

5 A fluid of density ρ flows at constant pressure along a flat plate. The velocity u , at a distance y from the plate, within the boundary layer is

$$\frac{u}{u_1} = 2\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2$$

where u_1 is the main stream velocity and δ is the boundary layer thickness.

(a) Define the terms displacement thickness δ^* and momentum thickness θ and show that $\theta = 2\delta/15$

(b) Explain, in outline only, the derivation of the following equation for the shear stress τ_o on the plate

$$\tau_o = \rho u_1^2 \left(\frac{d\theta}{dx} \right) \text{ where } x \text{ is the distance along the plate from its leading edge.}$$

(c) From the above relationships, show that

$$\frac{\delta}{x} = \left(\frac{30\mu}{\rho u_1 x} \right)^{0.5}$$

a) The flow rate within a boundary layer is less than that for a uniform flow of velocity u_1 . The reduction in flow is equal to the area under the curve in fig.2.3. If we had a uniform flow equal to that in the boundary layer, the surface would have to be displaced a distance δ^* in order to produce the reduction. This distance is called the displacement thickness and it is given by :

$$\text{flow reduction} = \int_0^{\delta} [u_1 - u] dy = u_1 \delta^*$$

If this is equivalent to a flow of velocity u_1 in a layer δ^* thick then :

$$\delta^* = \int_0^{\delta} \left[1 - \frac{u}{u_1} \right] dy$$

The momentum in a flow with a BL present is less than that in a uniform flow of the same thickness. The momentum in a uniform layer at velocity u_1 and height h is $\rho u_1^2 h$. When a BL exists this is reduced by $\rho u_1^2 \theta$. Where θ is the thickness of the uniform layer that contains the equivalent to the reduction. Using the same reasoning as before we get :

$$\theta = \int_0^{\delta} \left[\frac{u}{u_1} \right] \left[1 - \frac{u}{u_1} \right] dy$$

$$\frac{u}{u_1} = 2\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2 \quad \theta = \int_0^{\delta} \left[\frac{u}{u_1} \right] \left[1 - \frac{u}{u_1} \right] dy \quad \text{substitute } \eta = y/\delta \quad \delta d\eta = dy$$

$$\frac{u}{u_1} = 2\eta - \eta^2 \quad \theta = \delta \int_0^{\delta} [2\eta - \eta^2] [1 - 2\eta + \eta^2] d\eta \quad \theta = \delta \int_0^{\delta} [2\eta - 5\eta^2 + 4\eta^3 - \eta^4] d\eta$$

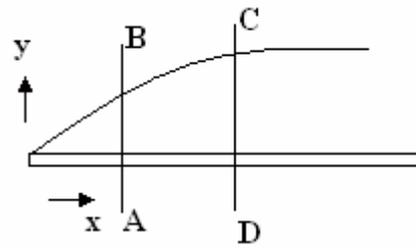
$$\theta = \delta \left[\eta^2 - \frac{5\eta^3}{3} + \eta^4 - \frac{\eta^5}{5} \right]_0^{\delta} = \delta \left[\left(\frac{y}{\delta}\right)^2 - \frac{5}{3}\left(\frac{y}{\delta}\right)^3 + \left(\frac{y}{\delta}\right)^4 - \frac{1}{5}\left(\frac{y}{\delta}\right)^5 \right]_0^{\delta}$$

$$\theta = \delta \left[(1)^2 - \frac{5}{3}(1)^3 + (1)^4 - \frac{1}{5}(1)^5 \right] - \delta [0^5] = \frac{2}{15} \delta$$

b) By considering the momentum and mass entering across BC it can be shown that

$$\tau_o = \rho u_1^2 \left(\frac{d\theta}{dx} \right)$$

$$C_f = \frac{2\tau_w}{\rho u^2} = 2 \frac{d\theta}{dx}$$



the

$$c) \frac{d\theta}{dx} = \frac{2}{15} \frac{d\delta}{dx} \quad \tau_o = \rho u_1^2 \frac{2}{15} \frac{d\delta}{dx}$$

$$\tau = \mu \frac{du}{dy} \quad \tau_w = \mu \left(\frac{du}{dy} \right)_{y=0}$$

$$u = u_1 \left[2 \frac{y}{\delta} - \left(\frac{y}{\delta} \right)^2 \right] \quad \frac{du}{dy} = u_1 \left[2\eta - (\eta)^2 \right] \text{ at } y = 0$$

$$\text{At } y = 0 \quad \frac{du}{dy} = \frac{2u_1}{\delta} \quad \tau_o = \mu u_1 \frac{2}{\delta}$$

$$\tau_o = \mu u_1^2 \frac{2}{15} \frac{d\delta}{dx} = \frac{\mu u_1^2}{\delta}$$

$$\delta d\delta = \frac{2\mu u_1}{\rho u_1^2} \frac{15}{2} dx$$

$$\frac{\delta^2}{2} = \frac{15\mu}{\rho u_1} x + C \quad \text{but at } x = 0, \delta = 0 \text{ so } C = 0$$

$$\delta^2 = \frac{30\mu}{\rho u_1} x$$

$$\delta^2 = \frac{30\mu}{\rho u_1} \frac{x^2}{x}$$

$$\frac{\delta^2}{x^2} = \frac{30\mu}{\rho u_1 x}$$

$$\frac{\delta}{x} = \left(\frac{30\mu}{\rho u_1 x} \right)^{0.5}$$