

MECHANICS OF SOLIDS - BEAMS

TUTORIAL 1

STRESSES IN BEAMS DUE TO BENDING

This is the first tutorial on bending of beams designed for anyone wishing to study it at a fairly advanced level. You should judge your progress by completing the self assessment exercises.

On completion of this tutorial you should be able to do the following.

- Define a beam.
- Recognise different types of beams.
- Define BENDING MOMENT
- Derive the BENDING FORMULAE for beams.
- Calculate the stress in a beam due to bending.
- Solve problems involving both bending and direct stress.
- Find the position of the neutral axis for combined stress situations.
- Solve problems involving simple composite beams.

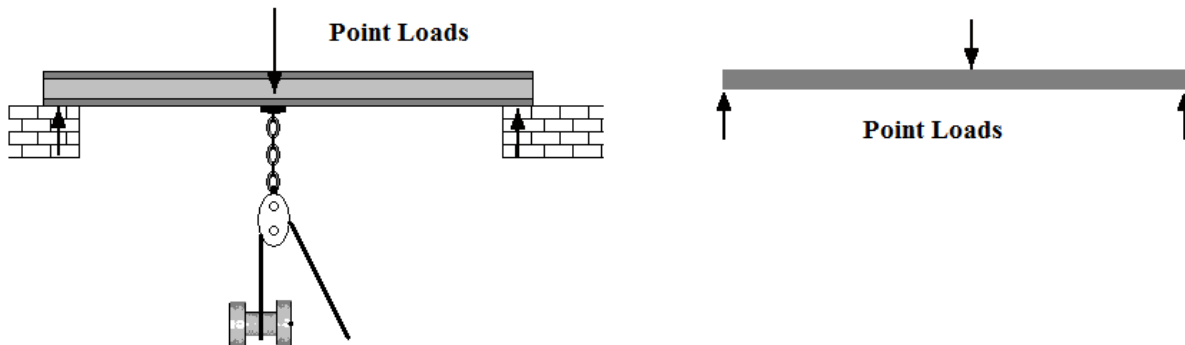
It is assumed that students doing this tutorial already understand the basic principles of moments, shear force, stress and moments of area.

Students must also be able to perform basic differentiation and calculus from their maths studies.

1 TYPES OF BEAMS

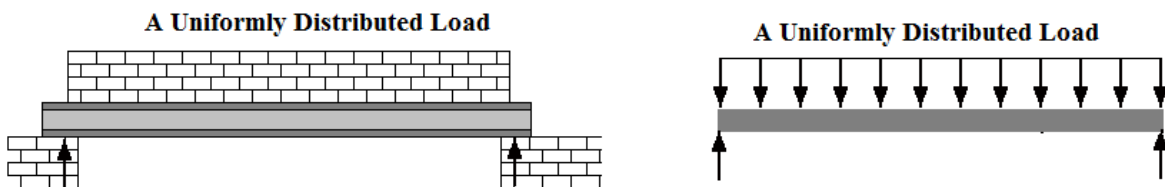
- A beam is a structure, which is loaded transversely (sideways).
- The loads may be point loads or uniformly distributed loads (udl).

The diagrams show the way that point loads and uniform loads are illustrated. A point load is a load or force that acts at a single point on a structure and it is depicted by a single arrow on diagrams.



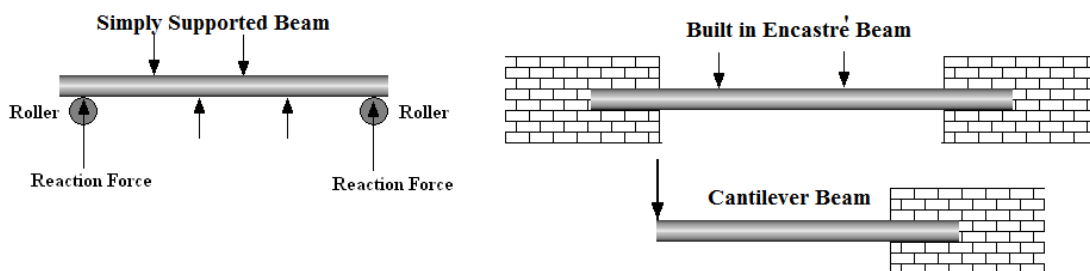
Figures 1a and b

A uniform load is one which is evenly distributed along a length such as the weight of the beam or a wall built on top of a beam. It is depicted by a series of arrows as shown. We usually denote the loading as w N/m.



Figures 1c and d

- The beam may be simply supported or built in.



Figures 1e to 1g

Transverse loading causes bending and bending is a very severe form of stressing a structure. The bent beam goes into tension (stretched) on one side and compression on the other.

The complete formula which describes all aspects of bending is $\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}$

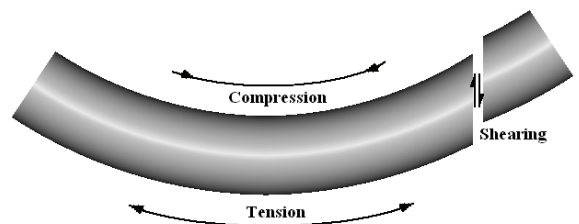


Figure 2

We will now look at the derivation and use of this formula.

2. THE BENDING FORMULA

2.1 NEUTRAL AXIS

This is the axis along the length of the beam which remains unstressed, neither compressed nor stretched when it is bent. Normally the neutral axis passes through the centroid of the cross sectional area. The position of the centroid is hence important.

Consider that the beam is bent into an arc of a circle through angle θ radians. AB is on the neutral axis and is the same length before and after bending. The radius of the neutral axis is R.

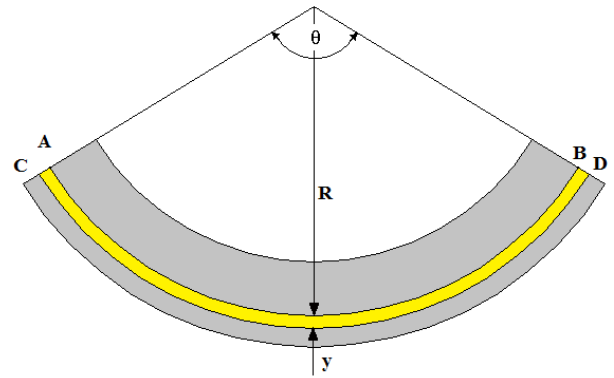


Figure 3

Remember the length of an arc is radius \times angle in radians

2.2 RADIUS OF CURVATURE

Normally the beam does not bend into a circular arc. However, whatever shape the beam takes under the sideways loads; it will basically form a curve on an $x - y$ graph. In maths, the radius of curvature at any point on a graph is the radius of a circle that just touches the graph and has the same tangent at that point.

2.3 RELATIONSHIP BETWEEN STRAIN AND RADIUS OF CURVATURE

The length of AB $AB = R \theta$

Consider figure 3. There is a layer of material distance y from the neutral axis and this is stretched because it must become longer. The radius of the layer is $R + y$.

The length of this layer is denoted by the line DC. $DC = (R + y)\theta$

This layer is strained and strain (ϵ) is defined as $\epsilon = \text{change in length} / \text{original length}$

$$\text{Substitute } AB = R \theta \text{ and } DC = (R + y)\theta \quad \epsilon = \frac{DC - AB}{AB} = \frac{(R + y)\theta - R\theta}{R\theta} = \frac{y}{R}$$

The modulus of Elasticity (E) relates direct stress (σ) and direct strain (ϵ) for an elastic material and the definition is as follows.

$$E = \frac{\text{stress}}{\text{strain}} = \frac{\sigma}{\epsilon}$$

$$\text{Substitute } \epsilon = \frac{y}{R} \text{ and } E = \frac{\sigma R}{y} \quad \frac{E}{R} = \frac{\sigma}{y}$$

It follows that stress and strain vary along the length of the beam depending on the radius of curvature.

We will now go on to show that the radius of curvature depends upon the bending moment M acting at any given point along the length of the beam.

2.4 RELATIONSHIP BETWEEN STRESS AND BENDING MOMENT

Consider a beam with a consistent shape along its length. An arbitrary oval shape is shown here. Think of the beam as being made of many thin layers of material running the length of the beam and held together by molecular forces.

Consider one such elementary layer at a given point along the length at a distance y from the neutral axis. When the cross section is viewed end on it appears as an elementary strip width b and thickness δy .

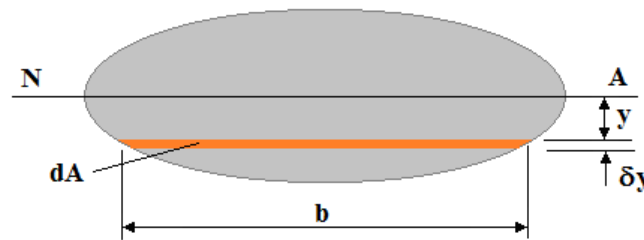


Figure 4

The cross sectional area is A .

The elementary strip is a small part of the total cross sectional Area and is denoted in calculus form as δA .

The strip may be regarded as a thin rectangle width b and height δy so $\delta A = b \delta y$

The stress on the strip is $\sigma = E y / R$

If the layer shown is stretched, then there is a small force δF pulling normal to the section trying to slide the layer out of the material in a lengthwise direction. This force must be the product of the stress and the area and is a small part of the total force acting on the section δF .

$$\delta F = \sigma \delta A \quad \text{Substitute } \sigma = \frac{E y}{R} \text{ and } \delta F = \frac{E y}{R} \delta A$$

Consider that the whole beam is made up of many such layers. Some are being stretched and pull normal to the section and some are compressed and push. The total force acting on the section is the sum of all these small forces.

$$F = \sum \delta F = \sum \frac{E y}{R} \delta A$$

In the limit as δy tends to zero, the number of strips to be summed tends to infinity. The small quantities δy and δA become the differential coefficient dy and dA . The total force is given by the integration

$$F = \int_{\text{bottom}}^{\text{top}} \frac{E y}{R} dA = \frac{E}{R} \int_{\text{bottom}}^{\text{top}} y dA$$

The expression $\int_{\text{bottom}}^{\text{top}} y dA$ is by definition the first moment of area of the shape about the centroid.

Evaluating this expression would give zero since any first moment of area is zero about the centroid.

The centroid in this case is on the neutral axis. The areas above and below the neutral axis are equal. Half the force is a compressive force pushing into the diagram, and half is tensile pulling out. They are equal and opposite so it follows that $F = 0$ which is sensible since cross sections along the length of a beam obviously are held in equilibrium.

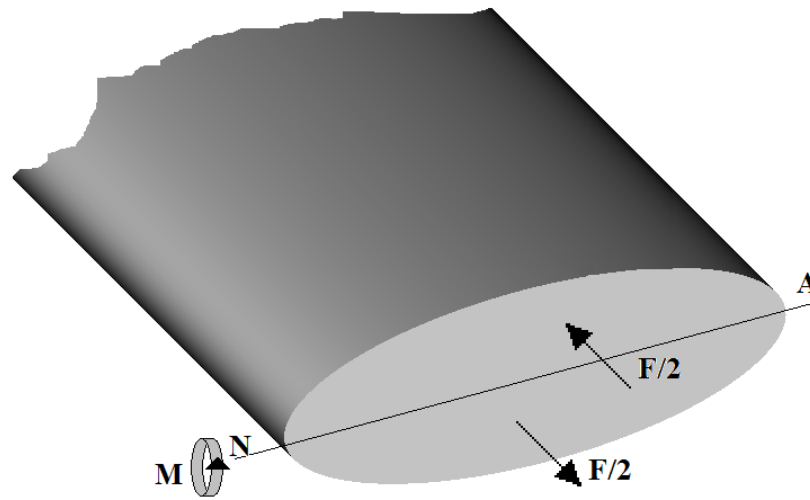


Figure 5

The diagram indicates that the two forces produce a turning moment about the neutral axis because half the section is in tension and half in compression. This moment must be produced by the external forces acting on the beam making it bend in the first place. This moment is called the bending moment M .

Consider the moment produced by the force on the elementary strip δF . It acts a distance y from the neutral axis so the moment produced is $\delta M = y \delta F$

In the limit as δy tends to zero the total moment is found by reverting to calculus again.

$$M = \sum y \delta F = \int_{\text{bottom}}^{\text{top}} y dF = \int_{\text{bottom}}^{\text{top}} y \frac{E y}{R} dA$$

$$M = \frac{E}{R} \int_{\text{bottom}}^{\text{top}} y^2 dA$$

The expression $\int_{\text{bottom}}^{\text{top}} y^2 dA$ is by definition the **second moment of area** about the neutral axis and this is not

zero but has a definite value. In general it is denoted by the symbol I .

$$I = \int_{\text{bottom}}^{\text{top}} y^2 dA$$

We may now write the moment as $M = \frac{E}{R} I$ and rearrange it to $\frac{M}{I} = \frac{E}{R}$

Combining $\frac{E}{R} = \frac{\sigma}{y}$ and $\frac{M}{I} = \frac{E}{R}$ we now have $\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}$

This is called the bending equation and it has 3 parts.

If the stress is required at a given point along the beam we use either $\sigma = \frac{My}{I}$ or $\sigma = \frac{Ey}{R}$ This indicates that the stress in a beam depends on the bending moment and so the maximum stress will occur where the bending moment is a maximum along the length of the beam. It also indicates that stress is related to distance y from the neutral axis so it varies from zero to a maximum at the top or bottom of the section. One edge of the beam will be in maximum tension and the other in maximum compression. In beam problems, we must be able to deduce the position of greatest bending moment along the length.

2.5 STANDARD SECTIONS

At this stage, don't worry about how M is found, it is covered later. For a given section the value of I may be determined by mathematics. Many beams are manufactured with standard sections. British Standard BS4 part 1 gives the properties of standard steel beams and joists. The areas and second moments of area are listed in the standards and since the distance y to the edge is also known they list a property called the ELASTIC MODULUS and this is defined as $z = \frac{I}{y}$. The stress at the edge of the beam is then found from

$$\text{the equation } \sigma = \frac{My}{I} = \frac{M}{Z}.$$

For standard shapes the second moment of area can be deduced. This is covered in the pre-requisite tutorial on moments of area. The following formulae apply to standard shapes.

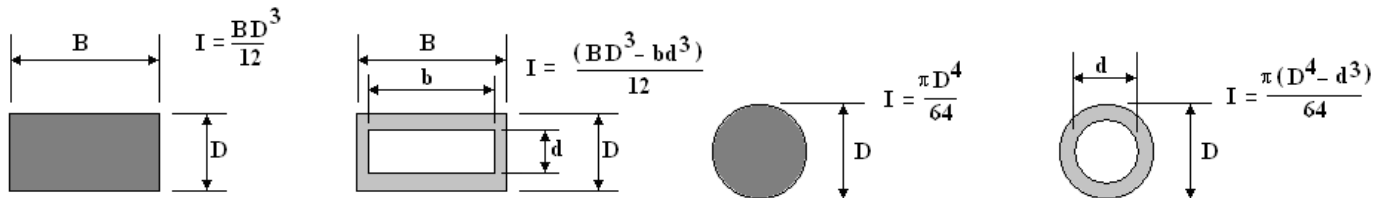


Figure 6

For more complex shapes such as TEE and U sections, you will need to study the pre-requisite level tutorial in order to solve the second moment of area. There are computer programmes for solving beam problems. The Archon Engineering web site has many such programmes.

WORKED EXAMPLE No.1

A beam has a rectangular cross section 80 mm wide and 120 mm deep. It is subjected to a bending moment of 15 kNm at a certain point along its length. It is made from metal with a modulus of elasticity of 180 GPa. Calculate the maximum stress on the section.

SOLUTION

$B = 80$ mm, $D = 120$ mm. It follows that the value of y that gives the maximum stress is 60 mm. Remember all quantities must be changed to metres in the final calculation.

$$I = \frac{BD^3}{12} = \frac{80 \times 100^3}{12} = 6.667 \times 10^6 \text{ mm}^4 = 6.667 \times 10^{-6} \text{ m}^4$$

$$\frac{M}{I} = \frac{\sigma}{y}$$

$$\sigma = \frac{My}{I} = \frac{15 \times 10^3 \times 0.05}{6.667 \times 10^{-6}} = 112.5 \times 10^6 \text{ N/m}^2$$

WORKED EXAMPLE No.2

A beam has a hollow circular cross section 40 mm outer diameter and 30 mm inner diameter. It is made from metal with a modulus of elasticity of 205 GPa. The maximum tensile stress in the beam must not exceed 350 MPa.

Calculate the following.

- (i) the maximum allowable bending moment.
- (ii) the radius of curvature.

SOLUTION

$$D = 40 \text{ mm}, d = 30 \text{ mm}$$

$$I = \pi(40^4 - 30^4)/64 = 85.9 \times 10^3 \text{ mm}^4 \text{ or } 85.9 \times 10^{-9} \text{ m}^4.$$

The maximum value of y is D/2 so y = 20 mm or 0.02 m

$$\frac{M}{I} = \frac{\sigma}{y}$$

$$M = \frac{\sigma I}{y} = \frac{350 \times 10^6 \times 85.9 \times 10^{-9}}{0.02} = 1503 \text{ Nm or } 1.503 \text{ MNm}$$

$$\frac{\sigma}{y} = \frac{E}{R}$$

$$R = \frac{Ey}{\sigma} = \frac{205 \times 10^9 \times 0.02}{350 \times 10^6} = 11.71 \text{ m}$$

SELF ASSESSMENT EXERCISE No.1

1. A beam has a bending moment (M) of 3 kNm applied to a section with a second moment of area (I) of $5 \times 10^{-3} \text{ m}^4$. The modulus of elasticity for the material (E) is $200 \times 10^9 \text{ N/m}^2$. Calculate the radius of curvature. (Answer 333.3 km).
2. The beam is Q1 has a distance from the neutral axis to the edge in tension of 60 mm. Calculate the stress on the edge. (Answer 36 kPa).
3. A beam under test has a measured radius of curvature of 300 m. The bending moment applied to it is 8 Nm. The second moment of area is 8000 mm^4 . Calculate the modulus of elasticity for the material. (Answer 300 GPa).

3. COMBINED BENDING AND DIRECT STRESS

3.1 NEUTRAL AXIS

When bending alone occurs in a member such as a beam, the neutral axis passes through the centroid. The stress varies from top to bottom over the structure from a maximum tensile on one edge to a maximum compressive on the other. The stress distribution is typically as shown. The stress is zero on the neutral axis and this passes through the centroid.

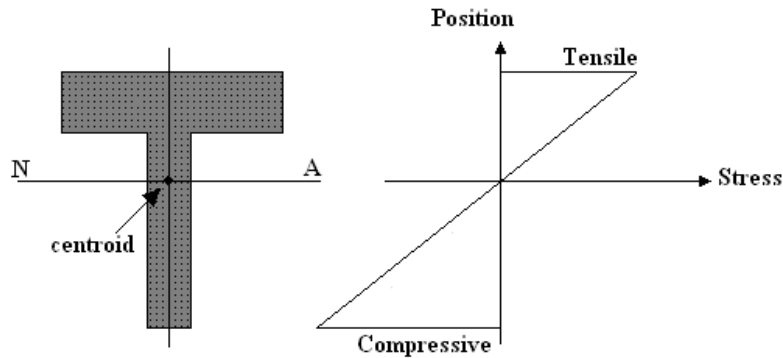


Figure 7

When a compressive stress is added to the bending stress, the stress everywhere is decreased by σ_B and the neutral axis moves away from the centroid towards the tensile edge as shown in figure 8. It is quite possible for the neutral axis to move beyond the edge.

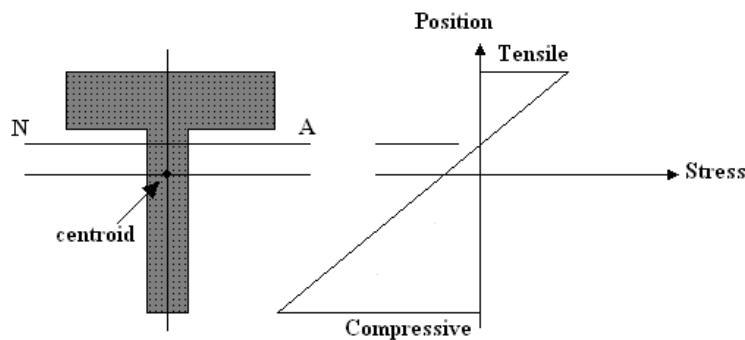


Figure 8

WORKED EXAMPLE No.3

Calculate the stress on the top and bottom of the section shown when the bending moment is 300 N m. Draw the stress distribution.

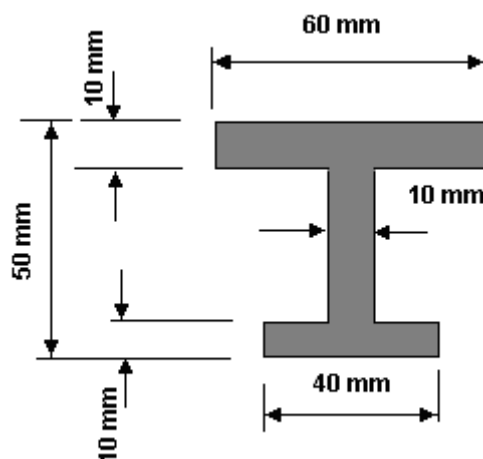


Figure 9

SOLUTION

First calculate the second moment of area using the tabular method that you should already know. Divide the shape into three sections A, B and C. First determine the position of the centroid from the bottom edge.

	Area	\bar{y}	$A\bar{y}$
A	600 mm ²	45 mm	27 000 mm ³
B	300 mm ²	25 mm	7500 mm ³
C	400 mm ²	5 mm	2000 mm ³
Totals	1300 mm ²		365000 mm ³

For the whole section the centroid position is $\bar{y} = 365000/1300 = 28.07$ mm

Now find the second moment of area about the base. Using the parallel axis theorem.

	$BD^3/12$	$A\bar{y}^2$	$I = BD^3/12 + A\bar{y}^2$
A	$60 \times 10^3/12 = 5000$ mm ⁴	$600 \times 45^2 = 1215000$	1220000 mm ⁴
B	$10 \times 30^3/12 = 22500$ mm ⁴	$300 \times 25^2 = 187500$	210000 mm ⁴
C	$40 \times 10^3/12 = 3333$ mm ⁴	$400 \times 5^2 = 10000$	13333 mm ⁴
			Total = 1443333 mm ⁴

The total second moment of area about the bottom is 1443333 mm⁴

Now move this to the centroid using the parallel axis theorem.

$$I = 1443333 - A\bar{y}^2 = 1443333 - 1300 \times 28.08^2 = 418300 \text{ mm}^4$$

Now calculate the stress using the well known formula $\sigma_B = My/I$

Top edge y = distance from the centroid to the edge = $50 - 28.08 = 21.93$ mm

$$\sigma_B = 300 \times 0.02192 / 418.300 \times 10^{-9} = 15.72 \times 10^6 \text{ Pa or } 15.72 \text{ MPa (Tensile)}$$

Bottom edge $y = \bar{y} = 28.07$ mm

$$\sigma_B = 300 \times 0.02808 / 418.300 \times 10^{-9} = 20.14 \text{ MPa (Tensile)}$$

The stress distribution looks like this.

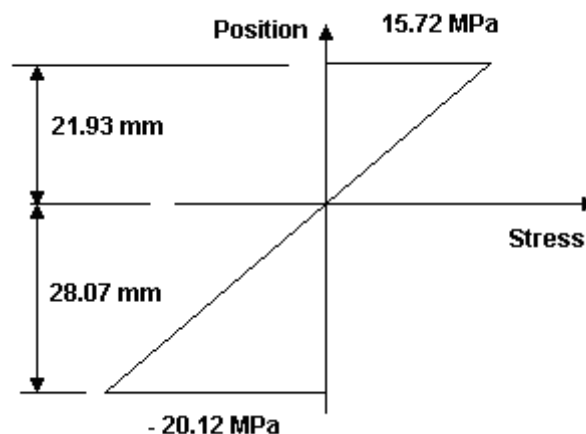


Figure 10

WORKED EXAMPLE No.4

The section solved in example 2 is subjected to a tensile force that adds a tensile stress of 10 MPa everywhere. Sketch the stress distribution and determine the new position of the neutral axis.

SOLUTION

The stress on the top edge will increase to 25.72 MPa and on the bottom edge it will decrease to -10.12 MPa. The new distribution will be as shown and the new position of the neutral axis may be calculated by ratios.

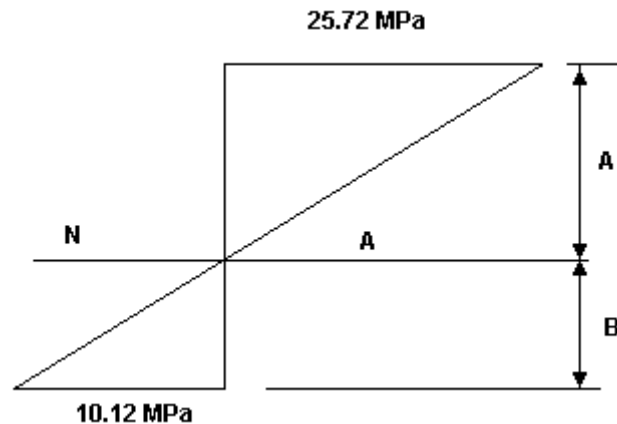


Figure 11

$$A + B = 50 \text{ mm} \quad \text{so } B = 50 - A$$

$$\text{By similar triangles } A/25.72 = B/10.12 \quad A = (25.72/10.12)B = 2.54 B$$

$$B = 50 - 2.54 B \quad 3.54 B = 50 \quad B = 14.12 \text{ mm} \quad A = 50 - 14.12 = 35.88 \text{ mm}$$

SELF ASSESSMENT EXERCISE No.2

1. A symmetrical I section beam is 60 mm deep with a second moment of area of $663 \times 10^{-9} \text{ m}^4$ and a cross sectional area of 1600 mm^2 . It is subject to a bending moment of 1.2 kNm and an axial force of 25 kN (tension). Find the position of the neutral axis.

(Answer the stresses are 69.92 MPa and -38.67 MPa and the neutral axis is 38.6 mm from the tensile edge)

4. COMPOSITE BEAMS

A simple composite beam might be a piece of timber clad with steel as shown. It must be assumed that they are firmly bonded so that the strain at the interface is the same for both materials.

The bending equation states $M/I = E/R = \sigma/y$

It has been shown that $\varepsilon = y/R = \sigma/E$

At the interface ε must be the same for both materials so it follows that $\sigma_t = \sigma_s E_t / E_s$ where t refers to timber and s to steel.

If the beam was made entirely out of steel, it would have to have the same flexural stiffness (EI) as the composite beam while retaining the same vertical dimension for the interface. Since $I = B \int y^3 dy$ it is necessary to maintain the same y values so B must be changed in the ratio of the values of E.

The equivalent steel section has a web width $w = B E_t / E_s$

Now the second moment of area about the centroid must be found and the stress can be evaluated for a given value of M.

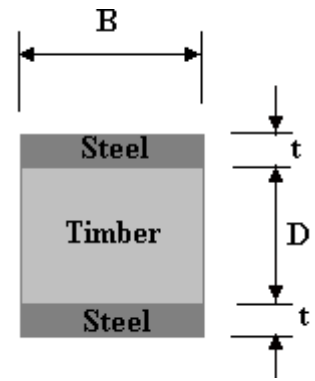


Figure 12

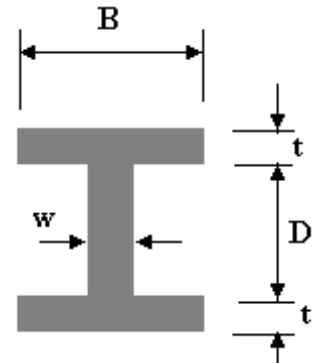


Figure 13

WORKED EXAMPLE No.5

A rectangular section timber beam is 50 mm wide and 75 mm deep. It is clad with steel plate 10 mm thick on the top and bottom. Calculate the maximum stress in the steel and the timber when a moment of 4 kNm is applied.

E for timber is 10 GPa and for steel 200 GPa

SOLUTION

The width of an equivalent steel web must be

$$t = 50 \times E_t / E_s = 50 \times 10/200 = 2.5 \text{ mm}$$

Now calculate I_{gg} for the equivalent beam.

This is easy because it is symmetrical and involves finding I for the outer box and subtracting I for the missing parts.

$$I_{gg} = 50 \times 95^3 / 12 - 47.5 \times 75^3 / 12$$

$$I_{gg} = 1.9025 \times 10^{-6} \text{ m}^4$$

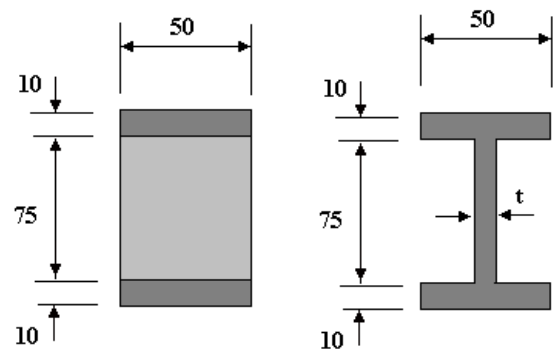


Figure 14

The stress at $y = 37.5 \text{ mm}$ $\sigma = My/I = 4000 \times 0.0375 / 1.9025 \times 10^{-6} = 78.845 \text{ MPa}$

The stress in the timber at this level will be different because of the different E value.

$$\sigma_t = \sigma_s E_t / E_s = 3.942 \text{ MPa}$$

The stress at $y = 47.5 \text{ mm}$ will be the stress at the edge of the steel.

$$\sigma_s = My/I = 4000 \times 0.0475 / 1.9025 \times 10^{-6} = 99.87 \text{ MPa}$$

WORKED EXAMPLE No.6

A symmetrical steel I section beam has a second moment of area $I_{gg} = 90 \times 10^{-6} \text{ m}^4$ and section area $6 \times 10^{-3} \text{ m}^2$. It has a vertical depth of 300 mm and forms part of a floor with concrete slabs firmly bonded to the top 1.2 m wide and 100 mm thick.

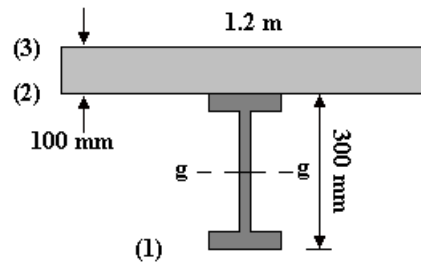


Figure 15

Calculate the stress in the steel at levels (1) and (2) and in the concrete at levels (2) and (3) when a bending moment of 50 kNm is applied to the section. The top layer is in compression. The modulus of elasticity is 200 GPa for steel and 20 GPa for concrete.

SOLUTION

First reduce the concrete to an equivalent width of steel. $B = 1.2 \times E_s/E_c$

$B = 1.2 \times 20/200 = 0.12 \text{ m}$. The equivalent steel beam is like this.

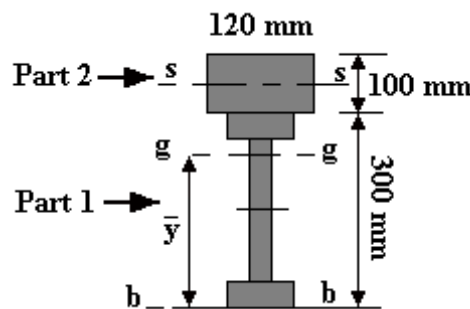


Figure 16

Now find the position of the centroid \bar{y} by finding the first moment of area about the base.

	Area	\bar{y}	$A\bar{y}$
Part 1	6×10^{-3}	0.15	0.9×10^{-3}
Part 2	12×10^{-3}	0.35	4.2×10^{-3}
Total	18×10^{-3}		5.1×10^{-3}

\bar{y} for the section is $5.1 \times 10^{-3} / 18 \times 10^{-3} = 0.283 \text{ m}$

Next find the second moment of area about the centroid axis $g-g$. Use the parallel axis theorem. H = distance from axis $g-g$ to centre of the part.

	Area	h	Ah^2	$BD^3/12$	I_{gg}
Part 2	12×10^{-3}	0.067	53.3×10^{-6}	10×10^{-6}	63.3×10^{-6}

Part 1 $I_{gg} = I + Ah^2 = 90 \times 10^{-6} + (6 \times 10^{-3} \times 0.133^2) = 196 \times 10^{-6}$

Total $I_{gg} = 63.3 \times 10^{-6} + 196 \times 10^{-6} = 259.3 \times 10^{-6} \text{ m}^4$

At level (1) $\sigma_s = My/I_{gg} = 50\,000 \times 0.283 / 259.3 \times 10^{-6} = 54.57 \text{ MPa}$

At level (2) $\sigma_s = My/I_{gg} = 50\,000 \times 0.017 / 259.3 \times 10^{-6} = 3.28 \text{ MPa}$

At level (2) $\sigma_c = 3.28 \times E_c/E_s = 3.28 \times (20/200) = 0.328 \text{ MPa}$

At level (3) $\sigma_s = My/I_{gg} = 50\,000 \times 0.117 / 259.3 \times 10^{-6} = 22.56 \text{ MPa}$

At level (3) $\sigma_c = 22.56 \times E_c/E_s = 22.56 \times (20/200) = 2.256 \text{ MPa}$

SELF ASSESSMENT EXERCISE No.3

1. A rectangular section timber beam is 60 mm wide and 100 mm deep. It is clad with steel plate 12 mm thick on the top and bottom. Calculate the maximum stress in the steel and the timber when a moment of 5 kNm is applied.
E for timber is 11 GPa and for steel 205 GPa.
(64.5 MPa and 28 MPa)
2. A symmetrical steel I section beam has a second moment of area $I_{gg} = 3391.3 \times 10^{-6} \text{ m}^4$ and section area $28.84 \times 10^{-3} \text{ m}^2$. It has a vertical depth of 851 mm and forms part of a floor with concrete slabs firmly bonded to the top 1.5 m wide and 1200 mm thick.

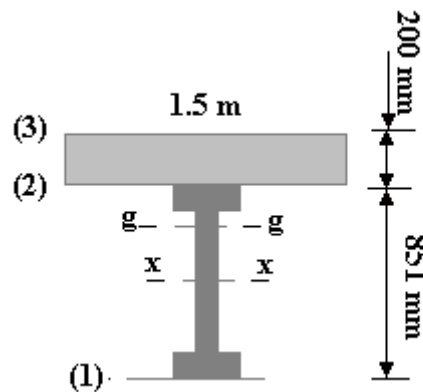


Figure 17

Calculate the stress in the steel at levels (1) and (2) and in the concrete at levels (2) and (3) when a bending moment of 50 kNm is applied to the section. The top layer is in compression. The modulus of elasticity is 205 GPa for steel and 18 GPa for concrete.

(4.64 MPa tensile, 1.2 MPa compressive, 0.105 MPa compressive and 0.226 MPa compressive)