This tutorial is essential pre-requisite material for anyone studying mechanical engineering. This tutorial uses the principle of learning by example. The approach is practical rather than purely mathematical and may be too simple for those who prefer pure maths.

Calculus is usually divided up into two parts, integration and differentiation. Each is the reverse process of the other. It is easier to explain integration first and a numerical approach helps explain what it is about.

On completion of this tutorial you should be able to do the following.

- Explain and use basic numerical integration.
- Apply Newton’s rules of integration to basic functions.
- Integrate between limits to find areas under graphs.
- Use integration to evaluate common engineering quantities.
- Apply the idea of integration to explain moments of area and inertia.
1. **NUMERICAL INTEGRATION**

The basic idea of the integral calculus is quite simple. All things are the sum of the individual parts and there is no limit to how small the parts can be.

In engineering we use the symbol $\Delta$ (capital delta) to mean “a change of” or a “part of” something so for example $\Delta T$ means a change in temperature and $\Delta t$ means a change of time and the change is significant.

The symbol $\delta$ (small delta) means a small but finite change so $\delta T$ and $\delta t$ means a very small change in temperature and time respectively.

In calculus you will find that we deal with changes so small that they approach zero and the symbol $d$ is used so $dT$ and $dt$ means an infinitesimal change in temperature and time respectively. These are also called the differential coefficients.

Let’s explain this with a simple example. A sheet of paper with dimensions $B$ and $D$ may be divided it up into many thin strips as shown. Each strip has a tiny area called $\delta A$. We call these elementary strips.

![Figure 1](image_url)

The distance from the bottom of the sheet is $y$. The height of the strip represents a small increase in the distance so we denote it $\delta y$. The area is hence $\delta A = B \delta y$. If we choose to put a small value to $\delta y$ we could work out $\delta A$.

Now consider that the sheet is made up of many strips. To find the total area we simply sum all up. We can write $A = \Sigma \delta A$ where the symbol $\Sigma$ (capital sigma) is used to mean “the sum of”. Since $\delta A = B \delta y$ we can write $A = \Sigma B \delta y$. In this case $B$ is the same for all strips so $A = B \Sigma \delta y$.

The length of the sheet is $D$ and this is found by adding all the values of $\delta y$. Clearly $\Sigma \delta y = D$ and the area of the sheet is $A = B D$.

If the value of $\delta y$ is the same for each strip and there are $n$ strips, we could find the area by multiplying. $A = n B \delta y$. This process is called Numerical Integration.

The example was very trivial but now consider what would happen if we try to apply the method to a more complicated shape such as right angle triangle.
The area of the elementary strip $\delta A = b \delta y$. The problem is that the width $b$ will be different for every strip and that it gets smaller as the distance $y$ increases. Suppose we know the height ($H$) and the angle ($\theta$). A bit of trigonometry tells us that:

$$B = \frac{H}{\tan(\theta)}$$

Suppose the height of each strip is $\delta y$ and that we will divide the area up into $n$ strips starting from the bottom.

The height of each strip must be $\delta y = \frac{H}{n}$

The area of each strip is $\delta A = b \delta y$.

The width of the strip $b$ is again found with trigonometry.

$$b = \frac{H - y}{\tan(\theta)}$$

The width of the $m^{th}$ strip would be $b = \frac{H - m \delta y}{\tan(\theta)}$ where $m$ is a number between 1 and $n$.

Substitute for $y$ and it follows that $\delta A = \frac{H - m \delta y}{\tan(\theta)} \delta y$

The total area is the sum of the strips

$$A = \sum_{m=1}^{n} \frac{H - m \delta y}{\tan(\theta)} \delta y$$

In order to actually evaluate the area, you would have to calculate the area of every elementary strip and add them up. The thinner the strips, the more there will be and the more calculations you have to do. The accuracy of the answer improves also. With the advent of computers, solving by doing many calculations has become easy and numerical integration has become a reality. We would write out our numerical integration as follows to show that we add up all the strips from the first to the $n^{th}$.

$$A = \sum_{m=1}^{n} \frac{H - m \delta y}{\tan(\theta)} \delta y$$

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The following shows how to enter this in MATHCAD®. Do this for an exercise if you have the software and try changing the number of samples, the height and the angle. See how it affects the accuracy.

Find the area of a right angle triangle with a height of 100 mm and angle 30 degrees.

From algebra we know \( B = \frac{H}{\tan 30} = 173.205 \) \( A = \frac{1}{2} BH = 866.25 \text{ mm}^2 \)

MATHCAD PROGRAMME FOR FINDING THE AREA OF A RIGHT ANGLE TRIANGLE BY NUMERICAL INTEGRATION.

First try with ten strips. The first line is the data.

\[
\begin{align*}
H &:= 100 \\
n &:= 10 \\
\delta y &:= \frac{H}{n} \\
\delta y &= 10 \\
\theta &:= 30\text{-deg}
\end{align*}
\]

The next line is the evaluation.

\[
\begin{align*}
B &:= \frac{H}{\tan(\theta)} \\
B &= 173.205 \\
A &:= \sum_{m=1}^{n} \frac{(H - m \delta y) \cdot \delta y}{\tan(\theta)} \\
A &= 7.794 \cdot 10^{3}
\end{align*}
\]

Now change to 1000 strips.

\[
\begin{align*}
H &:= 100 \\
n &:= 1000 \\
\delta y &:= \frac{H}{n} \\
\delta y &= 0.1 \\
\theta &:= 30\text{-deg}
\end{align*}
\]

\[
\begin{align*}
B &:= \frac{H}{\tan(\theta)} \\
B &= 173.205 \\
A &:= \sum_{m=1}^{n} \frac{(H - m \delta y) \cdot \delta y}{\tan(\theta)} \\
A &= 8.652 \cdot 10^{3}
\end{align*}
\]

A thousand strips give an accurate answer but I think you will agree that you would not want to work out the area of 1000 strips on a calculator and then add them up. The computer makes it easy.
2. **NEWTON’S METHOD**

Newton worked out a way of doing integration accurately without going through the laborious method of physically summing all the elements. If the elementary part can be written as a function of the variable, (an equation), then the total may be found by using his method.

Consider the simple function \( y = f(x) = x^2 \). A graph of this function looks like this.

![Graph of function y = x^2](image)

Suppose we wish to find the area under the graph. First we draw an elementary strip as shown. The height of the strip is \( y \) at any given point (this is a variable with \( x \)). The width of the strip is \( \delta x \). Over the range \( x = 0 \) to \( x = 10 \) the area is the sum of the strips that make up the total.

\[
\sum_{x=0}^{x=10} y \delta x = \sum_{x=0}^{x=10} x^2 \delta x
\]

In order to get an exact answer we must reduce the value of \( \delta x \) to an extremely small value and use millions of strips. To indicate that the size is now infinitesimally small we change from \( \delta \) (meaning a finite value) to \( d \) (meaning an infinitesimally small value). The answer will be perfectly correct when the small value \( dx \) is so small that it approaches the limit of zero. We say \( dx \) tends to zero and write it as \( dx \to 0 \).

The area is now expressed as follows.

\[
\int_{x=0}^{x=10} x^2 \, dx
\]

The elongated S is the integral sign and the limits of the integration are shown.

**THE RULE**

Newton worked out the rule that if we add one to the power of \( x \) and divide by the new power, we will get the exact answer. Carrying out the integration using this rule gives the following.

\[
A = \left[ \frac{x^3}{3} \right]_{x=0}^{x=10} = \frac{10^3}{3} - \frac{0^3}{3} = \frac{1000}{3}
\]

This is called integration between limits. Note the use of square brackets to indicate that the integration has been done but not evaluated.
In order to evaluate the area between \( x = 0 \) and \( x = 10 \) we do the following.

\[
A = \int_{x=0}^{x=10} x^2 \, dx = \left[ \frac{x^3}{3} \right]_0^{10} = \left[ \frac{10^3}{3} - \frac{0^3}{3} \right] = \frac{1000}{3} = 333.33
\]

If we wanted the area between \( x = 2 \) and \( x = 5 \) we would do it as follows.

\[
A = \int_{x=2}^{x=5} x^2 \, dx = \left[ \frac{x^3}{3} \right]_2^{5} = \left[ \frac{5^3}{3} - \frac{2^3}{3} \right] = \frac{1}{3} [125 - 8] = 39
\]

Note the order in which we evaluate the bracket: the last limit is evaluated first and then the second and this is subtracted from the first.

The rule for integrating powers may be written as follows.

\[
\int_{x^1}^{x^2} a x^n \, dx = a \left[ \frac{x^{n+1}}{n+1} \right]_{x^1}^{x^2}
\]

Note that no power shown against a variable (e.g. \( x \)), means \( x^1 \) and that this integrates as \( x^{3/2} \).

Anything raised to the power of zero is 1 so a number on its own integrates e.g. 2 could be written as \( 2x^0 \) and this integrates to \( 2x^1/1 = 2x \).

WORKED EXAMPLE No.1

Find the area under the graph of \( y = 2x^3 \) between the limits \( x = 4 \) and \( x = 6 \).

SOLUTION

The graph looks like this. The area of the elementary strip is \( y \, dx \)

\[
A = \int_{x=4}^{x=6} y \, dx = \int_{x=4}^{x=6} 2x^3 \, dx = 2 \left[ \frac{x^4}{4} \right]_4^6 = 2 \left[ \frac{6^4}{4} - \frac{4^4}{4} \right] = \frac{2}{4} [1296 - 256] = 520
\]
SELF ASSESSMENT EXERCISE No.1

1. Integrate the following without evaluating.
   
   i. \( 3x^6 \)

   ii. \( 5x \)

   iii. \( 6 \)

2. Find the area under the graph \( y = 5x^4 \) between the limits of \( x = 1 \) and \( x = 3 \).
   (Answer 242)

3. Find the area under the graph \( y = 1.5 \cdot x^{\frac{1}{6}} \) between the limits \( x = 2 \) and \( x = 4 \).
   (Answer 5.172)

4. Find the area under the graph \( y = 2x^3 \) between the limits 0 and 10
   (Answer 5000)

5. Find the area under the graph \( y = 12x \) between the limits 2 and 5
   (Answer 126)
In Engineering, graphs represent many things and the area under the graph represents real things. For example the area under a force – distance graph represents the work done or energy used.

**WORKED EXAMPLE No.2**

The force exerted on a mechanism is related to the distance moved by the equation: \( F = 2x^{1.5} \).

Determine the work done during the first 0.2 m of movement.

**SOLUTION**

\[
W = \int_{x=0}^{x=0.2} F \, dx = \int_{x=0}^{x=0.2} 2x^{1.5} \, dx = 2 \left[ \frac{x^{2.5}}{2.5} \right]_{0}^{0.2} = 2 \left[ \frac{0.2^{2.5}}{2.5} - 0 \right] = 0.0143 \text{ J}
\]

**SELF ASSESSMENT EXERCISE No.2**

1. The Energy stored in a capacitor is the area under the V – Q graph. The voltage over a given capacitor varies with charge stored by the law \( V = 2Q \). Calculate the energy stored when the charge reaches 10 Coulombs.
   (Answer 100 Joules)

2. The total charge passed into a battery is represented by the area under a graph of current against time by the law \( I = 2t^2 \). Determine the charge after 5 seconds.
   (Answer 83.3 Coulombs)

3. The velocity of a falling object is related to time by the law \( v = 9.81t \). The distance fallen is the area under the velocity – time graph. Calculate the distance fallen 5 seconds after release.
   (Answer 122.6 m)

4. The work done by a force \( F \) Newton moving a distance \( x \) metres is the area under the \( F – x \) graph. If the force is related to distance by the law \( F = 50x^{1.2} \) find the work done when it has moved a distance of 0.3 m.
   (Answer 1.61 Joules)
3. INTEGRATING SERIES EQUATIONS

Consider the integration \( y = \int (2x^3 + 4x^2 - 3x) \, dx \). This may be integrated by doing each part separately. The result is \( y = 2x^4/4 + 4x^3/3 - 3x^2/2 \).

**WORKED EXAMPLE No.3**

Evaluate \( F(x) = \int \left( \frac{4}{2} (2x^3 - 3x) \right) dx \)

\[
F(x) = \frac{4}{2} \int (2x^3 - 3x) \, dx = \left[ \frac{2x^4}{4} - \frac{3x^2}{2} \right] - \left[ \frac{2x^4}{4} - \frac{3x^2}{2} \right],
\]

\[
F(x) = \left[ (128 - 24) - (8 - 6) \right],
\]

\[
F(x) = 104 - 2 = 102
\]

**SELF ASSESSMENT EXERCISE No.3**

1. Integrate the following expressions.
   i. \( y = 3x^3 - x^2/2 \)
   ii. \( y = x^3 - x/2 \)

2. Evaluate the following.
   i. \( \int_0^4 (2x^2 - 3x) \, dx \) \hspace{1cm} (Answer 18.67)
   ii. \( \int_1^5 (4x^3 - 3x^2) \, dx \) \hspace{1cm} (Answer 500)
4. INTEGRATION WITHOUT LIMITS

The work covered so far shows how to do integration between limits. If the limits are not known then the answer to the integration must have a constant added and this can sometimes be evaluated. The reason for this becomes clear when you study differentiation in which a constant disappears. Integration being the reverse process, a constant reappears but may be zero. Integrating a power series without limits should be done as follows.

\[ \int ax^n = a \left[ \frac{x^{n+1}}{n+1} \right] + C \]

5. OTHER FORMS OF INTEGRALS

You have so far learned how to integrate power laws of the form \( y = ax^n \). There is a special case of this where the rule does not work. Integrating \( x^{-1} \) does not produce an answer. It can be shown that in this case the result is \( \ln x \).

Other functions can be integrated and a list is given below.

\[
\begin{align*}
\int x^{-1} \, dx &= \ln x + C \\
\int e^{ax} \, dx &= \frac{1}{a} e^{ax} + C \\
\int \sin ax \, dx &= -\frac{1}{a} \cos x + C \\
\int \cos ax \, dx &= \frac{1}{a} \sin x + C \\
\int \tan x \, dx &= -\ln x + C \\
\int \sin^2 x \, dx &= \frac{1}{2} x - \frac{1}{4} \sin 2x + C \\
\int \cos^2 x \, dx &= \frac{1}{2} x + \frac{1}{4} \sin 2x + C \\
\int \tan^2 x \, dx &= \tan x - x + C \\
\int \ln x \, dx &= x \ln x - x + C
\end{align*}
\]

Remember that if you are integrating between limits, you do not include the constant.
**WORKED EXAMPLE No.4**

Evaluate \( A = \int_0^\pi \sin x \, dx \)

**SOLUTION**

From the list of standard integrals we see \( \int \sin x = -\cos x \) so:

\[
A = \left[ -\cos x \right]_0^\pi = [-\cos \pi] - [-\cos 0] = -(1) - (-1) = 2
\]

**WORKED EXAMPLE No.5**

Evaluate \( A = \int_0^3 e^{2x} \, dx \)

**SOLUTION**

From the list of standard integrals we see \( \int e^{ax} = \frac{e^{ax}}{a} \) so:

\[
A = \left[ \frac{e^{2x}}{2} \right]_0^3 = \left[ \frac{e^4}{2} \right] - \left[ \frac{e^0}{2} \right] = 27.3 - 0.5 = 26.8
\]

**WORKED EXAMPLE No.6**

Evaluate \( A = \int_0^{\pi/2} \sin^2 x \, dx \)

**SOLUTION**

From the list of standard integrals we see \( \int \sin^2 x = \frac{1}{2} x - \frac{1}{4} \sin 2x \) so:

\[
A = \int_0^{\pi/2} \sin^2 x \, dx = \left[ \frac{x}{2} - \frac{\sin 2x}{4} \right]_0^{\pi/2} = \left[ \left( \frac{\pi}{4} - \frac{\sin \pi}{4} \right) - \left( \frac{0}{2} - \frac{\sin 0}{4} \right) \right] = [0.785 - 0] - [0 - 0] = 0.785
\]
SELF ASSESSMENT EXERCISE No.4

Solve the following integrals. All angles are in radian.

1. \[ W = 220 \int_{0.1}^{0.2} V^{-1} dV \] (Answer 152.49)

2. \[ A = 2 \int_{0}^{\pi} \sin(\theta) d\theta \] (Answer 4)

3. \[ A = \int_{0.5}^{1.5} \cos(\theta) d\theta \] (Answer 1.036)

4. \[ A = \int_{0}^{1} \sin^2(x) dx \] (Answer 0.545)
6. ENGINEERING CONCEPTS

Calculus is used to produce formulae for various engineering concepts. Some of these are explained in the following.

6.1 SECOND MOMENTS OF AREA

Second moment of area is an important mathematical concept used in beam theory and hydrostatics. Consider an area and an elementary strip as shown.

![Figure 6](image)

The area of the strip = \( \delta A = b \delta y \)
1st moment of area of strip = \( y \delta A = by \delta y \)
2nd moment of area of strip = \( y^2 \delta A = b y^2 \delta y \)

For the whole area, the 2nd moment of area is the sum of all the strips that make up the total area. This is found by integration, hence in the limit \( \delta y \to dy \to 0 \)

\[
I = \int b y^2 \, dy
\]

The limits of integration are from the bottom to the top of the area. This definition is important because in future work, when ever this expression is found, we may identify it and replace it with standard formula for I. We should now look at these.
WORKED EXAMPLE No. 7

Derive the standard formula for the second moment of area and radius of gyration for a rectangle of width \( B \) and depth \( D \) about an axis on its long edge.

**SOLUTION**

![Figure 7](image)

\[
I = \int_0^D dA \cdot y^2 = \int_0^D B \ dy \cdot y^2 \ = \ B \int_0^D y^2 \ dy = B \left[ \frac{y^3}{3} \right]_0^D = \frac{B}{3} [D^3 - 0] = \frac{BD^3}{3}
\]

WORKED EXAMPLE No. 8

Derive the standard formula for the second moment of area and radius of gyration for a rectangle of width \( B \) and depth \( D \) about an axis through its centroid and parallel to the long edge.

**SOLUTION**

![Figure 8](image)

\[
I = \int_{D/2}^{D/2} dA \cdot y^2 = \int_{D/2}^{D/2} B \ dy \cdot y^2 = B \left[ \frac{y^3}{3} \right]_{D/2}^{D/2} = \frac{B}{3} \left[ \left( \frac{D}{2} \right)^3 - \left( -\frac{D}{2} \right)^3 \right]
\]

\[
I = \frac{B}{3} \left[ \frac{D^3}{8} + \frac{D^3}{8} \right] = \frac{B}{3} \left[ \frac{D^3}{4} \right] = \frac{BD^3}{12}
\]
6.2 MOMENT OF INERTIA

Moment of inertia is an engineering concept similar to second moment of area. It is used in problems involving rotating bodies. It is in reality, the second moment of mass. Consider a small mass $\delta m$ rotating on a radius of $r$ metres.

$r \, \delta m =$ first moment of mass  
$r^2 \, \delta m =$ 2nd moment of mass.

Figure 9

This is usually called the moment of inertia. The symbol is $I$. It is an important property of a rotating body. The moment of inertia governs how easy or difficult it is to make a wheel speed up or slow down as it rotates. Unfortunately most rotating bodies do not have the mass concentrated at one radius and the moment of inertia is not calculated as easily as indicated.

Consider a plane disc rotating about its centre.

The disc may be considered as made up of lots of small masses at various radii. The moment of inertia for the whole disc may be found by summing up the individual parts such that

$I = \sum r^2 \, \delta m$

Figure 10

Another approach is to use an effective radius $k$. This is called the radius of gyration. If this is used then

$I = M \, k^2$ where $m$ is the mass of the disc. This may be applied to any wheel or rotor but the problem is finding $K$. For a simple plane disc, it may be shown that $k = 0.707 R$ where $R$ is the outer radius of the disc.

Now consider a thin elementary ring on the face of the disc. The disc is $b$ metres thick.

Length of ring $= 2\pi r$
Area $= 2\pi r \, dr$
Volume $= b2\pi r \, dr$
Mass $= \delta m = \rho b2\pi r \, dr$
Moment of inertia $= dI = \rho b2\pi r \times r^2 \, dr = \rho b2\pi r^3 \, dr$

Figure 11

The total moment of inertia is found by integration which is a way of summing all the rings that make up the disc.

$I = 2 \pi \rho \int_0^R r^3 \, dr = 2 \pi \rho \left[ \frac{r^4}{4} \right]_0^R = 2 \pi \rho \frac{b2\pi}{4} [R^4 - 0] = \frac{\pi \rho b R^4}{2}$

The mass of the disc is $M = \rho \pi b R^2$ Hence $I = MR^2/2$
SELF ASSESSMENT EXERCISE No.5

The last section was aimed at explaining concepts and self assessment is not required here but here are two numerical problems to go at.

1. An annular ring of metal has an outer diameter of 200 mm and an inner diameter of 100 mm and it is 50 mm thick. The density is 7800 kg/m$^3$. Find the moment of inertia of the ring. \(0.057 \text{ kg m}^2\)

2. Find the second moment of area of an annular ring about its diameter. The inner and outer diameters are 200 mm and 100 mm respectively. \((73.65 \times 10^6 \text{ mm}^4)\)