This tutorial is of interest to any student studying control systems and in particular the EC module D227 – Control System Engineering.

On completion of this tutorial, you should be able to do the following.

- Define a Laplace Transform.
- Transform some common functions of time.
- Use a table of transforms to solve problems.
- Conduct inverse transforms.
- Define the Fourier Transform.
- Transform equations into complex numbers.

Students should familiarise themselves with the tutorial on complex numbers.
1. **INTRODUCTION**

The Laplace transform is a method of changing a differential equation (usually for a variable that is a function of time) into an algebraic equation which can then be manipulated by normal algebraic rules and then converted back into a differential equation by inverse transforms.

This tutorial does not explain the proof of the transform, only how to do it.

2. **THE LAPLACE TRANSFORM**

The Laplace transform is used to convert various functions of time into a function of s. The Laplace transform of any function is shown by putting \( \mathbf{L} \) in front. Hence \( \mathbf{L} f(t) \) becomes \( f(s) \).

The transformation is achieved by solving the equation

\[
\mathbf{L} f(t) = f(s) = \int_0^\infty e^{-st} f(t) \, dt = f(s)
\]

The limits of integration for time is between 0 and \( t \) and for s it is between 0 and \( \infty \).

The first and possibly most difficult task is to find the Laplace transform of \( \frac{d\theta}{dt} \) where \( \theta \) is itself a function of time.

The reasons for this will become clear later.

\[
\mathbf{L} \frac{d\theta}{dt} = f(s) = \int_0^\infty e^{-st} \frac{d\theta}{dt} \, dt \text{ this can be done by integrating by parts.}
\]

\[
\int u \frac{dv}{dt} \, dt = uv - \int v \frac{du}{dt} \, dt \quad \text{so making } u = e^{-st} \text{ then } \frac{du}{dt} = -se^{-st} \text{ and making } \frac{dv}{dt} = \frac{d\theta}{dt} \text{ then } v = \theta
\]

Substituting and putting in the limits we have:

\[
\mathbf{L} \frac{d\theta}{dt} = f(s) = \left[ e^{-st} \theta \right]_0^\infty - \int_0^\infty \left( -se^{-st} \right) \, dt
\]

\[
\mathbf{L} \frac{d\theta}{dt} = f(s) = \left[ e^{-s\theta} - e^{-0\theta} \right]_0^\infty - \int_0^\infty \theta \left( -se^{-st} \right) \, dt
\]

\[
\mathbf{L} \frac{d\theta}{dt} = f(s) = \left[ 0 - e^{-0\theta} \right]_0^\infty - \int_0^\infty \theta \left( -se^{-st} \right) \, dt
\]

\[
\mathbf{L} \frac{d\theta}{dt} = f(s) = \left. \theta \right|_0^\infty - \int_0^\infty \theta \left( -se^{-st} \right) \, dt \quad \text{if } \theta \text{ is } 0 \text{ at } t = 0 \text{ then}
\]

\[
\mathbf{L} \frac{d\theta}{dt} = f(s) = -\int_0^\infty \theta \left( -se^{-st} \right) \, dt
\]

\[
\mathbf{L} \frac{d\theta}{dt} = f(s) = s \int_0^\infty \theta e^{-st} \, dt = s \mathbf{L} \theta
\]

The Laplace transform of the first derivative of \( \theta \) is the Laplace transform of \( \theta \) multiplied by the operator \( s \). It can be shown that the Laplace transform of the \( n \)th derivative of \( \theta \) is \( s^n \mathbf{L} \theta \).
3. **THE s OPERATOR**

The previous derivation leads us to conclude that a quick way of doing the transform of a derivative is to replace \( \frac{d}{dt} \) with the operator \( s \) so \( L(\frac{d\theta}{dt}) \) becomes \( s\theta \). By a similar derivation the Laplace transform of the \( n \)th derivative \( \frac{d^n\theta}{dt^n} \) becomes \( s^n\theta \). The transform of \( \theta \) on its own becomes \( s\theta \) and the integral of \( \theta \) becomes \( \frac{1}{s}\theta \) and so on. Here is a table that should make this clear:

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \int \theta dt )</th>
<th>( \int \theta dt )</th>
<th>( \theta )</th>
<th>( \frac{d\theta}{dt} )</th>
<th>( \frac{d^2\theta}{dt^2} )</th>
<th>( \frac{d^n\theta}{dt^n} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0 )</td>
<td>( s\theta )</td>
<td>( s\theta )</td>
<td>( s\theta )</td>
<td>( s\theta )</td>
<td>( s^2\theta )</td>
<td>( s^n\theta )</td>
</tr>
</tbody>
</table>

Now we should look at how to transform some other functions.

### WORKED EXAMPLE No.1

Find the Laplace transform \( L(H) \) when \( H \) is a constant.

**SOLUTION**

\[
L(H) = \int_{0}^{\infty} e^{-st} f(t) \, dt = \int_{0}^{\infty} e^{-st} H \, dt = H \left[ \frac{-e^{-st}}{s} \right]_{0}^{\infty} = H \left[ \frac{-1}{s} \right] = \frac{H}{s}
\]

For a unit step \( H = 1 \) and the Laplace transform is \( 1/s \)

### WORKED EXAMPLE No.2

Find the Laplace transform of \( e^{-at} \)

**SOLUTION**

\[
L(e^{-at}) = \int_{0}^{\infty} f(t) \, dt = \int_{0}^{\infty} e^{-st} e^{-at} \, dt = \int_{0}^{\infty} e^{-(s+a)t} \, dt = \left[ \frac{e^{-(s+a)t}}{s+a} \right]_{0}^{\infty} = \left[ \frac{0}{s+a} - \frac{1}{s+a} \right] = -\frac{1}{s+a}
\]

\[
L(e^{-at}) = f(s) = \frac{1}{s+a}
\]
SELF ASSESSMENT EXERCISE No.1

1. Find the Laplace transform for \( f(t) = ct \) and check your answer against the table.

2. Find the Laplace Transform of \( f(t) = 1 + 3e^{-at} \). (Answer \( 1/s + 3/(s+a) \))

3. Change the following differential equations into Laplace form.
   
   i. \( T \frac{d\theta}{dt} + \theta \)  
   (Answer \( \theta \{Ts + 1\} \))

   ii. \( T^2 \frac{d^2\theta}{dt^2} + 2\delta T \frac{d\theta}{dt} + \theta \)  
   (Answer \( \theta \{T^2s^2 + 2\delta Ts + 1\} \))

4. Using the table on the next page, find the Laplace Transform of the following time functions.
   
   i. \( k \sin(\omega t) \)

   ii. \( k \{ 1 - e^{-t/T} \} \)

4. INVERSE TRANSFORMS

Inverse transforms are simply the reverse process whereby a function of ‘s’ is converted back into a function of time. For example the reverse transform of \( k/s \) is \( k \) and of \( k/s^2 \) is \( kt \).

SELF ASSESSMENT EXERCISE No.2

Find the inverse transform of the following.

i. \( \frac{k}{s^2(Ts + 1)} \)

ii. \( \frac{k \omega^2}{s(s^2 + \omega^2)} \)
5. **TABLE OF COMMON LAPLACE TRANSFORMS**

Note that the use of the letters for constants is arbitrary and that often the solution is found by interchanging ‘a’ with ‘1/T’ and ‘ω’. H and k are arbitrary constants.

<table>
<thead>
<tr>
<th>Time domain f(t)</th>
<th>Frequency domain f(s)</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 e^{-at} f(t) dt</td>
<td>f(s + a)</td>
<td>Unit Impulse</td>
</tr>
<tr>
<td>2 δ</td>
<td>1</td>
<td>Step H</td>
</tr>
<tr>
<td>3 H</td>
<td>H/s</td>
<td>Ramp</td>
</tr>
<tr>
<td>4 ct</td>
<td>c/s^2</td>
<td>Delayed Step</td>
</tr>
<tr>
<td>5 H(t - T)</td>
<td>H e^{-sT}</td>
<td>Rectangular pulse</td>
</tr>
<tr>
<td>6 H(t - T)</td>
<td>1 - e^{-sT}</td>
<td>Sinusoidal</td>
</tr>
<tr>
<td>7 k e^{-at}</td>
<td>k/(s + a)</td>
<td>Exponential</td>
</tr>
<tr>
<td>8 kt e^{-at}</td>
<td>k/(s + a)^2</td>
<td>Co sinusoidal</td>
</tr>
<tr>
<td>9 K(e^{-at} - e^{-bt})</td>
<td>k(b - a)/(s + a)(s + b)</td>
<td>Damped sinusoidal</td>
</tr>
<tr>
<td>10 k sin (ωt)</td>
<td>kω/(s^2 + ω^2)</td>
<td>Damped cosinusoidal</td>
</tr>
<tr>
<td>11 k cos (ωt)</td>
<td>ks/(s^2 + ω^2)</td>
<td>Exponential growth</td>
</tr>
<tr>
<td>12 k e^{-at} sin (ωt)</td>
<td>kω/(s + a)^2 + ω^2</td>
<td></td>
</tr>
<tr>
<td>13 k e^{-at} cos (ωt)</td>
<td>s + a/(s + a)^2 + ω^2</td>
<td></td>
</tr>
<tr>
<td>14 k\left{1 - e^{-\frac{1}{T}}\right}</td>
<td>ka/(s(s + a))</td>
<td></td>
</tr>
<tr>
<td>15 k\left{t - \left(1 - e^{-\frac{1}{T}}\right)\right}</td>
<td>ka/s^2(s + a)</td>
<td></td>
</tr>
<tr>
<td>16 k(1-cosωt)</td>
<td>kω/s(s^2 + ω^2)</td>
<td></td>
</tr>
<tr>
<td>17 k sin (ωt + ϕ)</td>
<td>k{ω cos ϕ + s sin ϕ}/s^2 + ω^2</td>
<td></td>
</tr>
</tbody>
</table>
Suppose \( x = Ae^{j\omega t} \) then \( \frac{dx}{dt} = j\omega Ae^{j\omega t} \). In Laplace form \( s x = j\omega Ae^{j\omega t} = j\omega x \).

It seems that \( s x = j\omega x \), so the operator \( s \) is the same as \( j\omega \) and this substitution is the Fourier Transform. \( j \) is the complex operator \( j = \sqrt{-1} \). When this transform is done, \( G(s) \) is changed into \( G(j\omega) \).

**WORKED EXAMPLE No.3**

Transform \( G(s) = Ts + 1 \) into a complex number. Given \( T = 0.4 \) seconds, evaluate the magnitude of \( G(s) \) when \( \omega = 15 \) rad/s.

**SOLUTION**

Substitute \( s = j\omega \) and \( G(s) = Ts + 1 \) becomes \( G(j\omega) = j\omega T + 1 \) or \( G(j\omega) = 1 + j\omega T \). The usual form for a complex number is \( A + jB \). The complex number may be represented on an Argand Diagram as shown.

The magnitude is \( \sqrt{((\omega T)^2 + (1)^2)} = \sqrt{(15 \times 0.4)^2 + 1^2} = \sqrt{36 + 1} = 6.08 \).

**WORKED EXAMPLE No.4**

Transform \( G(s) = T^2s^2 + 2\delta Ts + 1 \) into a complex number. Given \( T = 0.4 \) seconds, evaluate the magnitude of \( G(s) \) when \( \omega = 15 \) rad/s and \( \delta = 0.5 \).

**SOLUTION**

Substitute \( s = j\omega \) and \( G(s) = (T^2s^2 + 2\delta Ts + 1) \) becomes
\[
G(j\omega) = T^2(j\omega)^2 + 2\delta T(j\omega) + 1 = -T^2\omega^2 + j2\delta T\omega + 1 = (1 - T^2\omega^2) + j2\delta T\omega
\]

The usual form for a complex number is \( A + jB \). The complex number may be represented on an Argand Diagram as shown.

The magnitude is \( \sqrt{((1 - T^2\omega^2)^2 + (2\delta T\omega)^2)} = \sqrt{(1 - 0.4^2 \times 15^2)^2 + (2 \times 0.5 \times 0.4 \times 15)^2} \)

The magnitude = \( \sqrt{(-35^2 + 6^2)} = \sqrt{(-35^2 + 6^2)} = 35.51 \)
WORKED EXAMPLE No.5

Convert $G(s) = 1/(T^2s^2 + 2\delta Ts + 1)$ into a complex number of the form $A + jB$

SOLUTION

\[
G(s) = \frac{1}{T^2s^2 + 2\delta Ts + 1}
\]

\[
G(j\omega) = \frac{1}{T^2(j\omega)^2 + 2\delta j\omega + 1}
\]

\[
G(j\omega) = \frac{1}{(1 - T^2\omega^2) + 2\delta j\omega}
\]

\[
G(j\omega) = \frac{1}{A + jB}
\]

where $A = (1 - T^2\omega^2)$ and $B = j2\delta\omega$

Multiply the top and bottom line by the conjugate number $A - jB$

\[
G(j\omega) = \frac{A - jB}{(A + jB)(A - jB)}
\]

\[
G(j\omega) = \frac{A - jB}{A^2 + B^2} = C - jD
\]

\[
C = \frac{A}{A^2 + B^2} = \frac{1 - \omega^2T^2}{(1 - \omega^2T^2) + (2\delta \omega T)^2}
\]

\[
D = \frac{B}{A^2 + B^2} = \frac{2\delta T\omega}{(1 - T^2\omega^2)^2 + (2\delta \omega T)^2}
\]

SELF ASSESSMENT EXERCISE No.3

Convert $G(s) = 1/(Ts + 1)$ into a complex number of the form $A + jB$.

The solution is shown on the next page.
SOLUTION TO S.A.E. 3

\[ G(s) = \frac{1}{Ts + 1} \]

For a sinusoidal response replace \( s \) with \( j\omega \)

\[ G(j\omega) = \frac{1}{j\omega T + 1} \]

This is converted into a complex number by multiplying the top and bottom by the conjugate number.

\[ G(j\omega) = \frac{(1 - j\omega T)}{(1 + j\omega T)(1 - j\omega T)} = \frac{(1 - j\omega T)}{(1 - \omega^2 T^2)} = \frac{1}{1 - \omega^2 T^2} - \frac{j\omega T}{1 - \omega^2 T^2} \]

\[ A = \frac{1}{1 + \omega^2 T^2} \]

\[ B = \frac{\omega T}{1 + \omega^2 T^2} \]