MECHANICS OF SOLIDS - BEAMS

TUTORIAL 1

STRESSES IN BEAMS DUE TO BENDING

This is the first tutorial on bending of beams designed for anyone wishing to study it at a fairly advanced level. You should judge your progress by completing the self assessment exercises.

On completion of this tutorial you should be able to do the following.

- Define a beam.
- Recognise different types of beams.
- Define BENDING MOMENT
- Derive the BENDING FORMULAE for beams.
- Calculate the stress in a beam due to bending.
- Solve problems involving both bending and direct stress.
- Find the position of the neutral axis for combined stress situations.
- Solve problems involving simple composite beams.

It is assumed that students doing this tutorial already understand the basic principles of moments, shear force, stress and moments of area.

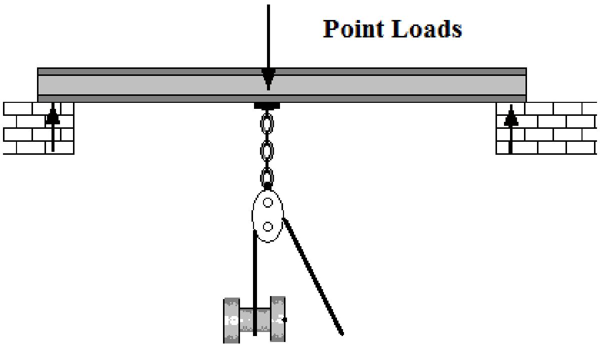
Students must also be able to perform basic differentiation and calculus from their maths studies.

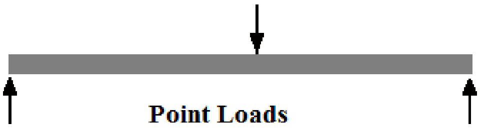
Contents

1 *Types of Beams*

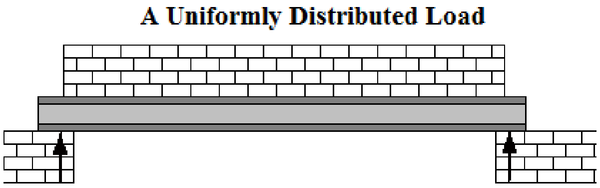
- A beam is a structure, which is loaded transversely (sideways).
- The loads may be point loads or uniformly distributed loads (udl).

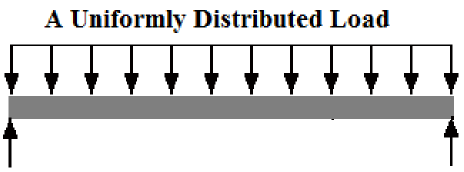
The diagrams show the way that point loads and uniform loads are illustrated. A point load is a load or force that acts at a single point on a structure and it is depicted by a single arrow on diagrams.



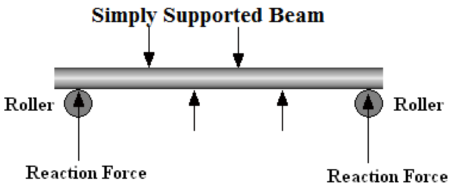


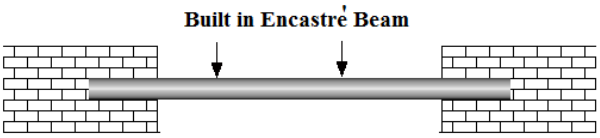
A uniform load is one which is evenly distributed along a length such as the weight of the beam or a wall built on top of a beam. It is depicted by a series of arrows as shown.
We usually denote the loading as w N/m.

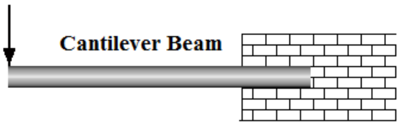




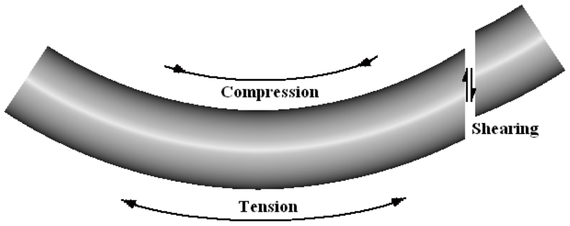
- **The beam may be simply supported or built in.**







Transverse loading causes bending and bending is a very severe form of stressing a structure. The bent beam goes into tension (stretched) on one side and compression on the other.



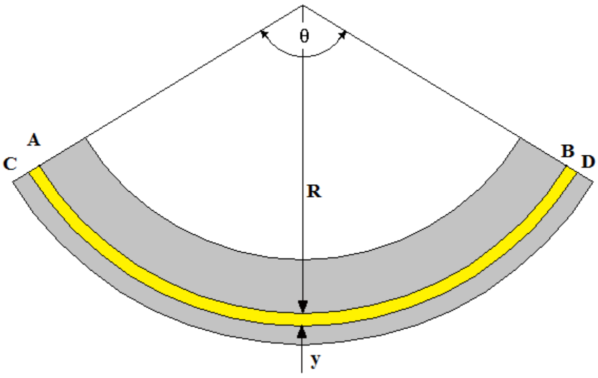
The complete formula which describes all aspects of bending is

$$\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}$$

We will now look at the derivation and use of this formula.

2. *The Bending Formula*

2.1 *Neutral Axis*



This is the axis along the length of the beam which remains unstressed, neither compressed nor stretched when it is bent. Normally the neutral axis passes through the centroid of the cross sectional area. The position of the centroid is hence important.

Consider that the beam is bent into an arc of a circle through angle θ radians. AB is on the neutral axis and is the same length before and after bending. The radius of the neutral axis is R.

Remember the length of an arc is radius x angle in radians

2.2 *Radius of Curvature*

Normally the beam does not bend into a circular arc. However, what ever shape the beam takes under the sideways loads; it will basically form a curve on an x – y graph. In maths, the radius of curvature at any point on a graph is the radius of a circle that just touches the graph and has the same tangent at that point.

2.3 *Relationship between Strain and Radius of Curvature*

The length of AB $AB = R\,\theta$

Consider the diagram above. There is a layer of material distance y from the neutral axis and this is stretched because it must become longer. The radius of the layer is R + y.

The length of this layer is denoted by the line DC. $DC = (R + y)\theta$

This layer is strained and strain (ε) is defined as ε= change in length/original length

Substitute $AB = R\,\theta$ and $DC = (R + y)\theta$

$$\varepsilon = \frac{DC - AB}{AB} = \frac{(R+y)\theta - R\theta}{R\theta} = \frac{y}{R}$$

The modulus of Elasticity (E) relates direct stress (σ) and direct strain (ε) for an elastic material and the definition is as follows.

$$E = \frac{\text{stress}}{\text{strain}} = \frac{\sigma}{\varepsilon} \quad \text{Substitute } \varepsilon = \frac{y}{R} \text{ and } E = \frac{\sigma R}{y} \quad \frac{E}{R} = \frac{\sigma}{y}$$

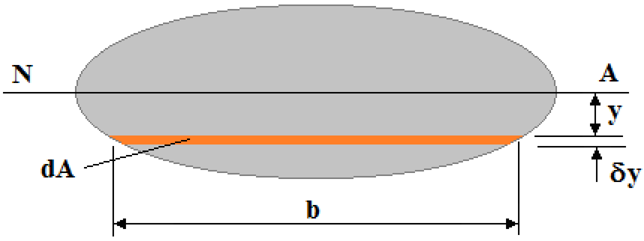
It follows that stress and strain vary along the length of the beam depending on the radius of curvature.

We will now go on to show that the radius of curvature depends upon the bending moment M acting at any given point along the length of the beam.

2.4 Relationship between Stress and Bending Moment

Consider a beam with a consistent shape along its length. An arbitrary oval shape is shown here. Think of the beam as being made of many thin layers of material running the length of the beam and held together by molecular forces.

Consider one such elementary layer at a given point along the length at a distance y from the neutral axis. When the cross section is viewed end on it appears as an elementary strip width b and thickness δy.



The cross sectional area is A.
The elementary strip is a small part of the total cross sectional Area and is denoted in calculus form as δA.
The strip may be regarded as a thin rectangle width b and height δy so **δA = b δy**
The stress on the strip is **σ = Ey/R**

If the layer shown is stretched, then there is a small force δF pulling normal to the section trying to slide the layer out of the material in a lengthwise direction. This force must be the product of the stress and the area and is a small part of the total force acting on the section δF.

$$\delta F = \sigma\,\delta A \quad \text{Substitute } \sigma = \frac{Ey}{R} \text{ and } \delta F = \frac{Ey}{R}\delta A$$

Consider that the whole beam s made up of many such layers. Some are being stretched and pull normal to the section and some are compressed and push. The total force acting on the section is the sum of all these small forces.

$$F = \sum \delta F = \sum \frac{Ey}{R}\delta A$$

In the limit as δy tends to zero, the number of strips to be summed tends to infinity. The small quantities δy and δA become the differential coefficient dy and dA. The total force is given by the integration

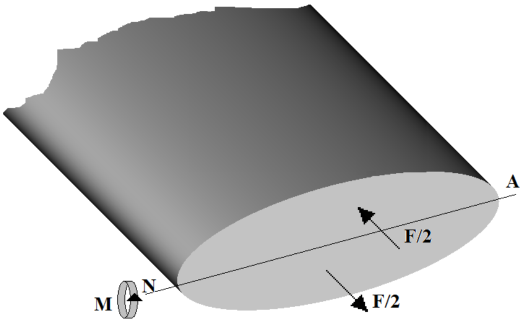
$$F = \int_{\text{bottom}}^{\text{top}} \frac{Ey}{R} dA = \frac{E}{R}\int_{\text{bottom}}^{\text{top}} y dA$$

$$\int_{\text{bottom}}^{\text{top}} y dA \ = \text{first moment of area about the neutral axis.}$$

Evaluating this expression would give zero since any first moment of area is zero about the centroid.

The centroid in this case is on the neutral axis. The areas above and below the neutral axis are equal. Half the force is a compressive force pushing into the diagram, and half is tensile pulling out. They are equal and opposite so it follows that F = 0 which is sensible since cross sections along the length of a beam obviously are held in equilibrium.

The diagram indicates that the two forces produce a turning moment about the neutral axis because half the section is in tension and half in compression. This moment must be produced by the external forces acting on the beam making it bend in the first place. This moment is called the bending moment M.



Consider the moment produced by the force on the elementary strip δF. It acts a distance y from the neutral axis so the moment produced is δM = y δF

In the limit as δy tends to zero the total moment is found by reverting to calculus again.

$$M = \sum y\,\delta F = \int_{\text{bottom}}^{\text{top}} y\,dF = \int_{\text{bottom}}^{\text{top}} y\frac{Ey}{R}dA = \frac{E}{R}\int_{\text{bottom}}^{\text{top}} y^2 dA$$

The second moment of area about the neutral axis is defined as $I = \int_{\text{bottom}}^{\text{top}} y^2 dA$

It follows that

$$M = \frac{EI}{R}$$

Rearrange and

$$\frac{M}{I} = \frac{E}{R}$$

Combining equations

$$\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}$$

This is called the bending equation and it has 3 parts. If the stress is required at a given point along the beam we use either

$$\sigma = \frac{My}{I} \text{ or } \frac{Ey}{R}$$

This indicates that the stress in a beam depends on the bending moment and so the ***maximum stress will occur where the bending moment is a maximum*** along the length of the beam. It also indicates that stress is related to distance y from the neutral axis so it varies from zero to a maximum at the top or bottom of the section. One edge of the beam will be in maximum tension and the other in maximum compression. In beam problems, we must be able to deduce the position of greatest bending moment along the length.

2.5 Standard Sections

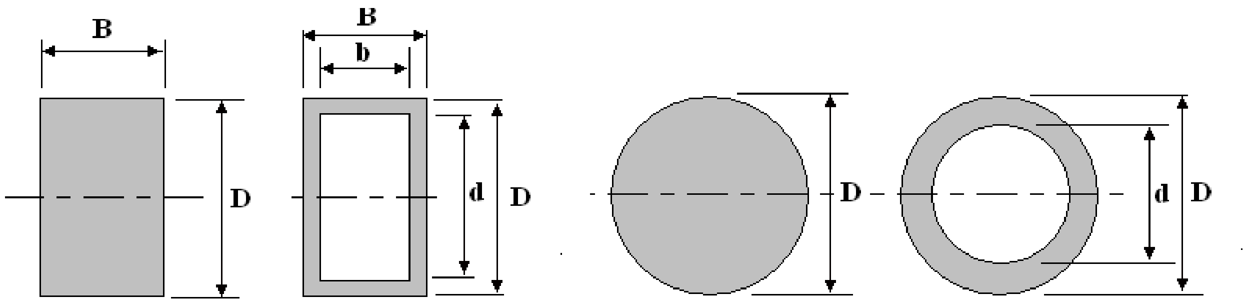
For a given section the value of I may be determined by mathematics. Many beams are manufactured with standard sections. British Standard BS4 part 1 gives the properties of standard steel beams and joists. The areas and second moments of area are listed in the standards and since the distance y to the edge is also known they list a property called the ***Elastic Modulus*** and this is defined as

$$Z = \frac{I}{y}$$

The stress at the edge of the beam is then found from the equation:

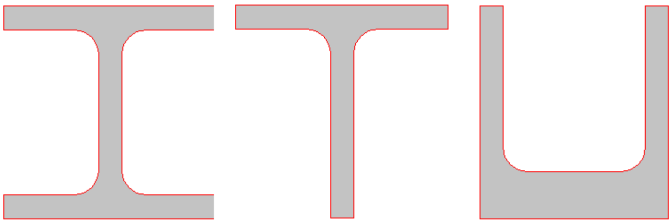
$$\sigma = \frac{My}{I} = \frac{M}{Z}$$

For standard shapes the second moment of area can be deduced. The following formulae apply to standard shapes. It is a good exercise to do the mathematical derivation.



$$I = \frac{BD^3}{12} \qquad I = \frac{BD^3 - bd^3}{12} \qquad I = \frac{\pi D^4}{64} \qquad I = \frac{\pi(D^4 - d^4)}{64}$$

For more complex shapes such as TEE and U sections, you will need further studies. Standard steel beams are known as Universal Beams and Rolled Steel Joists (RSJ). The properties of these are found in standard tables. “I”, “T” and “U” sections are commonly used as shown.



WORKED EXAMPLE No. 1

A beam has a rectangular cross section 80 mm wide and 120 mm deep. It is subjected to a bending moment of 15 kNm at a certain point along its length. It is made from metal with a modulus of elasticity of 180 GPa. Calculate the maximum stress on the section.

SOLUTION

B = 80 mm, D = 100 mm. It follows that the value of y that gives the maximum stress is 50 mm. Remember all quantities must be changed to metres in the final calculation.

$$I = \frac{BD^3}{12} = \frac{80 \text{ x } 100^3}{12} = 6.667 \times 10^6 \text{mm}^4$$

$$\sigma = \frac{My}{I} = \frac{15 \text{ x } 10^3 \times 0.05}{6.667 \times 10^6} = 112.5 \times 10^6 \text{ N/m}^2$$

WORKED EXAMPLE No. 2

A beam has a hollow circular cross section 40 mm outer diameter and 30 mm inner diameter. It is made from metal with a modulus of elasticity of 205 GPa. The maximum tensile stress in the beam must not exceed 350 MPa.

Calculate the following.

(i) the maximum allowable bending moment.
(ii) the radius of curvature.

SOLUTION

D = 40 mm, d = 30 mm

$$I = \frac{\pi(D^4 - d^4)}{64} = \frac{\pi(40^4 - 30^4)}{64} = 85.9 \times 10^3 \text{mm}^4 \text{ or } 85.9 \times 10^{-9} \text{ m}^4$$

The maximum value of y is D/2 so y = 20 mm or 0.02 m

$$M = \frac{\sigma I}{y} = \frac{350 \times 10^6 \times 85.9 \times 10^{-9}}{0.02} = 1\,503 \text{ Nm or } 1.503 \text{ MNm}$$

$$R = \frac{E\,y}{\sigma} = \frac{205 \times 10^9 \times 0.02}{350 \times 10^6} = 11.71 \text{ m}$$

SELF ASSESSMENT EXERCISE No. 1

1. A beam has a bending moment (M) of 3 kNm applied to a section with a second moment of area (**I**) of 5×10^{-3} m^4. The modulus of elasticity for the material (E) is 200×10^9 N/m^2.

   Calculate the radius of curvature. (Answer 333.3 km).

2. The beam is Q1 has a distance from the neutral axis to the edge in tension of 60 mm. Calculate the stress on the edge. (Answer 36 kPa).

3. A beam under test has a measured radius of curvature of 300 m.
   The bending moment applied to it is 8 Nm. The second moment of area is 8000 mm^4.

   Calculate the modulus of elasticity for the material. (Answer 300 GPa).

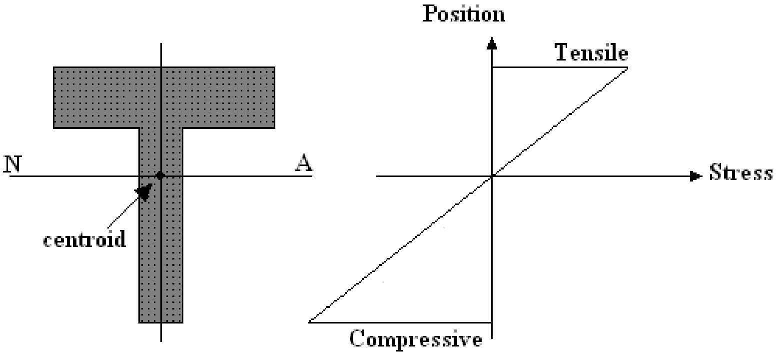
4. A beam is made from round tube 120 mm outer diameter and 100 mm inner diameter. If the bending moment at a given point is 6 kNm

   Determine the stress at the outer edge and the radius of curvature. Take E = 205 GPa
   (68.3 MPa and 180 m)

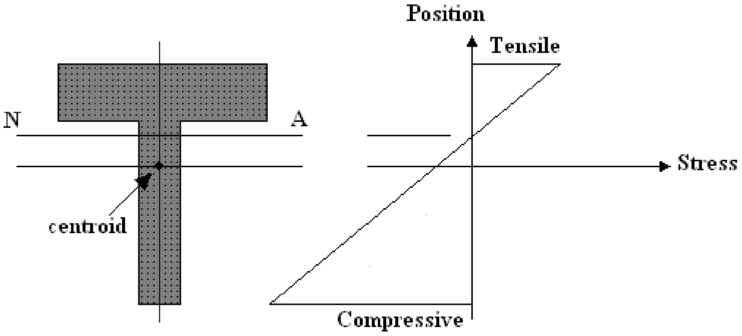
3. *Combined Bending and Direct Stress*

3.1 *Neutral Axis*

When bending alone occurs in a member such as a beam, the neutral axis passes through the centroid. The stress varies from top to bottom over the structure from a maximum tensile on one edge to a maximum compressive on the other. The stress distribution is typically as shown. The stress is zero on the neutral axis and this passes through the centroid.

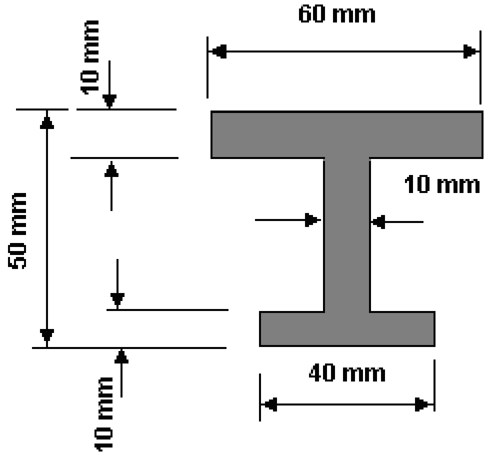


When a compressive stress is added to the bending stress, the stress everywhere is decreased by σ_B and the neutral axis moves away from the centroid towards the tensile edge as shown in below. It is quite possible for the neutral axis to move beyond the edge.



WORKED EXAMPLE No. 3

Calculate the stress on the top and bottom of the section shown when the bending moment is 300 N m. Draw the stress distribution.



SOLUTION

First calculate the second moment of area using the tabular method that you should already know. Divide the shape into three sections A, B and C. First determine the position of the centroid from the bottom edge.

	Area	$\bar{y}$	$A\bar{y}$
A	600 mm^2	45 mm	27 000 mm^3
B	300 mm^2	25 mm	7 500 mm^3
C	400 mm^2	5 mm	2 000 mm^3
Totals	1300 mm^2		365 000 mm^3

For the whole section the centroid position is

$$\bar{y} = \frac{365\ 000}{1\ 300} = 28.07 \text{ mm}$$

Now find the second moment of area about the base. Use the parallel axis theorem.

	$BD^3/12$	$A\bar{y}^2$	$I = BD^3/12 + A\bar{y}^2$
A	$60 \times 10^3/12 = 5\ 000$ mm^4	$600 \times 45^2 = 1215000$	1 220 000 mm^4
B	$10 \times 30^3/12 = 22\ 500$ mm^4	$300 \times 25^2 = 187500$	210 000 mm^4
C	$40 \times 10^3/12 = 3\ 333$ mm^4	$400 \times 5^2 = 10000$	13 333 mm^4
			Total = 1 443 333 mm^4

The total second moment of area about the bottom is 1443 333 mm4
Now move this to the centroid using the parallel axis theorem.

$$I = 1\ 443\ 333 - A\bar{y}^2 \quad = 1\ 443\ 333 - 1\ 300 \times 28.08^2 = 418\ 300 \text{ mm}^4$$

Now calculate the stress using the well known formula $\sigma_B = My/I$

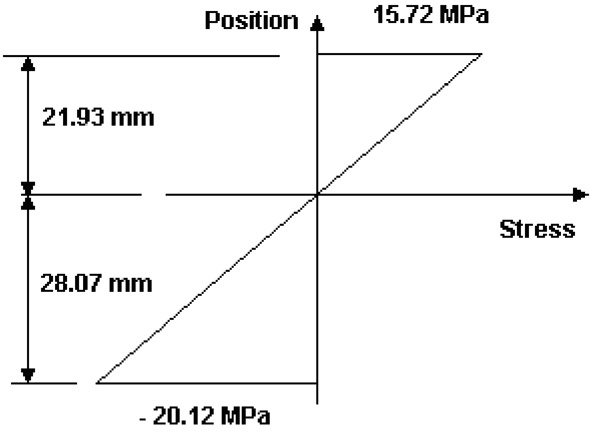
Top edge y = distance from the centroid to the edge = 50 – 28.08 = 21.93 mm

$$\sigma_B = \frac{300 \times 0.02192}{418.3 \times 10^{-9}} = 15.72 \times 10^6 \text{ Pa or } 15.72 \text{ MPa (Tensile)}$$

Bottom edge y = $\bar{y}$ =28.07 mm

$$\sigma_B = \frac{300 \times 0.02808}{418.3 \times 10^{-9}} = 20.14 \times 10^6 \text{ Pa or } 20.14 \text{ MPa (Compressive)}$$

The stress distribution looks like this.

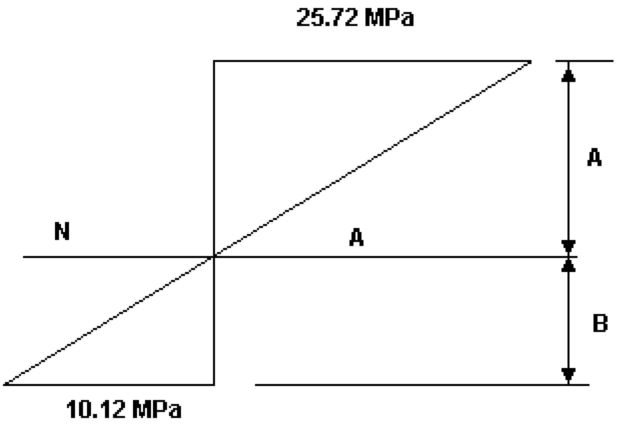


WORKED EXAMPLE No. 4

The section solved in example 2 is subjected to a tensile force that adds a tensile stress of 10 MPa everywhere. Sketch the stress distribution and determine the new position of the neutral axis.

SOLUTION

The stress on the top edge will increase to 25.72 MPa and on the bottom edge it will decrease to -10.12 MPa. The new distribution will be as shown and the new position of the neutral axis may be calculated by ratios.



A + B = 50 mm so B = 50 - A

By similar triangles A/25.72 = B/10.12 A = (25.72/10.12)B = 2.54 B

B = 50 - 2.54 B 3.54 B = 50 B = 14.12 mm A = 50 - 14.12 = 35.88 mm

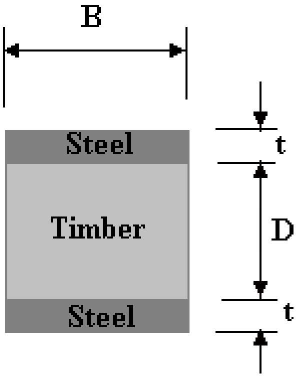
SELF ASSESSMENT EXERCISE No. 2

1. A symmetrical I section beam is 60 mm deep with a second moment of area of 663 × 10^{-9} m^4 and a cross sectional area of 1 600 mm^2. It is subject to a bending moment of 1.2 kN m and an axial force of 25 kN (tension).

   Determine the maximum and minimum stress and find the position of the neutral axis.

   (Answer the stresses are 69.92 MPa and -38.67 MPa and the neutral axis is 38.6 mm from the tensile edge)

4. *Composite Beams*



A simple composite beam might be a piece of timber clad with steel as shown. It must be assumed that they are firmly bonded so that the strain at the interface is the same for both materials.

The bending equation states

$$\frac{M}{I}=\frac{\sigma}{y}=\frac{E}{R}$$

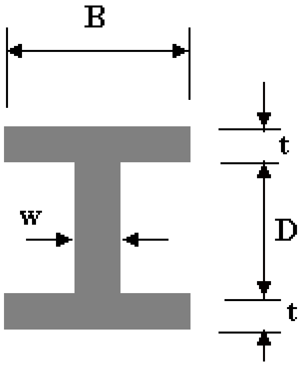
It has been shown that

$$\varepsilon=\frac{y}{R}=\frac{\sigma}{E}$$

At the interface ε must be the same for both materials so it follows that

$$\sigma_t=\sigma_s\frac{E_t}{E_s}$$

t refers to timber and s to steel.



If the beam was made entirely out of steel, it would have to have the same flexural stiffness (EI) as the composite beam while retaining the same vertical dimension for the interface. We defined I as follows

$$I=B\int y^3dy$$

It is necessary to maintain the same y values so B must be changed in the ratio of the values of E.

The equivalent steel section has a web width

$$w=B\frac{E_t}{E_s}$$

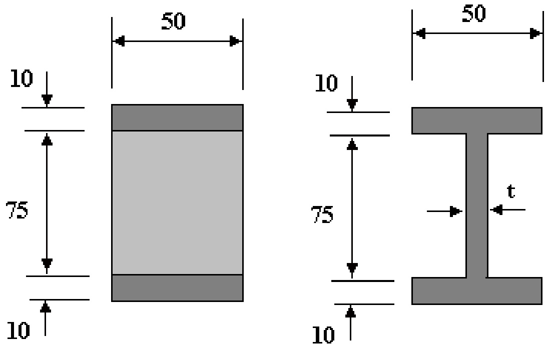
Now the second moment of area about the centroid must be found and the stress can be evaluated for a giver value of M.

WORKED EXAMPLE No. 5

A rectangular section timber beam is 50 mm wide and 75 mm deep. It is clad with steel plate 10 mm thick on the top and bottom.
Calculate the maximum stress in the steel and the timber when a moment of 4 kN m is applied.

E for timber is 10 GPa and for steel 200 GPa

SOLUTION



The width of an equivalent steel web must be

$$t = 50 \times \frac{10}{200} = 2.5 \text{ mm}$$

Now calculate I_{gg} for the equivalent beam. This is easy because it is symmetrical and involves finding I for the outer box and subtracting I for the missing parts.

$$I_{gg} = \frac{50 \times 95^3}{12} - \frac{47.5 \times 75^3}{12} = 1.9025 \times 10^{-6} \text{ m}^4$$

The stress at y = 37.5 mm

$$\sigma = \frac{My}{I} = \frac{4\,000 \times 0.0375}{1.9025 \times 10^{-6}} = 78.845 \text{ MPa}$$

The stress in the timber at this level will be different because of the different E value.

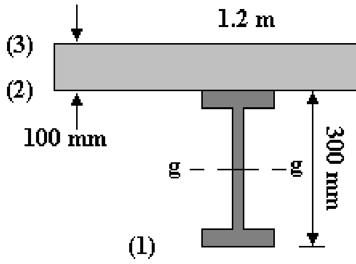
$$\sigma_t = \sigma_s \frac{E_t}{E_s} = 3.942 \text{ MPa}$$

The stress at y = 47.5 mm will be the stress at the edge of the steel.

$$\sigma_s = \frac{My}{I} = \frac{4\,000 \times 0.0475}{1.9025 \times 10^{-6}} = 99.87 \text{ MPa}$$

WORKED EXAMPLE No. 6

A symmetrical steel I section beam has a second moment of area $I_{gg} = 90 \text{ x } 10^{-6} \text{ m}^4$ and section area $6 \text{ x } 10^{-3} \text{ m}^2$. It has a vertical depth of 300 mm and forms part of a floor with concrete slabs firmly bonded to the top 1.2 m wide and 100 mm thick.

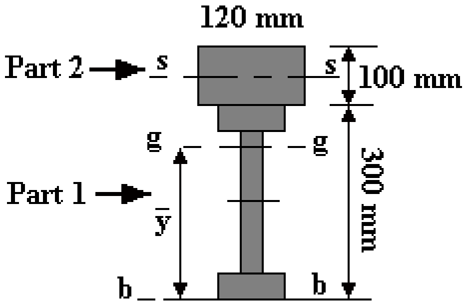


Calculate the stress in the steel at levels (1) and (2) and in the concrete at levels (2) and (3) when a bending moment of 50 kNm is applied to the section. The top layer is in compression. The modulus of elasticity is 200 GPa or steel and 20GPa for concrete.

SOLUTION

First reduce the concrete to an equivalent width of steel. $B = 1.2 \text{ x } E_s/E_c$

$B = 1.2 \text{ x } 20/200 = 0.12$ m. The equivalent steel beam is like this.



Now find the position of the centroid $\bar{y}$ by finding the first moment of area about the base.

	Area	$\bar{y}$	$A\bar{y}$
Part 1	6×10^{-3}	0.15	0.9×10^{-3}
Part 2	12×10^{-3}	0.35	4.2×10^{-3}
Total	18×10^{-3}		5.1×10^{-3}

$\bar{y}$ for the section is $5.1 \times 10^{-3}/18 \times 10^{-3} = 0.283$ m

Next find the second moment of area about the centroid axis g g. Use the parallel axis theorem. H = distance from the axis gg to centre of the part.

	Area	h	Ah^2	$BD^3/12$	I_{gg}
Part 2	12×10^{-3}	0.067	53.3×10^{-6}	$10 \text{ x } 10^{-6}$	63.3×10^{-6}

Part 1 $I_{gg} = I + Ah^2 = 90 \times 10^{-6} + (6 \times 10^{-3} \times 0.133^2) = 196 \times 10^{-6}$

Total $I_{gg} = 63.3 \times 10^{-6} + 196 \times 10^{-6} = 259.3 \times 10^{-6} \text{ m}^4$

At level (1) $\sigma_s = My/I_{gg} = 50\ 000 \times 0.283/ 259.3 \times 10^{-6} = 54.57$ MPa

At level (2) $\sigma_s = My/I_{gg} = 50\ 000 \times 0.017/ 259.3 \times 10^{-6} = 3.28$ MPa

At level (2) $\sigma_c = 3.28 \times E_c/E_s = 3.28 \times (20/200) = 0.328$ MPa

At level (3) $\sigma_s = My/I_{gg} = 50\ 000 \times 0.117/\ 259.3 \times 10^{-6} = 22.56$ MPa

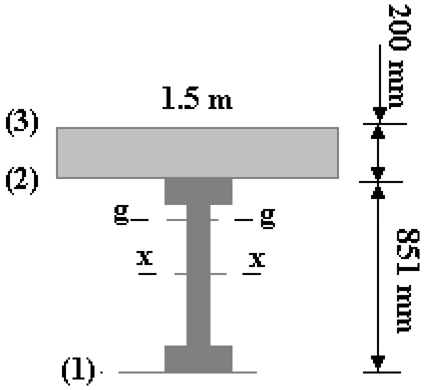
At level (3) $\sigma_c = 22.56 \times E_c/E_s = 22.56 \times (20/200) = 2.256$ MPa

SELF ASSESSMENT EXERCISE No. 3

1. A rectangular section timber beam is 60 mm wide and 100 mm deep. It is clad with steel plate 12 mm thick on the top and bottom. Calculate the maximum stress in the steel and the timber when a moment of 5 kNm is applied.

   E for timber is 11 GPa and for steel 205 GPa.
   (64.5 MPa and 2.8 MPa)

2. A symmetrical steel I section beam has a second moment of area $I_{gg} = 3\ 391.3 \times 10^{-6}$ m^4 and section area 28.84×10^{-3} m^2. It has a vertical depth of 851 mm and forms part of a floor with concrete slabs firmly bonded to the top 1.5 m wide and 1 200 mm thick.



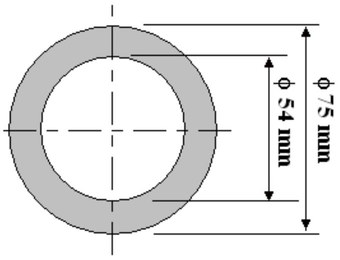
   Calculate the stress in the steel at levels (1) and (2) and in the concrete at levels (2) and (3) when a bending moment of 50 kNm is applied to the section. The top layer is in compression. The modulus of elasticity is 205 GPa or steel and 18GPa for concrete.
   (4.64 MPa tensile, 1.2 MPa compressive, 0.105 MPa compressive and 0.226 MPa compressive)

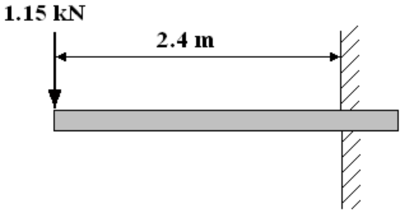
5. *Safety Factor*

It is normal to design any structure so that the stress does not exceed the maximum allowed. This might be the yield stress or the ultimate stress or some other figure. The safety factor is defined as the ratio of the maximum allowable stress to the actual stress.

WORKED EXAMPLE No. 7

A simple cantilever has a single point load at the end as shown and a cross section as shown. The maximum allowable tensile stress is 120 MPa. Calculate the safety factor.





SOLUTION

First it should be fairly obvious that the maximum bending moment in the beam is at the wall and is :

$M = 1150 \times 2.4 = 2760$ N m.

Next calculate the second moment of area.

$$I = \frac{\pi(D^4 - d^4)}{64} = \frac{\pi(75^4 - 54^4)}{64} = 1.136 \times 10^6 \text{mm}^4 \text{ or } 1.136 \times 10^{-6} \text{ m}^4$$

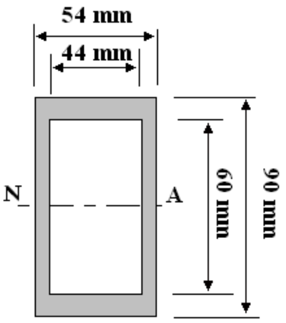
Next calculate the stress.

$$\sigma = \frac{My}{I} = \frac{2760 \times 0.0375}{1.136 \times 10^{-6}} = 91.128 \times 10^6 \text{N/m}^2 \text{ or } 91.128 \text{ MPa}$$

The safety factor is 120/91.128 = 1.317

WORKED EXAMPLE No. 8

A beam is made from a hollow box section as shown. The maximum allowable stress is 450 MPa. Calculate the maximum allowable bending moment about the neutral axis (NA) if a safety factor of 3 is to be used.



SOLUTION

First calculate I

$$I = \frac{BD^3 - bd^3}{12} = \frac{54 \text{ x } 90^3 - 44 \text{ x } 60^3}{12} = 2.489 \times 10^6 \text{mm}^4 \text{or } 2.489 \times 10^{-6} \text{m}^4$$

The working stress is 450/3 = 150 MPa y = 90/2 = 45 mm

Next calculate the bending moment.

$$M = \frac{\sigma I}{y} = \frac{150 \text{ x } 10^6 \times 2.489 \text{ x } 10^{-6}}{0.045} = 8.295 \times 10^3 \text{ N m or } 8.295 \text{ kN m}$$

SELF ASSESSMENT EXERCISE No. 4

1. A vertical pole is made of alloy tube 50 mm outer diameter and 38 mm inner diameter. It is placed in a ground socket and is free standing. If the yield stress is 550 MPa determine the bending moment that will make it fail. If the safety factor to be used is 2.0 determine the maximum bending moment allowed. (4.5 kN m and 2.225 kN m)

2. A beam is made from a hollow box section 50 mm × 75 mm.
   The inside dimensions are 40 mm × 55 mm.
   The maximum allowable stress is 300 MPa.
   Calculate the maximum allowable bending moment about the neutral axis parallel with the shorter edge. A safety factor of 2 is to be used. The beam is symmetrical about the centre line. (4.813 kN m)