This the fourth and final tutorial on bending of beams. You should judge your progress by completing the self assessment exercises.

On completion of this tutorial you should be able to do the following.

- Define the relationship between bending moment and shear stress in a beam.
- Define complementary shear stress.
- Define shear flow.
- Determine how shear stress is distributed over the cross section of a beam.
- Determine how shear stress is distributed over the flanges and webs of a beam.
- Define and shear centre.
- Calculate the position of the shear centre.

*It is assumed that students doing this tutorial already understand the basic principles of moments, shear force, stress and moments of area.*

*Students must also be able to perform basic differentiation and calculus from their maths studies.*

You will find other good tutorial material at the following web addresses (currently available in 2007)

http://www.me.mtu.edu/~mavable/Spring03/chap6.pdf  
http://gaia.ecs.csus.edu/~ce113/steel-shear.pdf  
http://www.ae.msstate.edu/~masoud/Teaching/exp/A14.8_ex1.html  
http://ctsm.umd.edu/assakkaf/Courses/ENES220/Lectures/Lecture15.pdf
1. THE RELATIONSHIP BETWEEN BENDING MOMENT AND SHEAR FORCE

Consider a beam subject to bending and transverse shear. At some distance along the x direction further consider a short length $\delta x$. Over this length the bending moment increases by $dM$ and the shear force increases by $dF$.

![Diagram of beam with forces and moments](image1.png)

Figure 1

If we take this section out of the beam we must add the forces and moments that were previously exerted by the material in order to maintain equilibrium. Equilibrium of forces and moments exists at all points so it is convenient to look at the equilibrium of moments at the bottom right corner.

$$F \delta x + M = M + \delta M$$

$$F = \frac{\delta M}{\delta x}$$ and in the limit as $\delta x \to 0$ we have

$$F = \frac{dM}{dx}$$
2. **COMPLEMENTARY SHEAR STRESS**

If a transverse shear force $F_y$ acts at a point on a beam in the vertical (y) direction, transverse shear stress is created. Studies of two dimensional stress and strain in elastic materials (see the tutorial on Mohr’s circle of stress and strain) show that shear stress cannot exist in just one direction but it is always accompanied by an equal and opposite shear stress on the plane normal to it. It follows that if a shear stress exists on the cross sectional area of a beam, there must be an equal shear stress acting in the x direction. This is called the complementary shear stress.

3. **SHEAR STRESS DISTRIBUTION**

Consider a simple cantilever beam. Bending Moment $M = F_y x$

The direct stress at distance $y$ from the centre is $\sigma = \frac{M y}{I_z} = \frac{F_y x y}{I_z}$

The force acting on the area $dA$ is $dF = \sigma \, dA = \sigma \, b \, dy = \frac{F_y \, x \, y \, dA}{I_z} \text{ or } \frac{F_y \, x \, b \, y \, dy}{I_z}$

If we integrate between $y$ and the $y_m$, we get the force acting normal to the yz plane. This must be balanced by a force $F_x$ at level $y$ on the xz plane $F_x = \frac{F_y \, x \, y_m}{I_z} \int y \, dA$

Suppose the cross section does not have a uniform width $b$. The shear stress on the xz plane is found by dividing $F_x$ by the area $b y_x$ $\tau_x = \frac{F_y}{b y_z} \int y \, dA \text{ or } \frac{F_y}{b y_z} \int by \, dy$

This shear stress must be complimentary to the shear stress on the yz plane so it follows that

$\tau_x = \tau_y = \frac{F_y}{b y_z} \int y \, dA \text{ or } \frac{F_y}{b y_z} \int y \, dA$ is the first moment of area $A \, \bar{y}$

$\tau_x = \frac{F_y}{b y_z} A \, \bar{y}$ Note the term ‘$A \, \bar{y}$’ refers to the shaded area on the diagram. If the shape is rectangular it is easy to calculate the shear stress. The equation indicates that $\tau$ is constant at all values of $x$ if $I$ and $A$ are constant.
4. SHEAR FLOW and SHEAR FORCE

The shear force should be equal to the integration of the shear stress over the section.

\[ dF = \tau \, dA = \tau \, b \, dy \]

\[ F = \int_{\text{bottom}}^{\text{top}} \tau \, b \, dy \]

The product \( \tau \, b \) is called the shear flow and denoted \( q \).

\[ F = \int_{\text{bottom}}^{\text{top}} q \, dy \]

For standard sections such as ‘I’, ‘T’, ‘U’ and ‘L’, there is vertical shear flow in the flange and horizontal shear flow in the web in the horizontal direction. For this reason ‘t’ is often used for the thickness of the flange or web.

The importance of this work will be clear later in the tutorial. The units of \( q \) are N/m. If the cross section is not a uniform shape, e.g. a Tee section, \( t \) or \( b \) is not constant and the maximum shear stress occurs where \( \tau \, y / b \) is a maximum.

WORKED EXAMPLE No.1

Derive an equation for the shear stress and shear flow distribution on a rectangular cross section. Compare the average and maximum shear stress. Show that the shear force is obtained by integrating the shear flow over the section.

SOLUTION

Let \( D/2 = C \). In this case the shaded area is \( B \) \((C - y) \) and \( \bar{y} = (C + y)/2 \) \quad \( b_y = B \)

\[ L = BD^3/12 \quad A = BD \]

\[ \tau_y = \frac{F_y B (C - y)(C + y)}{2BL_z} = \frac{12 F (C - y)(C + y)}{2(BD^3)} = \frac{6 F (C^2 - y^2)}{AD^2} = \frac{6 F (C^2 - y^2)}{4AC^2} = \frac{3F}{2A} \left(1 - \frac{y^2}{C^2}\right) \]

Plotting \( \tau_y \) against \( y \) we find the shear stress varies as a parabola from zero at the bottom to a maximum at the centroid and zero at the top.

The maximum shear stress occurs at \( y = 0 \)

\[ \tau_{\text{max}} = \frac{3F}{2A} \]

The transverse shear force on the section is \( F_y \) and the mean shear stress is \( F/A \) so \( \tau_{\text{max}} = \tau_{\text{mean}} \times 3/2 \)

The shear flow is easily obtained since the width is constant.

\[ q_y = \tau_y B = \frac{3F}{4C} \left(1 - \frac{y^2}{C^2}\right) \]

\[ F = \int_{\text{bottom}}^{\text{top}} q_y \, dy = \frac{3F}{4C} \int_{-C}^{C} \left(1 - \frac{y^2}{C^2}\right)^C \frac{3F}{4C} \left[\frac{y^3}{3C^2}\right]_{-C}^{C} = \frac{3F}{4C} \left[C - \frac{C^3}{3C^2}\right] \left[-C + \frac{C^3}{3C^2}\right] = F \]
5 OTHER SECTIONS

5.1 CIRCULAR SECTION

A circular section can be solved with some difficulty in the same way. For a circular section of radius R:
\[
\tau_y = \frac{4F(r^2 - y^2)}{3\pi R^4} \quad \tau_{\text{max}} = \frac{4F}{3\pi R^2} = \frac{4}{3}\tau_{\text{mean}}
\]

5.2 NON UNIFORM SECTIONS

If the shape is hollow or does not have a constant width b, the problem is more complex. For example consider a triangular section.
\[
\tau_y = \frac{F_y A y}{I_z b_y}
\]

Because b is a function of y, the maximum shear stress does not occur at the centroid but at the point shown.

6 SUDDEN DISCONTINUITIES

With ‘T’, ‘I’, ‘U’ and ‘L’ sections, the width ‘b’ or ‘t’ suddenly changes at the junctions of the web and flange so the shear stress suddenly changes as the ratio of the widths. This is best illustrated with a worked example.

WORKED EXAMPLE No.2

Determine the shear stress at the junction of the top flange and web for the section shown when a shear force of 40 kN acts vertically down on the section.

SOLUTION

First calculate the second moment of area. The tabular method is used here. Divide the shape into three sections A, B and C. First determine the position of the centroid from the bottom edge.

<table>
<thead>
<tr>
<th>Area</th>
<th>(\bar{y})</th>
<th>A (\bar{y})</th>
</tr>
</thead>
<tbody>
<tr>
<td>A 600 mm^2</td>
<td>45 mm</td>
<td>27 000 mm^3</td>
</tr>
<tr>
<td>B 300 mm^2</td>
<td>25 mm</td>
<td>7500 mm^3</td>
</tr>
<tr>
<td>C 400 mm^2</td>
<td>5 mm</td>
<td>2000 mm^3</td>
</tr>
<tr>
<td>Totals 1300 mm^2</td>
<td>5 mm</td>
<td>365000 mm^3</td>
</tr>
</tbody>
</table>

Figure 7

Figure 8
For the whole section the centroid position is $\bar{y} = \frac{365000}{1300} = 28.07$ mm

Now find the second moment of area about the base using the parallel axis theorem.

\[
\begin{align*}
\text{BD}^3/12 & \quad A \bar{y}^2 \\
\text{A} & \quad 60 \times 10^3/12 = 5000 \text{ mm}^4 \\
& \quad 600 \times 45^2 = 1215000 \quad 1220000 \text{ mm}^4 \\
\text{B} & \quad 10 \times 30^3/12 = 22500 \text{ mm}^4 \\
& \quad 300 \times 25^2 = 187500 \quad 210000 \text{ mm}^4 \\
\text{C} & \quad 40 \times 10^3/12 = 3333 \text{ mm}^4 \\
& \quad 400 \times 5^2 = 10000 \quad 13333 \text{ mm}^4 \\
\text{Total} & \quad \end{align*}
\]

The total second moment of area about the bottom is $1443333 \text{ mm}^4$

Now move this to the centroid using the parallel axis theorem.

\[
I = 1443333 - A \bar{y}^2 = 1443333 - 1300 \times 28.07^2 = 418300 \text{ mm}^4
\]

Now calculate the stress using the well known formula $\sigma_B = \frac{My}{I}$

Top edge $y = \text{distance from the centroid to the edge} = 50 - 28.07 = 21.93$ mm

\[
\tau = \frac{FA \bar{y}}{Ib}
\]

At the junctions of the flange and web, this will change suddenly as the value of $b$ changes.

Consider the junction of the top flange with the web.

![Diagram of the section](image)

\[
\begin{align*}
A & = 600 \text{ mm}^2 \\
I_{gs} & = 418300 \text{ mm}^4 \\
\bar{y} & = 16.92 \text{ mm}
\end{align*}
\]

\[
\tau = \frac{FA \bar{y}}{Ib} = \frac{40 \times 10^3 \times 600 \times 10^{-6} \times 16.92 \times 10^{-3}}{418300 \times 10^{-12} \times b} = \frac{970.8 \times 10^3}{b}
\]

Where $b = 10$ mm the shear stress is $\tau = 97$ MPa

Where $b = 60$ mm the shear stress is $\tau = 16.18$ MPa
WORKED EXAMPLE No.3

Determine the shear stress distribution for the ‘T’ section shown when the transverse shear force is 200 kN

Figure 10

SOLUTION

First find $\bar{y}$ by taking first moments of area about the base $s – s$.

Area = $(150 \times 250) – (130 \times 230) = 7600 \text{ mm}^2$

$1^{st}$ Mom of Area = $150 \times 250^2/2 – 130 \times 230^2/2 = 1249000 \text{ mm}^3$

$\bar{y} = 1249000/7600 = 164.3 \text{ mm}$

Next find $I_{gg}$

$I_{ss} = (150 \times 250^3/3) – (130 \times 230^3/3) = 254 \times 10^6 \text{ mm}^4$

$I_{gg} = I_{ss} - A \bar{y}^2 = 254 \times 10^6 – 7600 \times 164.3^2 = 48.75 \times 10^6 \text{ mm}^4$

Now examine the shear stress. The bottom half is the easiest so start there.

Figure 11

$\tau = \frac{FA \bar{y}}{I b} = \frac{200 \times 10^3 \times A\bar{y}}{48.75 \times 10^{-6} \times b}$ $b = t = 0.020$

$A\bar{y} = \left[(164.3 – y) \times 20 \times \left(82.15 + \frac{y}{2}\right)\right] \times 10^{-9}$
Evaluate at various values of $y$ and we get

<table>
<thead>
<tr>
<th>$y$ (mm)</th>
<th>0</th>
<th>20</th>
<th>40</th>
<th>80</th>
<th>120</th>
<th>160</th>
<th>164.3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau$ (MPa)</td>
<td>55.4</td>
<td>52.1</td>
<td>42.3</td>
<td>25.9</td>
<td>2.9</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

The top half is more difficult. For $y = 65.66$ to $y = 85.66$ mm $b = 150$ mm.

Figure 12

\[
A\bar{y} = 150 \times (85.66 - y) \left( 65.66 + \frac{85.66 - y}{2} \right)
\]

\[
\tau = \frac{FA\bar{y}}{1b} = \frac{200 \times 10^3 \times A\bar{y}}{48.75 \times 10^{-6} \times b} \quad b = t = 0.150
\]

Finally for $y = 0$ to $y = 65.66$ mm

\[
A\bar{y} = 150 \times (85.66 - y) \left( \frac{85.66 + y}{2} \right) - 130 \times (65.66 - y) \left( \frac{y + 65.66}{2} \right)
\]

\[
\tau = \frac{FA\bar{y}}{1b} = \frac{200 \times 10^3 \times A\bar{y}}{48.75 \times 10^{-6} \times b} \quad b = t = 0.020
\]

Evaluate at various values of $y$ and we get

<table>
<thead>
<tr>
<th>$y$ (mm)</th>
<th>0</th>
<th>20</th>
<th>40</th>
<th>65.66</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau$ (MPa)</td>
<td>55.4</td>
<td>54.6</td>
<td>52.1</td>
<td>46.5</td>
</tr>
</tbody>
</table>

The distribution is like this.
SELF ASSESSMENT EXERCISE No.1

1. Calculate the maximum shear stress and the shear stress at the junction of the bottom flange and web for the same case as worked example No.2

(104 MPa, 22 MPa and 88.42 MPa)

2. A ‘T’ section beam has a transverse shear force of 80 kN at a point on its length. The flange and web are 12 mm thick. The flange is 80 mm wide and the overall depth is 100 mm. Determine the maximum shear stress and the shear stresses at the junction of the flange and web.

(94.3 MPa, 86 MPa and 12.9 MPa)

7. SHEAR DISTRIBUTION IN THE FLANGES

AUTHOR’S NOTE. I admit that I don’t fully understand why the horizontal shear stress in the flanges is determined in the manner described below. Rigid proofs should be sought elsewhere. I have attempted to place a simple explanation for shear flow in the flanges. The result is valid and used in the section following on shear centre.

The shear stress distribution in non uniform sections such as ‘T’ and ‘U’ sections cannot be quite as simple as indicated previously for the following reasons. The shear stress at a free surface can only be parallel to the surface and not normal. It follows that the vertical shear stress in the outer regions of the flanges cannot be the same as at the junction of the flange and web.

We should also consider how the shear force in the vertical direction is applied along the flange. Assuming a perfectly elastic material the force will be spread along the flange and some of it will cause bending. This will induce shear stress in the z and y directions.

The result is that a shear flow $q_w$ is set up in the web and $q_f$ in the flanges as shown when the transverse shear force is vertical ($F_y$). In the flanges the flow varies linearly with $z$. 

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In the flange the shear flow is in the z direction so there is a shear force \( F_z \) as shown. As the total force in the z direction is zero, the forces in the two halves are equal and opposite.

Here follows the bit I have a problem with because the formula was derived for vertical shear and I can’t understand why it applies to horizontal shear in the flange.

We have defined shear flow as \( q = \tau b \) for the web. To calculate the shear flow in the flange, ‘b’ becomes the thickness ‘\( t \)’.

\[
\tau_z = \frac{F_y A \bar{y}}{I_z t} \\
A\bar{y} = \int t \, dz = \int \bar{y} \, dz
\]

\[
\tau_z = \frac{F_y}{I_z} \frac{z_{end}}{t} \int dz = \frac{F_y}{I_z} \bar{y} (z_{end} - z) \text{ N/m}^2
\]

\[
q_z = \tau_z t = \frac{F_y}{I_z} t \bar{y} (z_{end} - z) \text{ N/m}
\]

**WORKED EXAMPLE No.4**

Determine the shear stress and shear flow distribution in the ‘T’ section shown when a transverse shear force of 1 kN is applied vertically. Calculate the horizontal force in the flange and determine the fraction of the shear force carried by the web.
SOLUTION

First determine the position of the centroid by taking first moments of area about the base.

\[ \bar{y} = \frac{\sum A_y y}{A} = \frac{(40 \times 10 \times 45) + (40 \times 10 \times 20)}{(40 \times 10) + (40 \times 10)} = 32.5 \text{mm} \]

Next determine \( I_z \)

\[ I_z = \sum \left( \frac{BD^3}{12} + A\bar{y}^2 \right) = \left[ \frac{40 \times 10^3}{12} + 40 \times 10(45 - 32.5)^2 \right] + \left[ \frac{10 \times 40^3}{12} + 10 \times 40(32.5 - 20)^2 \right] \]

\[ I_z = 65.83 \times 10^3 + 115.8 \times 10^3 = 181.7 \times 10^3 \text{ mm}^4 \]

Next calculate the shear stress in the flange for \( y = 7.5 \) to \( 17.5 \). \( \tau_{yx} = \frac{F_y A \bar{y}}{I_z b} \)

The first moment of area is \( A \bar{y} = \int y dy = 40 \int_{y=7.5}^{17.5} y dy = 20\int_{y=7.5}^{17.5} [7.5^2 - y^2] dy \)

\[ \tau_{yx} = \frac{F_y A \bar{y}}{I_z b} = \frac{1000}{181.7 \times 10^3 x 40} \int_{y=7.5}^{17.5} y^2 dy = 2.75 \times 10^{-3} \int_{y=7.5}^{17.5} [7.5^2 - y^2] N/mm^2 \]

\[ q_y = \tau_{y} b = 40[2.75 \times 10^{-3} (17.5^2 - y^2)] \]

Next calculate the shear flow in the web for \( y = -32.5 \) to \( 7.5 \)

To make life easier, take the area between \( y = -32.5 \) and \( y \)

The first moment of area is \( A \bar{y} = \int_{-32.5}^{y} y dy = -10 \int_{y=-32.5}^{y} y dy = -5\int_{y=-32.5}^{y} [y^2 - 32.5^2] y \)

\[ \tau_{yx} = \frac{F_y A \bar{y}}{I_z b} = \frac{-1000 \times 5}{181.7 \times 10^3 x 10} \int_{y=-32.5}^{32.5} [y^2 - 32.5^2] dy = 2.75 \times 10^{-3} \int_{y=-32.5}^{32.5} [y^2 - 32.5^2] y \]

Evaluating the shear stress and plotting we get the following result.

![Figure 19](image)

Now let's examine the distribution in the flange. Consider the shaded area on the diagram.
For $z = 5$ to $20$ mm
\[
\tau_z = \frac{F_y \bar{y} (z_{end} - z)}{I_z} = \frac{1000}{181.7 \times 10^3} 12.5 \times 10^{-3} (20 - z) = 68.8 \times 10^{-3} (20 - z)
\]
At $z = 5$ this is $1.032 \text{ N/mm}^2$ and at $z = 20$ mm it is zero.
$q_z = \tau_z t = 688 \times 10^{-3} (20 - z)$ and at $z = 5$ mm this is $10.32 \text{ N/m}$
This is a linear variation from $0$ at $z = 20$ mm and a maximum of $1.032 \text{ N/mm}^2$ at $z = 5$ mm

The force in the $z$ direction is found is the area under the $q - z$ graph so $F_z = 10.32 \times 15/2 = 77.4 \text{ N}$
There is an equal and opposite force in the left side. Now consider the shear flow in the web below the flange.
\[
q_y = \tau_y x 10 \quad q_y = 27.5 \times 10^{-3} \left[ \frac{y^2}{3} - 32.5^2 \right]_{32.5}^7.5
\]
\[
F_y = \int q_y \, dy = 27.5 \times 10^{-3} \left[ \frac{y^3}{3} - 32.5y \right]_{32.5}^{7.5} = 843.3 \text{ N}
\]
Next consider the vertical shear flow in the flange.
\[
q_y = \tau_y b = 40 \times 2.75 \times 10^{-3} \left( 17.5^2 - y^2 \right) = 110 \times 10^{-3} \left( 17.5^2 - y^2 \right)
\]
\[
F_y = \int q_y \, dy = 110 \times 10^{-3} \left[ \frac{17.5^2}{7.5} - y^2 \right]_{7.5}^{17.5} = 155.8 \text{ N}
\]
The total vertical force is $843.3 + 155.8 = 9991 \text{ N}$
This is quite close to the expected figure of $1000 \text{ N}$
The web carries about $84\%$ of the shear force and the flange about $16\%$

Important note – the thinner the flange, the closer the figure becomes to $100\%$ in the web.
8. **SHEAR CENTRE**

Consider a cantilever beam with a point load that acts vertically but not through the centroid. The beam bends but in addition it twists as shown.

![Figure 22](image.png)

This produces additional torsional stress in the beam. In the case of a symmetrical beam like that shown, the solution is simply to apply the load so that it acts through the centroid. In other cases we must apply the load so that it acts through some other point that results in no twisting and this point is called the centre of shear or shear centre.

The shear centre is that point through which the loads must act if there is to be no twisting, or torsion.

The **shear centre is always located on the axis of symmetry**; therefore, if a member has two axes of symmetry, the shear centre will be the intersection of the two axes. If there is only one axis of symmetry, the shear centre is somewhere on that axis.

Here are some examples of sections that are symmetrical in two axis.

![Figure 23](image.png)

A ‘U’ section is a good example of one where the shear centre is difficult to find. As the following example shows, it occurs off the section altogether. Note that when the sections are made from thin sheets of material, the problems are easier to resolve and most of this work concerns the shear stress in thin sections.
**WORKED EXAMPLE No.5**

Determine the position of the shear centre in terms of dimension ‘a’ for the ‘U’ section shown made from thin metal sheet of thickness ‘t’. The force is applied vertically.

![Diagram](image)

Figure 24

The centre of shear for the ‘U’ channel is to the left of the section as drawn. First calculate the position of the centroid. This must be on the horizontal centre line so we need to calculate the position from the vertical edge.

<table>
<thead>
<tr>
<th>Area</th>
<th>( z )</th>
<th>( A )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>at ( a/2 )</td>
<td>( a^2t/2 )</td>
</tr>
<tr>
<td>B</td>
<td>at ( a/t )</td>
<td>( a^2t/2 )</td>
</tr>
<tr>
<td>C</td>
<td>2at</td>
<td>( t/2 )</td>
</tr>
<tr>
<td>Total</td>
<td>4at</td>
<td>( a^2t + at^2 )</td>
</tr>
</tbody>
</table>

For the section \( z = (a^2t + at^2)/4at \) if \( t \) is small the \( t^2 \) term may be ignored so \( z = a^2t/4at = a/4 \)

Next we calculate the shear stress in the section due to transverse shear force \( F \)

\[
\tau = \frac{FAy}{I_z b}
\]

\( I_z \) is the second moment of area of the section about the axis.

For the vertical section \( I = t(2a)^3/12 = (2/3)a^3t \)
For the flanges we may approximate with \( I = A \times a^2 \) where \( A = at \) so \( I = 2 \times a^3t \)

Adding we get \( I_z = (2/3)a^3t + 2 \times a^3t \)

\[
I_z = \frac{8}{3}a^3t \quad \tau = \frac{FAy}{I_z b} = \frac{3FAy}{8a^3tb}
\]

**SHEAR DISTRIBUTION IN THE FLANGE**

![Diagram](image)

Figure 25

Area \( A = (a-z)t \quad y = a \quad \tau = \frac{3FAy}{8a^3tb} \) and in this case \( b = t \) the thickness of the flange.

\[
\tau = \frac{3F(a-z)}{8a^2t} \quad \text{and the shear flow is} \quad q = \tau t = \frac{3F(a-z)}{8a^2}
\]
The maximum shear stress and shear flow occurs when \( z = 0 \) and are:
\[
\tau_{\text{max}} = \frac{3F}{8at} \quad q_{\text{max}} = \frac{3F}{8a}
\]
The minimum values are zero at \( z = a \). In between the variation is linear. The distribution in the bottom flange is the same but negative.

**SHEAR DISTRIBUTION IN THE WEB**

![Figure 26](image)

Next find the expression for the shear stress distribution in the vertical section. \( \bar{y} \) is harder to find.

<table>
<thead>
<tr>
<th>Area</th>
<th>( y )</th>
<th>A ( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>( at )</td>
<td>( a^2t )</td>
</tr>
<tr>
<td>B</td>
<td>((a-y)t)</td>
<td>((a+y)/2)</td>
</tr>
</tbody>
</table>

Total Area = \( at + (a-y)t = t(2a-y) \)
Total 1st moment = \( a^2t + (a^2-y^2)t/2 = (3a^2-y^2)t/2 \)

\( \bar{y} \) for the shaded section is \( (3a^2-y^2)t/2 + t(2a-y) = (3a^2-y^2)/2(2a-y) \) \( I_z = 8/3a^3t \)

\[
\tau = \frac{FA\bar{y}}{I_zt} = \frac{3Ft(2a-y)(3a^2-y^2)}{8a^3t^2 2(2a-y)} = \frac{3F(3a^2-y^2)}{16a^3t} \quad q = \tau t = \frac{3F(3a^2-y^2)}{16a^3}
\]

When \( y = 0 \) \( \tau = \frac{9F}{16at} \quad q = \frac{9F}{16a} \) When \( y = a \) \( \tau = \frac{3F}{8at} \quad q = \frac{3F}{8a} \)

The shear stress distribution is like this. The stress on the top and bottom flanges falls linearly to zero at the edges.

![Figure 27](image)

The forces in the flange are the area under the graphs. \( F = \frac{3F}{8a} \times \frac{a}{2} = \frac{3F}{16} \)

These forces acts horizontally and are equal and opposite in direction in the top and bottom flanges.
Now we need the force in the web. This is twice the force in one half so

\[
F' = 2 \int_0^a q \, dy = 2x \frac{3F}{16a^3} \int_0^{2a} (3a^2 - y^2) \, dy = \frac{3F}{8a} \left[ \frac{3a^2 y - \frac{y^3}{3} }{3} \right]_0^a = \frac{3F}{8a} \cdot \frac{3a^3}{8}
\]

F' = F

This is not totally unexpected since we have assumed very thin sections. The forces are like this.

[Diagram showing forces and moments]

Figure 28

Balancing moment about the centroid we have:

\[
F \left( e + \frac{a}{4} \right) = \frac{3F}{16} a + \frac{3F}{16} a + F \frac{a}{4}
\]

\[
\left( e + \frac{a}{4} \right) = \frac{3a}{8} + \frac{a}{4}
\]

\[
e = \frac{3a}{8}
\]
SELF ASSESSMENT EXERCISE No.2

1. A ‘U’ section is made from thin plate of thickness \( t = 2 \) mm with outer dimensions \( B = 120 \) mm and \( D = 100 \) mm as shown. A transverse shear force of \( 60 \) kN is applied. Calculate the position of the shear centre.

\[
\begin{align*}
\text{Answer: } z &= 43.14 \text{ mm}, \\
I_z &= 1.3 \times 10^6 \text{ mm}^4 \\
\text{Flange force} &= 29.9 \text{ kN each and web force} = 60 \text{ kN} \\
\text{The shear centre is} &\text{ 48.8 mm to the left of the section.}
\end{align*}
\]

2. Repeat the problem but with \( t = 10 \) mm and comment on the differences.

\[
\begin{align*}
\text{Answer: } z &= 46.25 \text{ mm}, \\
I_z &= 5.3 \times 10^6 \text{ mm}^4 \\
\text{Flange force} &= 33.6 \text{ kN each and web force} = 59.6 \text{ kN} \\
\text{The shear centre is} &\text{ 50.2 mm to the left of the section. The problems with thicker material is the distribution and force in the junction areas and what precise values to use for the dimensions.}
\end{align*}
\]