculate the stress and strain.

   Assuming that the material has remained within the elastic limit, determine the modulus of elasticity.

   If the fail stress is 250 MPa calculate the safety factor.

   (Answers 191 MPa, 900με , 212.2 GPa and 1.31.)

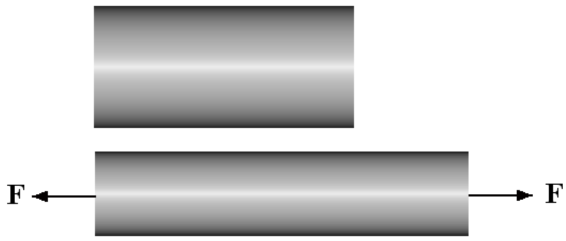
2. The maximum safe stress in a steel bar is 300 MPa and the modulus of elasticity is 205 GPa. The bar is 80 mm diameter and 240 mm long. If a factor of safety of 2 is to be used, determine the following.
   i. The maximum allowable stress. (150 MPa)
   ii. The maximum tensile force allowable. (754 kN)
   iii. The corresponding strain at this force. (731.7 με)
   iv. The change in length. (0.176 mm)

3. A circular metal column is to support a load of 500 Tonne and it must not compress more than 0.1 mm. The modulus of elasticity is 210 GPa. The column is 2 m long. Calculate the following.
   i. The cross sectional area($0.467\ m^2$)
   ii. The diameter. (0.771 m)

   Note 1 Tonne is 1000 kg.

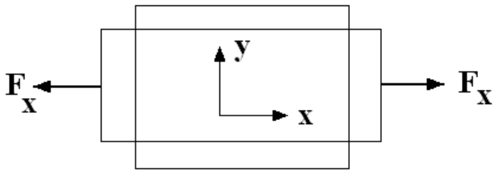
7. *Poisson's Ratio*

Consider a bar or tie that is stretched with a force F as shown. The bar will not only get longer in the direction it is stretched but it will also get thinner as shown.



Let's reduce this to two dimensions x and y.

The stress in the x direction is σ_x and there is no stress in the y direction. When it is stretched in the x direction, it causes the material to get thinner in all the other directions at right angles to it. This means that a negative strain is produced in the y direction. This is called the lateral strain. For elastic materials it is found that the lateral strain is always directly proportional to the applied such that



$$\frac{\varepsilon_y}{\varepsilon_x} = -\nu$$

ν (Nu) is an elastic constant called Poisson's' ratio.

The strain produced in the y direction is: $\varepsilon_y = -\nu\varepsilon_x$

If stress is applied in the y direction then the resulting strain in the x direction would similarly be $\varepsilon_x = -\nu\varepsilon_y$

NOTE that a force in the x direction acts on a plane in the y direction and many text books take σ_x and ε_x to mean the stress and strain on the x plane. Here we take it to mean the stress and strain in the x direction.

NOTE that we do not have to confine ourselves to the x and y directions and that the formula works for any two stresses at 90° to each other. In general we use σ_1 and σ_2 with corresponding strains ε_1 and ε_2. In the following examples we use σ_L and σ_D to mean in the direction of the length and any diameter.

WORKED EXAMPLE No. 4

A bar is 500 mm long and is stretched to 500.45 mm. The bar is 10 mm diameter. Given that ν = 0.23, calculate the new diameter.

SOLUTION

$$\varepsilon_L = \frac{\Delta L}{L} = \frac{0.45}{500} = 900\ \mu\varepsilon$$

$$\varepsilon_D = -\nu\sigma_L = -0.23 \times 900 = -207\ \mu\varepsilon$$

$$\Delta D = \varepsilon_D \times D = -207 \times 10^{-6} \times 10 = -2.07 \times 10^{-3}\text{ mm}$$

The new diameter is 10 - 0.00207 = 9.99793 mm

WORKED EXAMPLE No. 5

A tie bar 2 m long and 10 mm diameter is stretched with a stress of 2 MPa. Given the elastic constants are E = 205 GPa and ν = 0.27, calculate the strains in both the longitudinal and diametral directions. Calculate the change in length and change in diameter.

SOLUTION

$$\varepsilon_L = \frac{\sigma_L}{E} = \frac{2 \times 10^6}{205 \times 10^9} = 9.756\ \mu\varepsilon$$

$$\varepsilon_D = -\nu\sigma_D = -0.27 \times 9.756 = -2.634\ \mu\varepsilon$$

$$\Delta D = \varepsilon_D \times D = -207 \times 10^{-6} \times 10 = -2.07 \times 10^{-3}\ \text{mm}$$

$$\Delta L = \varepsilon_L L = 9.756 \times 10^{-6} \times 2\,000 = 0.0196\ \text{mm}$$

$$\Delta D = \varepsilon_D D = -2.634 \times 10^{-6} \times 10 = -26.34 \times 10^{-6}\text{mm}$$

WORKED EXAMPLE No. 6

A metal bar 0.5 m long and 0.2 m diameter is compressed by an axial load of 800 kN. Given the elastic constants are E = 200 GPa and ν = 0.25, calculate the stresses and strains in both the longitudinal and diametral directions. Calculate the change in length and change in diameter.

SOLUTION

$A = \pi D^2/4 = 31.42 \times 10^{-3}\ \text{m}^2$

$\sigma_L = -F/A = 800 \times 10^3/31.42 \times 10^{-3} = -25.462$ MPa (Compressive)

$$\varepsilon_L = \frac{\sigma_L}{E} = \frac{-25.462 \times 10^6}{200 \times 10^9} = -127.3\ \mu\varepsilon$$

$$\varepsilon_D = -\nu\sigma_D = -0.25 \text{ x } (-127.3\ \mu\varepsilon) = 31.826\mu\varepsilon$$

$$\Delta L = \varepsilon_L L = -127.3 \times 10^{-6} \times 500 = -63.65\ \text{mm}$$

$$\Delta D = \varepsilon_D D = 31.826 \times 10^{-6} \times 200 = 6.365 \times 10^{-3}\text{mm}$$

SELF ASSESSMENT EXERCISE No. 3

1. A tie is 1 m long and 5 mm diameter is stretched by 0.3 mm. Given that ν = 0.25, calculate the new diameter. (4.99962 mm)

2. A metal bar 0.2 m long and 0.1 m diameter is compressed by an axial load of 1 MN. Given the elastic constants are E = 200 GPa and ν = 0.25, calculate the stresses and strains in both the longitudinal and diametral directions. Calculate the change in length and change in diameter. (-0.1273 mm and 0.015913 mm)

8. *Stress in Two Mutually Perpendicular Directions.*

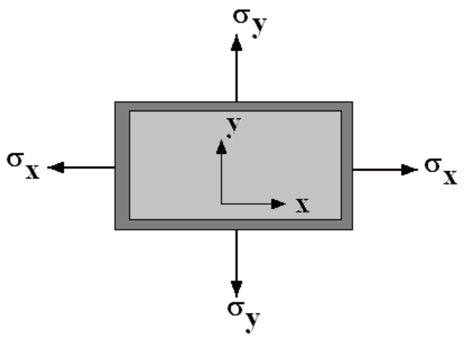
Now consider that the material has an applied stress in both the x and y directions. The resulting strain in any one direction is the sum of the direct strain and the lateral strain. Hence:

$$\varepsilon_x = \frac{\sigma_x}{E} = -\nu\sigma_y = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E}$$

$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\sigma_y\right) \ldots\ldots (A)$$

Similarly

$$\varepsilon_y = \frac{1}{E}\left(\sigma_y - \nu\sigma_x\right) \ldots\ldots (B)$$



The modulus E must be the same in both directions and such a material is not only elastic but ***Isotropic.***

WORKED EXAMPLE No. 7

A material has stresses of -2 MPa in the x direction and 3 MPa in the y direction. Given the elastic constants E = 205 GPa and ν= 0.27, calculate the strains in both direction.

SOLUTION

$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\sigma_y\right) = \frac{1}{205 \text{ x } 10^9}(-2 \text{ x } 10^6 - 0.27 \text{ x } 3 \text{ x } 10^6\,) = 13.7\ \mu\varepsilon$$

$$\varepsilon_y = \frac{1}{E}\left(\sigma_y - \nu\sigma_x\right) = \frac{1}{205 \text{ x } 10^9}\{3 \text{ x } 10^6 - 0.27(-2 \text{ x } 10^6)\} = 17.3\ \mu\varepsilon$$

WORKED EXAMPLE No. 8

A material has stresses of 2 MPa in the x direction and 3 MPa in the y direction. Given the elastic constants E = 205 GPa and ν = 0.27, calculate the strains in both direction.

SOLUTION

$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\sigma_y\right) = \frac{1}{205 \times 10^9}(2 \times 10^6 - 0.27 \times 3 \times 10^6\,) = 5.8\ \mu\varepsilon$$

$$\varepsilon_y = \frac{1}{E}\left(\sigma_y - \nu\sigma_x\right) = \frac{1}{205 \times 10^9}\{3 \times 10^6 - 0.27 \times 2 \times 10^6\} = 12\ \mu\varepsilon$$

WORKED EXAMPLE No. 9

A thin flat plate is 200 mm x 100 mm forms part of a structure and when in service a stress of 100 MPa is produced in the long direction and 150 MPa in the short direction. Given the elastic constants E = 205 GPa and ν = 0.25, calculate the strains in both direction. Calculate the change in dimensions ands area of the plate.

SOLUTION

$$\varepsilon_L = \frac{1}{E}(\sigma_L - \nu\sigma_S) = \frac{1}{205 \times 10^9}(100 \times 10^6 - 0.25 \times 150 \times 10^6) = 304.9\ \mu\varepsilon$$

$$\varepsilon_S = \frac{1}{E}(\sigma_S - \nu\sigma_L) = \frac{1}{205 \times 10^9}\{150 \times 10^6 - 0.25 \times 150 \times 10^6\} = 609.8\ \mu\varepsilon$$

Change in length = $\varepsilon_L \times 200 = 60.976 \times 10^{-3}$ mm

Change in side = $\varepsilon_S \times 100 = 60.976 \times 10^{-3}$ mm

Change in area = $60.976 \times 10^{-3} \times 60.976 \times 10^{-3} = 3.718 \times 10^{-3}$ mm

SELF ASSESSMENT EXERCISE No. 4

1. Solve the strains in both directions for the case below.

   E = 180 GPa ν =0.3 σ_1= - 3 MPa σ_2= 5 MPa

   (Answers 32.78 με and -25 με)

2. A large cylindrical pressure vessel is constructed from thin plate. When pressurised a tensile stress of 60 MPa is produced in the longitudinal direction and 120 MPa in the direction around the circumference. Calculate the strain in these directions.
   Part of the shell is a plate 100 mm x 100 mm and this may be treated as a flat plate. Calculate the change in dimensions and area.

   Take E = 205 GPa and ν = 0.25

   (146.3 με and 512.2 με)
   (14.364×10^{-3} mm and 51.22×10^{-3} mm)
   (6.555 mm^2)

9. *Three Dimensional Stress and Strain*

Equations A and B were derived for a 2 dimensional system. Suppose a material to be stressed in 3 mutually perpendicular directions x, y and z. The strain in any one of these directions is the direct strain plus the lateral strain from the other two directions. It follows that:

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y - \nu\sigma_z) = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] \ldots\ldots(C)$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x - \nu\sigma_z) = \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)] \ldots\ldots(D)$$

$$\varepsilon_Z = \frac{1}{E}(\sigma_z - \nu\sigma_x - \nu\sigma_y) = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)] \ldots\ldots(E)$$

If the stress is the same in all directions, then the strains are the same.

WORKED EXAMPLE No. 10

A solid cube of metal has sides of 200 mm. It is compressed by a pressure of 80MPa on all its faces. Determine the change in length of each side and the reduction of volume.
E is 71 GPa and Poisson's ratio ν is 0.34.

SOLUTION

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] = \frac{1}{71 \times 10^9}[80 \times 10^6 - 0.34(80 \times 10^6 + 80 \times 10^6)]$$

$$\varepsilon_x = 360.6\ \mu\varepsilon$$

The strain will be the same in the y and z directions

Change in length = $\delta = \varepsilon_x \times L = 360.6 \times 10^{-6} \times 200 = 72.113 \times 10^{-3}$ mm

The change in volume in each direction is $\delta \times L^2 = 72.113 \times 10^{-3} \times 200^2 = 2\,885$ mm^3

The total change in volume is $3 \times 2\,885 = 8\,654$ mm^3

Note this is not quite true because we should also subtract the corner common to all three which is $(72.113 \times 10^{-3})^3 = 0.000\,375$ mm^3 and is negligible.

SELF ASSESSMENT EXERCISE No. 5

1. A cube is stressed in 3 mutually perpendicular direction x, y and z. The stresses in these directions are

   σ_x = 50 MPa σ_y = 80 MPa σ_z = -100 MPa

   Determine the strain in each direction.
   ν is 0.34 and E is 71 GPa.

   (Answer 800×10^{-6}, 1.366×10^{-9} and -2.031×10^{-3})

2. A cube of metal of side 30 mm is placed inside a pressure vessel and the pressure is raised to 20 MPa. Given that E = 205 GPa and ν = 0.25 determine the change in volume of the cube.
   (Answer 3.951 mm^3)

3. Repeat question 2 for a solid sphere 30 mm diameter. (-2.069 mm^3)