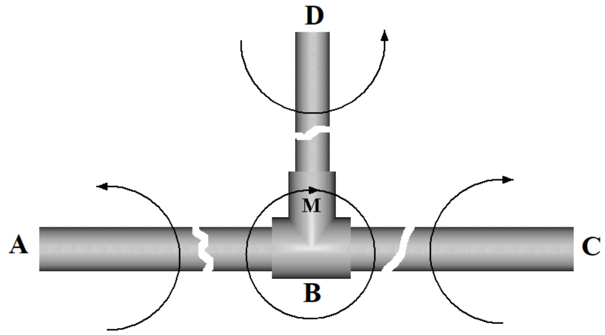
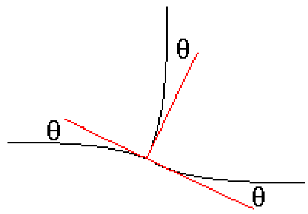
SOLID MECHANICS
STATICS - BEAMS
TUTORIAL 5 - MOMENT DISTRIBUTION METHOD

This is a 'stand-alone' tutorial for students studying structures and is set at NQF Level 5. It is about a method of finding the bending moment in beams that cannot easily be solved by other methods. The theory is due to the work of Professor Hardy Cross - the very same man who evolved the theory for solving pipe networks. The work can be used to solve the bending moment in frames with rigid joints. We will only be solving beams in this tutorial but we will start by considering the stiffness of a rigid joint with 3 members attached as shown below.

When a moment M is applied to the joint the moment is distributed to the connected members. The amount distributed to each depends on the stiffness of the member. Identify each member with letters as shown.

- The first point is that the joint will rotate a tiny angle θ and this is the same for all the members at the joint.





- The second point is that the moments in all the members must add up to total M.
- The fraction of M that is distributed to each member is called the ***Distribution Factor***
- All the fractions must add up to 1. It is normal to use decimal numbers rather than actual fractions.

The distribution factor depends on the stiffness of each member as seen at the joint. This depends on:

- The length, material and cross section (L, E and I)
- The way the other end of the member is fixed.

Now consider how to calculate the stiffness of each member.

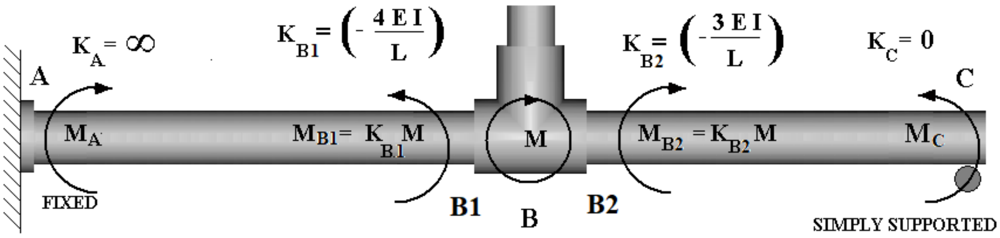
Rotational Stiffness K

Click here for the full derivation→ ROTATIONAL STIFFNESS

Rotational Stiffness K for each member is the ratio

$$K = \frac{M}{\theta}$$

M is the moment in the member at the joint. Use a suitable notation to identify K with the end of a member as shown.



At end B1 of AB the other end at A is rigidly fixed so

$$K_{B1} = \frac{M}{\theta} = -\frac{4EI}{L}$$

At end B2 of BC the other end is simply supported so

$$K_{B2} = \frac{M}{\theta} = -\frac{3EI}{L}$$

Ends A and C are both a joint also and has a stiffness factor.

The stiffness at a free end is zero – for example at C

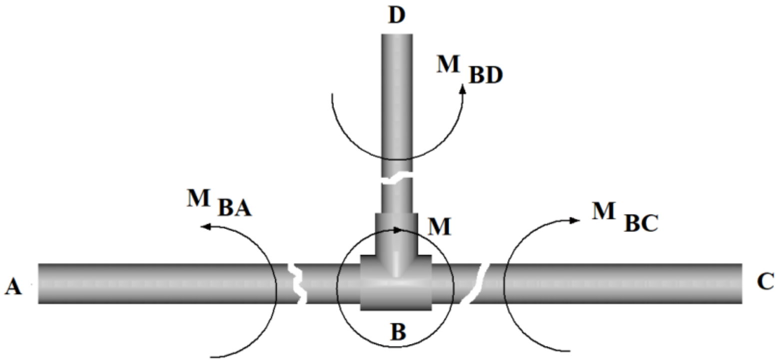
$$K_C = 0$$

The stiffness at a rigidly fixed end is infinity – for example at A.

$$K_A = \infty$$

Moment Distribution Factor

Click here for the derivation → DISTRIBUTION FACTOR



The moment M applied at the joint is distributed to the members in a proportion that depends on the stiffness of the member as seen from the joint. The moment at B is distributed to each member as shown.

$$M = M_{BA} + M_{BC} + M_{BD}$$

The proportion of M distributed to a member is called the ***Distribution Factor*** which we will designate K_d.

$$M_{BA} = (K_d)_{BA} M \quad M_{BC} = (K_d)_{BC} M \quad M_{BD} = (K_d)_{BD} M$$

If you study the derivation for K_D you will see that the possible values are given as

$$K_d = \frac{K}{\sum K}$$

Carry Over Factor

When a portion of M is distributed to a member, a moment is produced at the opposite end of the member. The proportion is called the ***Carr Over Factor*** and we will define it as K_{CF}.

For example the carryover from B to A is

$$M_A = (K_{CF})_A \times M_B$$

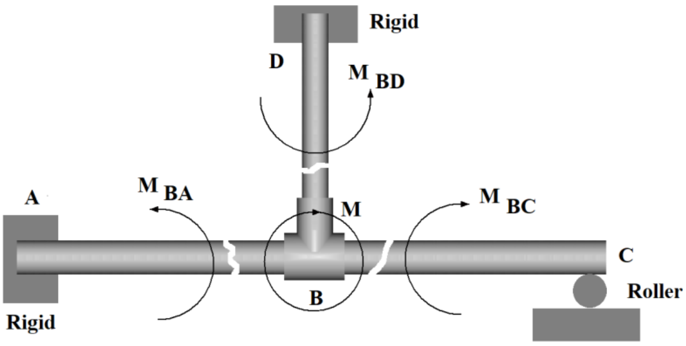
The derivation shows that when the other end is rigidly fixed K_{CF} = ½ of the moment at the joint end.

When the other end is simply supported K_{CF} = 0 (no moment possible at a simple support or pin joint).

WORKED EXAMPLE No. 1

Calculate the moments at each joint as a portion of M given that the EI values are the same for all members. The lengths of each member are:

BA = 1.5 m   BC = 1.2 m   BD = 1 m



Solution

First determine the stiffness of each section as seen from B

Member AB- The stiffness seen from B to a fixed end at A is

$$K_{BA} = -\frac{4EI}{L} = -\frac{4EI}{1.5} = -2.667\ EI$$

Member BC - The stiffness seen from B to a free end at C is

$$K_{BC} = -\frac{3EI}{L} = -\frac{3EI}{1.2} = -2.5\ EI$$

Member BC- The stiffness seen from B to a fixed end at D is

$$K_{BD} = -\frac{4EI}{L} = -\frac{4EI}{1} = -4\ EI$$

Total Stiffness

$$\sum K = -2.667\ EI - 2.5\ EI - 4\ E = 9.167EI$$

Calculate the Distribution Factors

At end B of BA

$$(K_d)_{BA} = -\frac{K_{BA}}{\sum K} = \frac{-2.667\ EI}{9.167\ EI} = 0.291$$

At end B of BC

$$(K_d)_{BC} = -\frac{K_{BC}}{\sum K} = \frac{-2.5\ EI}{9.167\ EI} = 0.273$$

At end B of BD

$$(K_d)_{BD} = -\frac{K_{BD}}{\sum K} = \frac{-4\ EI}{9.167\ EI} = 0.436$$

Note and check that this adds up to 1

Calculate the moments

$$M_{BA} = M(K_d)_{BA} = 0.291\ M$$

$$M_{BC} = M(K_d)_{BC} = 0.273\ M$$

$$M_{BD} = M(K_d)_{BD} = 0.436\ M$$

The total is M

Calculate the moment carried over to the end of each member

End A is rigidly fixed so $K_{CF} = ½$ $M_A = 0.29 \times M \times ½ = 0.1444$ M

End D is rigidly fixed so $K_{CF} = ½$ $M_D = 0.273 \times M \times ½ = 0.1365$ M

End C is free so $K_{CF} = 0$ $M_C = 0$

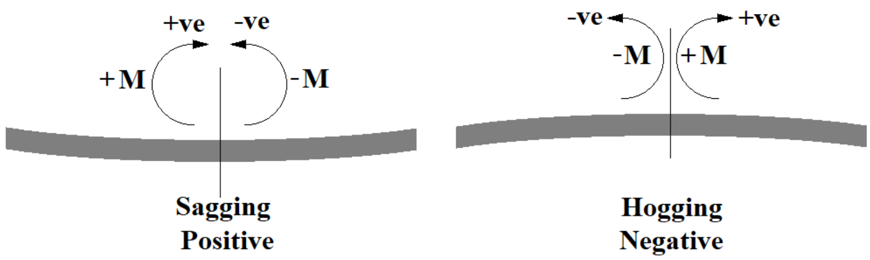
Application to Beams

The moment distribution method may be used to solve difficult problems that cannot be solved by other means because there are too many unknowns (Indeterminate beam or structure).

In this tutorial we will only study beams with joints that are either fixed or free to rotate such as the ones shown.

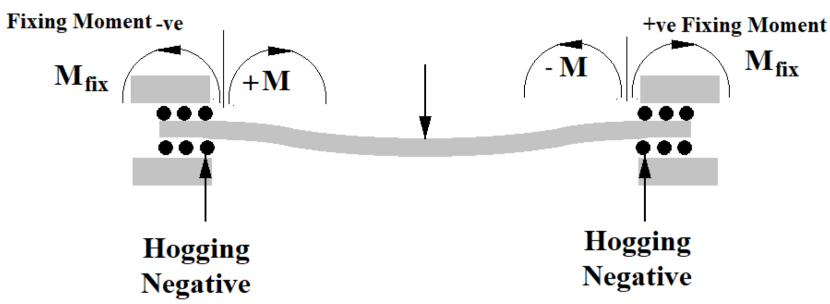
Polarity of Moments

It is necessary to understand which moments are positive and which negative in the following work.



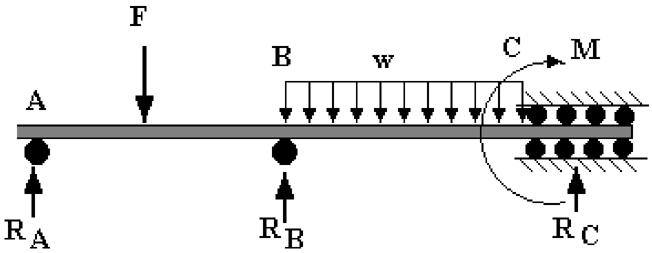
Sagging is positive and hogging is negative. At any point on the beam the bending is produced by equal and opposite moments M. Clockwise is positive and anticlockwise is negative. A sagging bending moment hence consists of +M on the left of the section and –M on the right. A hogging moment is –M on the left and +M on the right.

Consider a beam rigidly fixed at both ends.



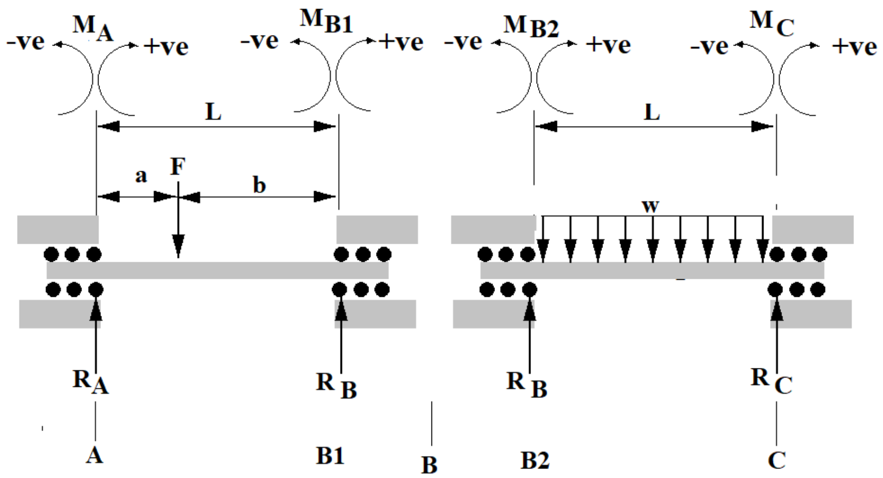
The beam is hogging at both ends. The fixing moment is the moment counteracting the bending moment in the beam. This is negative on the left end and positive on the right end. This will be used in the following examples.

In the following examples of an indeterminate beam it is rigidly held at the wall C and simply supported at A and B. There will be three vertical reactions due to the point load F and the uniform load w. These are too many unknowns to solve by normal means.



It might be that the lengths AB and BC have a different flexural stiffness EI and this can be accommodated.

We start be treating the beam as two separate parts AB_1 and B_2C as shown. We clamp each end rigidly as shown by applying fixing moments until there is no deflection or rotation. This enables us to tackle each section separately to begin.



In this example the moment must be hogging at all the ends. These are the ones we use in the calculations so the fixing moments are -M on the left and +M on the right of each section.

It can be shown that for a single point load F the bending moment at the wall is

$$M_A = -\frac{Fb^2a}{L^2} \text{ and } M_{B1} = \frac{Fa^2b}{L^2}$$

And for a single uniformly distributed load w

$$M_{B2} = -\frac{wL^2}{12} \text{ and } M_C = \frac{wL^2}{12}$$

For any other combination of loading, the principle of superposition may be used.

Click here for the derivation of the above formulae→ FIXING MOMENT

Those joints that in reality are not rigidly fixed must now be released one at a time. The bending moment at that point must equalise but depending on the stiffness of each member the difference is distributed by the ratio determined by the distribution factor.

Consider what happens when joint B is released.

The moments at B_1 and B_2 must equalise. Sum up the two moments M_{B1} and M_{B2} (which could include carry over moments if one or both ends have been released first).

Next calculate the distribution to each

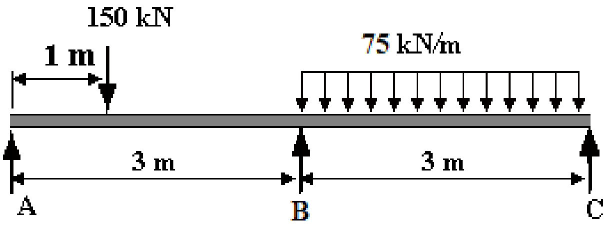
$$\text{Distribution} = K_d \times (M_{B1} + M_{B2})$$

The polarity of the distributed moment must be such that M_{B1} and M_{B2} are equal and opposite.

Let us do a numerical example to illustrate all this. These are the steps to be taken in the calculations for this example.

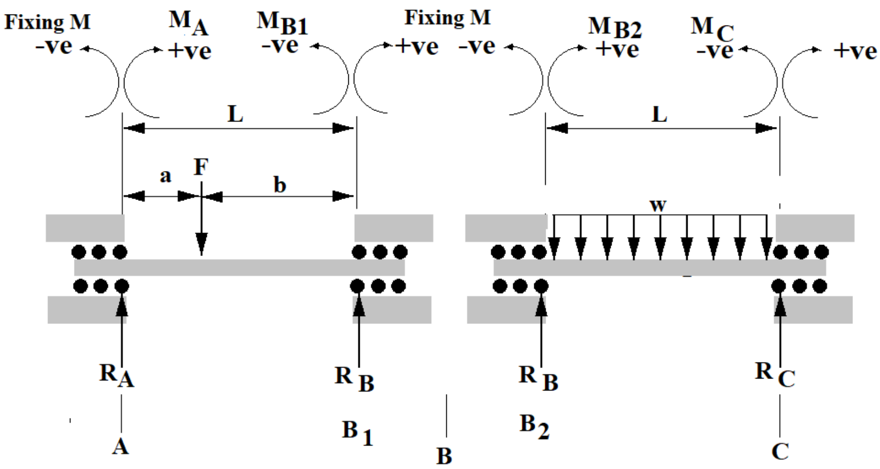
WORKED EXAMPLE No. 2

Find the bending moment at A, B and C for the beam shown. The EI values are the same for both sections.



Solution

First treat the beam as two separate sections with rigid ends.



Calculate Moments and Stiffness Factors

Member AB

$$M_A = \frac{Fb^2a}{L^2} = \frac{150 \times 10^3 \times 2^2 \times 1}{3^2} = 66.67 \times 10^3$$

$$M_{B1} = -\frac{Fa^2b}{L^2} = \frac{150 \times 10^3 \times 1^2 \times 2}{3^2} = -33.33 \times 10^3$$

$$K_{BA} = -\frac{3EI}{L} = -\frac{3EI}{3} = -1\ EI$$

Member BC

$$M_{B2} = \frac{wL^2}{12} = \frac{75 \times 10^3 \times 3^2}{12} = 56.25 \times 10^3 \qquad M_C = -\frac{wL^2}{12} = -56.25 \times 10^3$$

$$K_{BC} = -\frac{3EI}{L} = -\frac{3EI}{3} = -1\ EI$$

Calculate the Distribution Factors

At end B of BA

$$K_d = -\frac{K_{BA}}{\sum K} = \frac{-1\ EI}{-1\ EI - 1\quad EI} = 0.5$$

At end B of BC

$$K_d = -\frac{K_{BC}}{\sum K} = \frac{-1\ EI}{-1\ EI - 1\ EI} = 0.5$$

Determine the ***Carry Over Factor***

When a balancing moment is applied at a point, ½ of the moment is carried over to a fixed end and zero to a free end. If we released the joints that were fixed we to equalise the moments by working out the difference and distributing it using K_d.

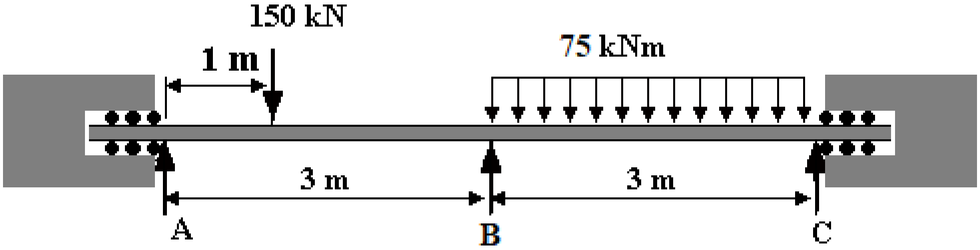
First release A then C and finally B as shown below.

	A	**B_1**	**B**	**B_2**	**C**
Distribution Factor K_d		0.5		0.5	
Moment M_A, M_{B1}, M_{B1}, M_A	66.66667	-33.3333		56.25	-56.25
Release A	-66.6667				
Carry Over Half	→	-33.3333			
Release C					56.25
Carry over half				28.125	←
Totals	0	-66.6667		84.375	0
Release B					
Difference at B			17.71		
Distribute /Equalise	0	-8.85	←→	-8.85	0
Carry Over to free ends A and B	0				0
Final Totals	0	-75.52		75.52	

At B we have achieved equal and opposite moments. The moment at A and C are zero. The moment at B is 75.52 kN m (Hogging).

WORKED EXAMPLE No. 3

Repeat the last example for the case both ends are rigidly fixed.



Solution

First treat the beam as two separate sections with rigid ends.

The moments, stiffness factors and distribution factors are the same as before.

Member AB

$$M_A = 66.67 \times 10^3 \qquad M_{B1} = -33.33 \times 10^3 \quad K_{BA} = -\frac{4EI}{L} = -\frac{4EI}{3} = -1.333\ EI$$

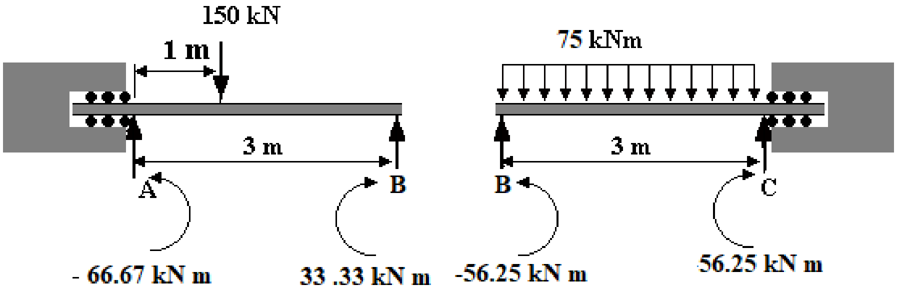
Member BC

$$M_{B2} = 56.25 \times 10^3 \quad M_C = -56.25 \times 10^3 \quad K_{BC} = -\frac{4EI}{L} = -\frac{4EI}{3} = -1.333\ EI$$

At end B of BA and BC $K_d = 0.5$

Carry Over Factors -When a balancing moment is applied at a point, ½ of the moment is carried over to both rigid ends. ***To solve this case we only need to release B***.

When B is released carry over moments are created to the opposite ends. Since these are fixed we carry over half.



	A	B_1	B	B_2	C
K_d		0.5		0.5	
Moment M	66.67	-33.33		56.25	-56.25
Release B and equalise M_{B1} M_{B2}			22.92		
Distribute Difference		-11.456	←→	-11.46	
Carry Over 1/2	-5.73				-5.73
Total	60.94	-44.79		44.79	-61.98

At B we have achieved equal and opposite moments of 44.8 at B (Hogging)
The moment at A is 60.9 kN m and at C it is 61.9 kN m (Both Hogging)

WORKED EXAMPLE No. 4

Repeat the last example but this time section BC has twice the flexural stiffness (EI) than section AB.

Solution

The difference in flexural stiffness affects the distribution factor K_d only so these must be recalculated.

Member AB

$$M_A = 66.67 \times 10^3 \qquad M_{B1} = -33.33 \times 10^3 \quad K_{BA} = -\frac{4EI}{L} = -\frac{4EI}{3} = -1.333\ EI$$

Member BC

$$M_{B2} = 56.25 \times 10^3 \quad M_C = -56.25 \times 10^3 \quad K_{BC} = -\frac{4(2EI)}{L} = -\frac{8EI}{3} = -2.667\ EI$$

Calculate the Distribution Factors

At end B of BA

$$K_d = -\frac{K_{BA}}{\sum K} = \frac{-1.333\ EI}{-1.333\ EI - 2.667\quad EI} = 0.333$$

At end B of BC

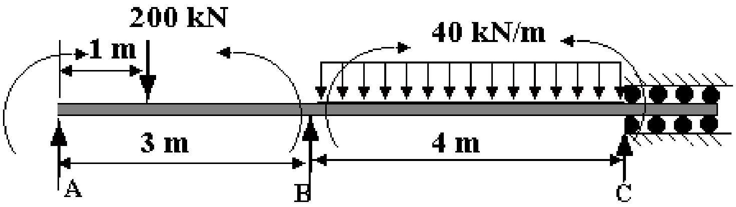
$$K_d = -\frac{K_{BC}}{\sum K} = \frac{-2.667\ EI}{-1.333\ EI - 2.667\ EI} = 0.667$$

	A	**B1**	**B**	**B2**	**C**
K_d		0.333		0.667	
Moment M	66.67	-33.33		56.25	-56.25
Release B and equalise M_{B1} M_{B2}			22.92		
Distribute Difference		-7.639	←→	-15.278	
Carry Over 1/2	-3.819				-7.639
Total	62.85	-40.97		40.97	-63.89

The bending moment at A is 62.85 kN m Hogging
The bending moment at B is 40.97 kN m Hogging
The bending moment at C is 63.89 kN m hogging

WORKED EXAMPLE No. 5

Find the moments at A, B and C for the beam below. The section AB has an EI value twice that of section BC.



Calculate Fixing Moments and Stiffness Factors.

MEMBER AB

$$M_A = \frac{Fb^2a}{L^2} = \frac{200 \times 10^3 \times 2^2 \times 1}{3^2} = 88.888 \times 10^3$$

$$M_{B1} = -\frac{Fa^2b}{L^2} = -\frac{200 \times 10^3 \times 1^2 \times 2}{3^2} = -44.444 \times 10^3$$

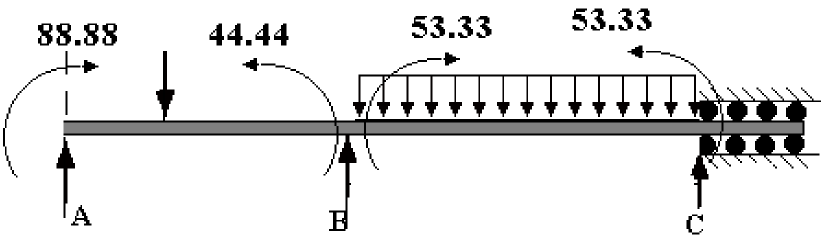
$$K_{BA} = -\frac{3 \times 2EI}{L} = -\frac{3 \times 2EI}{3} = -2EI$$

MEMBER BC

$$M_{B2} = \frac{wL^2}{12} = \frac{40 \times 10^3 \times 4^2}{12} = 53.333 \times 10^3$$

$$M_C = -\frac{wL^2}{12} = -53.333 \times 10^3$$

$$K_{BC} = -\frac{4EI}{L} = -\frac{4EI}{4} = -1EI$$



Calculate the Distribution Factors

At end B of BA

$$K_d = -\frac{K_{BA}}{\sum K} = \frac{-2\ EI}{-2\ EI - 0.75\ EI} = 0.727$$

At end B of BC

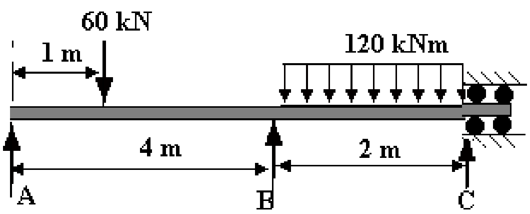
$$K_d = -\frac{K_{BC}}{\sum K} = \frac{-1.0\ EI}{-2\ EI - 0.75\ EI} = 0.273$$

Joint	A	B1	B	B2	Joint C
Fixing Moment	88.889	-44.444		53.333	-53.333
Balance A	-88.889				
K_d		0.667		0.333	
Carry over A to B	-88.889 × ½→	-44.444			
Totals	0	-88.888		53.333	-53.333
Imbalance at B			-35.555		
Balance B			+35.555		
Distribute using K_D		23.7		11.85	
Carry Over	0	←23.7 × 0		11.85 × ½→	5.93
Total	**0**	**-65**		**65**	**-47.4**

At B we have achieved equal and opposite moments so M_B = 65 kN m (Hogging).
The moment at A is zero
The at the wall (C) -47.4 kN m.

SELF ASSESSMENT EXERCISE No. 1

1. Solve the moments at A B and C for the beam shown. Section AB has an EI value three times that of BC.

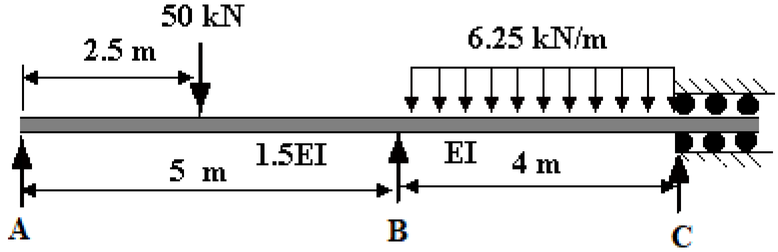


Answers 0, 34.4 kN m and 42.8 kN m

2. Repeat problem 1 if section BC has an EI value three times that of AB.

(Answers 0, -40 kN m and -40 kN m)

3. Solve the bending moment at the 3 supports for the beam shown.



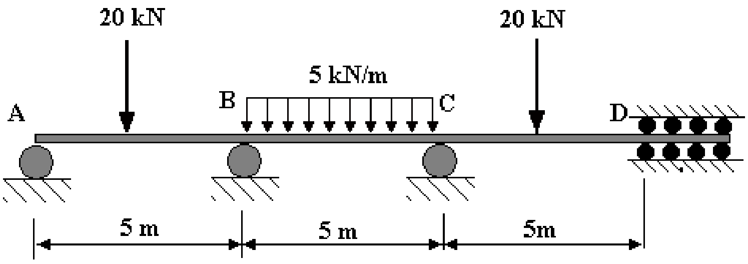
Answers 0, 0.31 kN m and -12.3 kN m

Beams with Three Sections

In the section we need to use more correction cycles. This is best explained with an worked example.

WORKED EXAMPLE No. 6

Calculate the bending moment in the beam at A, B, C and D. The point loads at mid span. The middle span (BC) has an EI value twice that of the other two spans.



Calculate Fixing Moments and Stiffness Factors For B.

SOLUTION

This time we must use 3 sections rigidly fixed at its ends.

Member AB

$$M_A = \frac{Fb^2a}{L^2} = \frac{20 \times 10^3 \times 2.5^2 \times 2.5}{5^2} = 12.5 \times 10^3$$

$$M_B = -\frac{Fa^2b}{L^2} = -\frac{20 \times 10^3 \times 2.5^2 \times 2.5}{5^2} = -12.5 \times 10^3$$

$$K_{BA} = -\frac{3 \times EI}{L} = -\frac{3EI}{5} = -0.6EI$$

Member CB

$$M_B = \frac{wL^2}{12} = \frac{5 \times 10^3 \times 5^2}{12} = 10.417 \times 10^3$$

$$M_C = -\frac{wL^2}{12} = -10.417 \times 10^3$$

$$K_{BC} = -\frac{4 \times (2EI)}{L} = -\frac{4 \times (2EI)}{5} = -1.6EI$$

Member CD

$$M_C = \frac{Fb^2a}{L^2} = \frac{20 \times 10^3 \times 2.5^2 \times 2.5}{5^2} = 12.5 \times 10^3$$

$$M_D = -\frac{Fa^2b}{L^2} = -\frac{20 \times 10^3 \times 2.5^2 \times 2.5}{5^2} = -12.5 \times 10^3$$

$$K_{CD} = -\frac{4 \times EI}{L} = -\frac{4EI}{5} = -0.8EI$$

Calculate the Distribution Factors

$$K_{dB1} = \frac{K_{BA}}{\sum K} = \frac{-0.6EI}{-0.6EI - 1.2EI} = 0.333$$

$$K_{dB2} = \frac{K_{BC}}{\sum K} = \frac{-1.2EI}{-0.6EI - 1.2EI} = 0.667$$

$$K_{dC1} = \frac{K_{BC}}{\sum K} = \frac{-1.2EI}{-1.2EI - 0.8EI} = 0.6$$

$$K_{dC2} = \frac{K_{BC}}{\sum K} = \frac{-0.8EI}{-1.2EI - 0.8EI} = 0.4$$

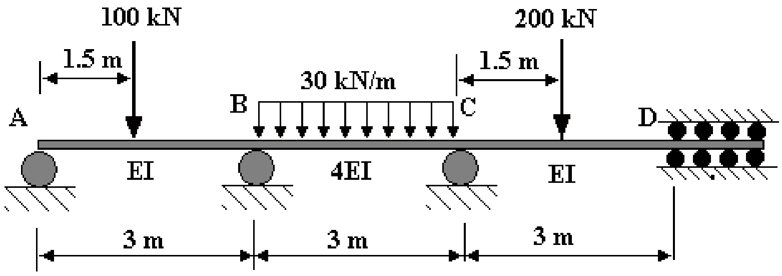
The following table shows the process of balancing and correcting until we get very close to zero difference at B and C. This shows that the moments at A, B, C and D are 0, 15.4 kNm (Hogging) 10.3 kNm (Hogging), and 13.6 kNm (Hoggng). The table was produced using a spread sheet.

	A			**B**				**C**			**D**
Kd			0.3333		0.6666		0.6		0.4		
M	12.5		-12.5		10.4166		-10.4167		12.5		-12.5
Balance A	-12.5	>	-6.25								
Total	0		-18.75		10.4166						
Difference				-8.3333							
Distribute	0	<	2.777778		5.5555	>	2.7777				
Total	0		-15.9722		15.97222		-7.6388		12.5		
Difference								-4.8611			
Distribute					-1.4583	<	-2.9166		-1.9444	>	-0.9722
Total	0		-15.9722		14.5138		-10.555		10.555		-13.472
Balance B				-1.458							
Dist B	0	<	0.486111		0.9722	>	0.4861				
Totals	0		-15.4861		15.4861		-10.069		10.5555		-13.4722
Balance C								-0.4861			
Dist C					-0.14583	<	-0.2916		-0.1944	>	-0.0972
Totals	0		-15.4861		15.3402		-10.361		10.361		-13.569
Balance B				-0.145							
Dist B	0	<	0.04861		0.09722	>	0.04861				
Totals	0		-15.4375		15.4375		-10.312		10.361		-13.569
Balance C								-0.0486			
Dist C					-0.01458	<	-0.0291		-0.0194	>	-0.0097
Totals	0		-15.4375		15.4229		-10.341		10.341		-13.579

SELF ASSESSMENT EXERCISE No. 2

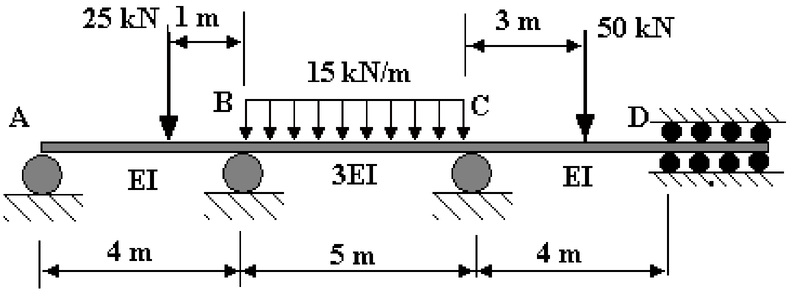
Find the bending moments in the beams shown at A, B, C and D.

1.



Answers 0, 45.8 kN, 59 kN and 83 kN

2.

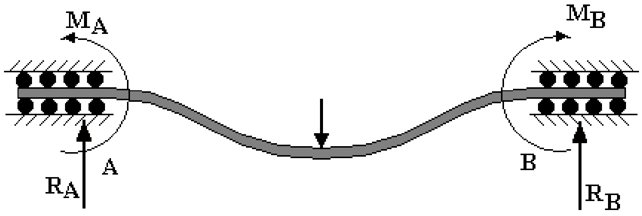


Answers 0, 19.1 kN, 18 kN and 23.8 kN

Fixing Moments

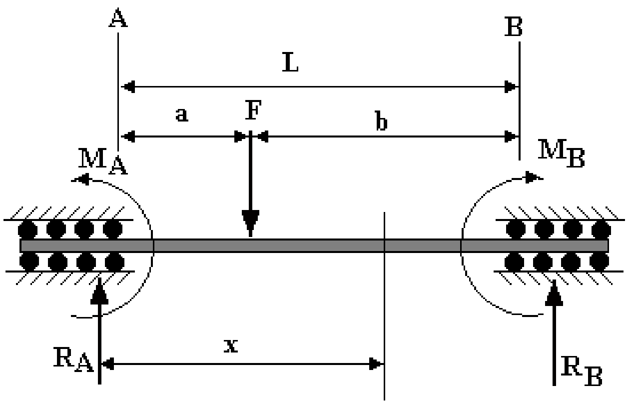
Fixing moments have been covered in the tutorial on deflection of beams under the heading *Encastré Beams*.

A fixing moment is the moment at the end of a beam where it is rigidly fixed to prevent any deflection or rotation. Finding them can be quite complicated depending on the loading of the beam. For those wishing to get to grips with the derivation the following is given. Let the beam span be designated A to B and the moments at the ends M_A and M_B. There will also be reactions R_A and R_B.



We shall use Macaulay's method to solve the slope and deflection and start with the derivation for a single point load. It might be deduced that the bending moments at the ends for the case shown are both hogging and therefore negative. Generally it is unknown so in the derivation assume they are positive but they should turn out to negative when solved.

Point Load



The bending moment at distance x from the left end is

$$M = EI\frac{d^2y}{dx^2} = R_A x - F[x-a] + M_A$$

Integrate

$$EI\frac{dy}{dx} = R_A\frac{x^2}{2} - \frac{F[x-a]^2}{2} + M_A x + A$$

The slope at both ends is zero so using x = 0 we find as always A = 0

$$EI\frac{dy}{dx} = R_A\frac{x^2}{2} - \frac{F[x-a]^2}{2} + M_A x \ldots\ldots (1)$$

Integrate again

$$EIy = R_A\frac{x^3}{6} - \frac{F[x-a]^3}{6} + M_A\frac{x^2}{2} + B$$

The deflection is zero at both ends so put y = 0 and x = 0 and find B = 0

$$EIy = R_A\frac{x^3}{6} - \frac{F[x-a]^3}{6} + M_A\frac{x^2}{2} \ldots\ldots (2)$$

(Note that the constants of integration A and B are always zero for an encastré beam).

Equations 1 and 2 give the slope and deflection. Before they can be solved, the fixing moment must be found by using another boundary condition. Remember the slope and deflection is zero at both ends of the beam so we have two more boundary conditions to use.

From Equation (1) with x = L

$$EI(0) = R_A\frac{x^2}{2} - \frac{F[x-a]^2}{2} + M_A x$$

$$EI(0) = R_A\frac{L^2}{2} - \frac{F[L-a]^2}{2} + M_A x$$

Substitute b = L – a

$$0 = R_A\frac{L^2}{2} - \frac{Fb^2}{2} + M_A L$$

$$0 = R_A L^2 - Fb^2 + 2M_A L$$

$$2M_A = \frac{Fb^2}{L} - R_A L \ldots\ldots (3)$$

Using equation (2) with y = 0 at x = L

$$EI(0) = R_A\frac{x^3}{6} - \frac{F[x-a]^3}{6} + M_A\frac{x^2}{2}$$

$$0 = R_A\frac{L^3}{6} - \frac{F[L-a]^3}{6} + M_A\frac{L^2}{2}$$

$$0 = R_A L^3 - F[L-a]^3 + 3L^2 M_A$$

$$3M_A = \frac{Fb^3}{L^2} - R_A L \ldots\ldots (4)$$

Subtract (3) from (4)

$$M_A = \frac{Fb^3}{L^2} - \frac{Fb^2}{L} = \frac{Fb^2}{L^2}[b-L] = \frac{Fb^2}{L^2}[-a] = -\frac{Fb^2 a}{L^2}$$

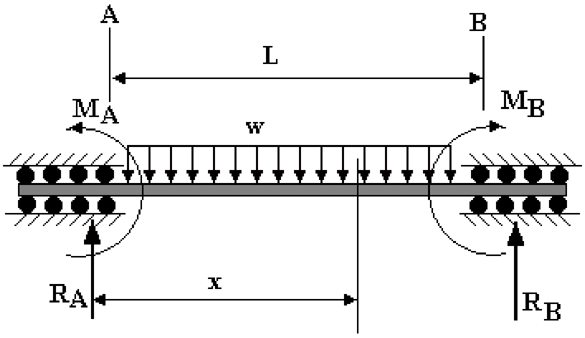
It can be shown that

$$M_B = +\frac{Fba^2}{L^2}$$

Both fixing moments are hogging

__Uniformly Distributed Loads__

A uniform load across the entire span will produce equal fixing moments at both ends both negative and hogging.



In this case the reactions are $R_A = R_B = wL/2$

The bending moment at distance x from the left end is

$$M = EI\frac{d^2y}{dx^2} = R_A x - \frac{wx^2}{2} + M_A$$

$$EI\frac{d^2y}{dx^2} = \frac{wLx}{2} - \frac{wx^2}{2} + M_A$$

Integrate

$$EI\frac{dy}{dx} = \frac{wLx^2}{4} - \frac{wx^3}{6} + M_A x + A$$

The slope is zero at both ends so put x = 0 and find A = 0

$$EI\frac{dy}{dx} = \frac{wLx^2}{4} - \frac{wx^3}{6} + M_A x \ldots\ldots (1)$$

Integrate again

$$EIy = \frac{wLx^3}{12} - \frac{wx^4}{24} + \frac{M_A x^2}{2} + B$$

Since the deflection is zero at both ends put y = 0 and x = 0 and find B = 0

$$EIy = \frac{wLx^3}{12} - \frac{wx^4}{24} + \frac{M_A x^2}{2} \ldots\ldots (2)$$

As in the other case, A and B are zero but we must find the fixing moment by using the other boundary condition of y = 0 when x = L

$$0 = \frac{wL^4}{12} - \frac{wL^4}{24} + \frac{M_A L^2}{2} = \frac{wL^4}{24} + \frac{M_A L^2}{2}$$

$$M_A = -\frac{wL^2}{12}$$

If we worked out M_B we would get

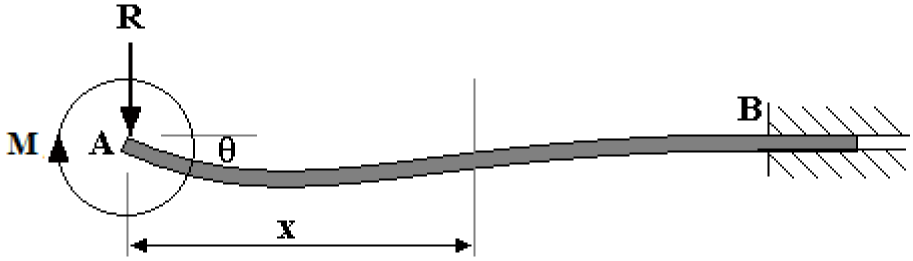
$$M_B = +\frac{wL^2}{12}$$

Rotational Stiffness and Carry Over

When a moment is applied to the end of a beam without any deflection, the end will rotate a small angle θ. The ratio M/θ is the stiffness and this depends on the values of E, I and L. It also depends how the other end of the beam is fixed.

Fixed End

Consider a section of beam with a moment M at the left end and fixed rigidly at the right end. The beam will deflect as shown.



The left end must not deflect so a force R must be applied also. The bending moment any distance x from the end is

$$M_x = EI\frac{d^2y}{dx^2} = M - Rx$$

Integrate

$$EI\frac{dy}{dx} = Mx - \frac{Rx^2}{2} + A$$

At x = L the slope is zero.

$$0 = ML - \frac{RL^2}{2} + A \qquad A = -ML + \frac{RL^2}{2}$$

$$EI\frac{dy}{dx} = Mx - \frac{Rx^2}{2} - ML + \frac{RL^2}{2}$$

Integrate again.

$$EIy = \frac{Mx^2}{2} - \frac{Rx^3}{6} - MLx + \frac{RL^2x}{2} + B$$

The deflection at x = 0 hence B = 0

The deflection at x = L is zero hence

$$0 = \frac{ML^2}{2} - \frac{RL^3}{6} - ML^2 + \frac{RL^3}{2} = -\frac{ML^2}{2} + \frac{RL^3}{3} \qquad R = \frac{3M}{2L}$$

$$EI\frac{dy}{dx} = Mx - \frac{Rx^2}{2} - ML + \frac{RL^2}{2}$$

The gradient is the rotation θ and at x = 0

$$EI\frac{dy}{dx} = EI\theta = -ML + \frac{RL^2}{2}$$

Substitute for R

$$EI\theta = -ML + \frac{3M}{2L} \times \frac{L^2}{2} = -ML + \frac{3ML}{4} = -\frac{ML}{4}$$

The rotational stiffness is hence

$$\frac{M}{\theta} = -\frac{4EI}{L}$$

Carry Over Factor

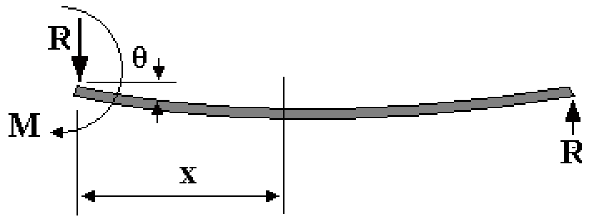
At end B the moment is M_B = - M + R L

$$M_B = -M + \frac{3M}{2} \qquad M_B = \frac{M}{2}$$

The fraction ½ is called the ***Carry Over Factor***

Simply Supported End

Consider a section of beam with a moment M at the left end and fixed rigidly at the right end. The beam will deflect as shown.



The reaction at the support must be equal and opposite to R. It follows that M = RL

$$M_x = EI\frac{d^2y}{dx^2} = M - Rx = M - \frac{Mx}{L}$$

Integrate

$$EI\frac{dy}{dx} = Mx - \frac{Mx^2}{2L} + A$$

Integrate again

$$EIy = \frac{Mx^2}{2} - \frac{Mx^3}{6L} + Ax + B$$

The deflection at x = 0 must be zero hence B = 0

$$EIy = \frac{Mx^2}{2} - \frac{Mx^3}{6L} + Ax$$

The deflection at x = L is zero hence

$$0 = \frac{ML^2}{2} - \frac{ML^3}{6L} + AL = \frac{ML^2}{2} - \frac{ML^2}{6} + AL$$

Divide by L

$$0 = \frac{ML}{2} - \frac{ML}{6} + A \qquad A = -\frac{ML}{3}$$

$$EI\frac{dy}{dx} = \frac{Mx^2}{2} - \frac{Mx^3}{6L} - \frac{MLx}{3}$$

The gradient at the end is the rotation θ and at x = 0 we have

$$EI\theta = -\frac{ML}{3}$$

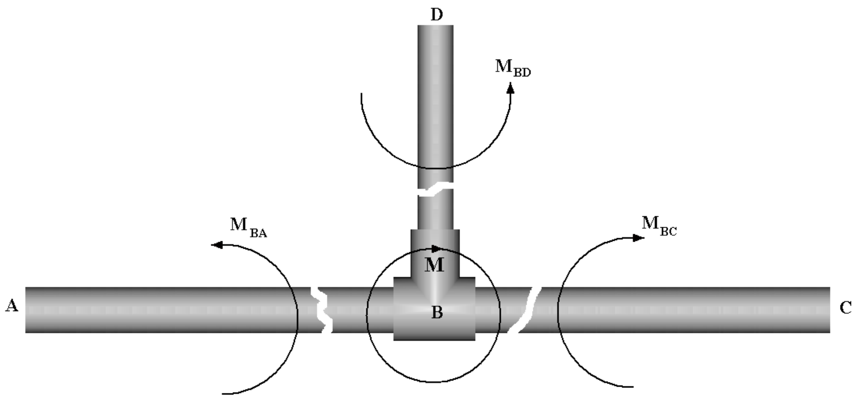
The rotational stiffness is hence

$$\frac{M}{\theta} = -\frac{3EI}{L}$$

Carry Over Factor

The moment at the simply supported end is zero so the carry over factor is zero.

Distribution Factor



The diagram shows three members AB, DB and CB rigidly joined at B.
When a moment M is applied at B the joint will twist an angler θ. The stiffness of each member as seen from B is

$$K = \frac{M}{\theta}$$

The moment distributed to each member is M_{BA}, M_{BC} and M_{BD}

$$(K)_{BA} = \frac{M_{BA}}{\theta} \qquad (K)_{BC} = \frac{M_{BC}}{\theta} \qquad (K)_{BD} = \frac{M_{BD}}{\theta}$$

At the joint the sum of all the moments is zero so it follows

$$M = M_{BA} + M_{BC} + M_{BD} = (K)_{BA}\theta + (K)_{BC}\theta + (K)_{BD}\theta$$

$$\frac{M}{\theta} = (K)_{BA} + (K)_{BC} + (K)_{BD} = \sum K \qquad \theta = \frac{M}{\sum K}$$

$$M_{BA} = (K)_{BA}\theta \quad \text{substitute and } M_{BA} = M\frac{(K)_{BA}}{\sum K}$$

Similarly

$$M_{BC} = M\frac{(K)_{BC}}{\sum K} \quad \text{and} \quad M_{BD} = M\frac{(K)_{BD}}{\sum K}$$

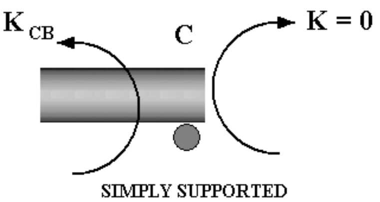
The moment is distributed to each member by the distribution factor

$$K_d = \frac{K}{\sum K}$$

$$M_{BA} = (K_d)_{BA}M \quad M_{BC} = (K_d)_{BC}M \quad M_{BD} = (K_d)_{BD}M$$

At a free end, (e.g. C)

$$K_d = \frac{K_{CB}}{\sum K} = \frac{K_{CB}}{K_{CB} + 0} = 1$$



At a rigidly fixed end (e.g. A)

$$K_D = \frac{K_{AB}}{\sum K} = \frac{K_{AB}}{K_{AB} + \infty} = 0$$