MATHEMATICS FOR ENGINEERING

ADVANCED VECTORS

On completion of this tutorial you should be able to do the following.

• Explain the Scalar or Dot Product.

• Explain the Vector or Cross Product.

• Explain the Triple Scalar Product.

• Explain the Triple Vector Product

The tutorial ‘Vectors’ should be studied before starting on this tutorial. It is presumed that students are already proficient at algebra and trigonometry.
1. VECTOR MULTIPLICATION

Multiplying vectors is not straightforward and there are two different ways of doing it producing two different results called the Scalar Product and Vector Product. Both have their uses. Throughout this work a vector is denoted by bold and underline characters and magnitude or scalar by an ordinary character.

2. SCALAR or DOT PRODUCT

Consider two vectors \( A \) and \( B \) on the \( x – y \) plane as shown. Let’s look at the result of multiplying the \( x – y \) components to give \( (a_x b_x + a_y b_y) \)

The way we write this kind of multiplication is with a dot so we are finding \( A \cdot B = (a_x b_x + a_y b_y) \)

The magnitude of \( A \) is \( \sqrt{a_x^2 + a_y^2} \)

The magnitude of \( B \) is \( \sqrt{b_x^2 + b_y^2} \)

The angle of \( A \) to the x axis is \( \theta_a = \tan^{-1}(a_y/a_x) \)

The angle of \( B \) to the x axis is \( \theta_b = \tan^{-1}(b_y/b_x) \)

The angle between the vectors is \( \theta = \theta_a - \theta_b \)

Without proof

\[ A \cdot B = (a_x b_x + a_y b_y) = A B \cos \theta \]

This form of multiplication has come about by multiplying the vector components in the \( x \) and \( y \) directions and then adding them. The result is more easily calculated by multiplying the magnitudes and the cosine of the angle between them. The result is NOT A VECTOR but a scalar so it has no direction. This may be demonstrated by the following practical example.

Consider a body moving under the action of a force \( F \) as shown. Suppose we wish to know the work done when the body moves a distance ‘s’ in the \( x \) direction. Note that distance ‘s’ and force ‘F’ are both vector quantities.

The component of this acting in the \( x \) direction is \( F \cos \theta \)

The work done in the \( x \) direction is \( W_x = F s \cos \theta \)

Work is a scalar quantity. We write the Dot product as \( F \cdot s \)

\[ W_x = F s \cos \theta = F \cdot s \]

WORKED EXAMPLE No.1

Two vectors have the Cartesian coordinates \( A \ (4, 4) \) and \( B \ (6, 3) \). Find the dot product.

SOLUTION

\[
\begin{align*}
  a_x &= 4 & a_y &= 4 & b_x &= 6 & b_y &= 3 \\
  \theta_a &= \tan^{-1}(a_y/a_x) = 45^\circ \\
  \theta_b &= \tan^{-1}(b_y/b_x) = 26.56^\circ \\
  \theta &= 45^\circ - 26.56^\circ = 18.43^\circ \\
  \text{Magnitudes} & \quad A = \sqrt{4^2 + 4^2} = 5.657 \quad B = \sqrt{6^2 + 3^2} = 6.708 \\
  A \cdot B &= A B \cos \theta = (5.657)(6.708)\cos18.43^\circ = 36 \\
  A \cdot B &= (a_x b_x + a_y b_y) = (4)(6) + (4)(3) = 36
\end{align*}
\]

Both ways gives the result 36 with no direction.
3. PROPERTIES OF THE DOT PRODUCT

The scalar produced by a Dot product is the same no matter which is Dotted with which, so :-

\[ \mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \]

If either vector is multiplied by a scalar \( \lambda \) then it follows

\[ \lambda (\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \cdot \lambda \mathbf{B} = \lambda \mathbf{A} \cdot \mathbf{B} \]

\[ \lambda \mathbf{A} \cdot \mathbf{B} = \lambda \mathbf{A} \mathbf{B} \cos \theta \]

\[ \lambda \mathbf{A} \cdot \mathbf{B} = \lambda \mathbf{A} \mathbf{B} \cos \theta \]

Consider the example again. This time the body is moved by two forces \( F_1 \) and \( F_2 \). The work done in the x direction is:-

\[ W_x = F_1 \cdot s + F_2 \cdot s \]

We get the same result by Dotting the resultant of \( F_1 + F_2 \) with \( s \) so:

\[ (F_1 + F_2) \cdot s = F_1 \cdot s + F_2 \cdot s \]

In general for three vectors we have the rule

\[ (\mathbf{A} + \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot \mathbf{C} + \mathbf{B} \cdot \mathbf{C} \]

Note when \( \theta = 90^\circ \) the result is zero.

The Dot product of a vector \( \mathbf{A} \) with itself is \( \mathbf{A} \cdot \mathbf{A} \cos 0 = A^2 \)

4. EXTENSION TO THREE DIMENSIONS

In three dimensional coordinates the dot product of two vectors is

\[ \mathbf{A} \cdot \mathbf{B} = (a_x b_x + a_y b_y + a_z b_z) = A B \cos \theta \]

From the tutorial on coordinate systems we found the following.

For vector \( \mathbf{A} \) the magnitude is \( A = \sqrt{(a_x^2 + a_y^2 + a_z^2)} \)

The angle to the z – x plane is \( \theta = \tan^{-1} a_y/(a_x^2 + a_z^2)^{1/2} \)

The angle rotated from the x – y plane is \( \phi = \tan^{-1}(a_z/a_x) \)

The Cartesian coordinates are:

\[ a_x = A \cos \theta \cos \phi \quad a_y = R \sin \theta \quad a_z = A \cos \theta \sin \phi \]

WORKED EXAMPLE No.2

Two vectors have coordinates:- \( \mathbf{A} (-5, 7, 4) \) and \( \mathbf{B} (2, -2, 3) \). Determine the angle between them.

SOLUTION

\[ \mathbf{A} \cdot \mathbf{B} = (a_x b_x + a_y b_y + a_z b_z) = (-5)(2) + (7)(-2) + (4)(3) = -12 \]

\[ A^2 = a_x^2 + a_y^2 + a_z^2 = (-5)^2 + (7)^2 + (4)^2 = 90 \]

\[ B^2 = b_x^2 + b_y^2 + b_z^2 = (2)^2 + (-2)^2 + (3)^2 = 17 \]

\[ \mathbf{A} \cdot \mathbf{B} = -12 = A B \cos \theta \]

\[ -12 = \sqrt{90} \sqrt{17} \cos \theta = 39.11 \quad \cos \theta = -12/39.11 = -0.306 \quad \theta = 107.8^\circ \]

© D.J.Dunn 3
SELF ASSESSMENT EXERCISE No. 1

1. Determine the scalar product of the vector pair A (5, 10) and B (3, 6)  (Answer 45)
2. Determine the scalar product of the vector pair A (12 ∠80°) and B (7, ∠30°)  (Answer 54)
3. Determine the scalar product of the vector pair A (5, 4, 5) and B (2, 3, 5)  (Answer 47)
4. Determine the magnitude of A and B and angle between them for (3). (Equate to find angle)  (Answer 8.124, 6.164 and 20.2°)
5. Two vectors have coordinates:- A (7, -3, 6) and B (5, -2, 10). Calculate the angle between them. (Answer 23.48°)

5. VECTOR OR CROSS PRODUCT

This was developed to help define a moment of force in three dimensions.

The cross product of two vectors \( \mathbf{A} \) and \( \mathbf{B} \) is denoted \( \mathbf{A} \times \mathbf{B} \) and the result is a third vector \( \mathbf{C} \) in a direction normal (orthogonal) to the plane formed by \( \mathbf{A} \) and \( \mathbf{B} \).

We shall show that \( \mathbf{C} = \mathbf{A} \times \mathbf{B} = \mathbf{A} \mathbf{B} \sin \theta \mathbf{n} \)

\( \theta \) is the angle between the vectors and \( \mathbf{n} \) is a unit vector normal to the plane created by \( \mathbf{A} \) and \( \mathbf{B} \).

The direction of \( \mathbf{C} \) is determined by the right hand rule as follows. Point the index fingure of the right hand in the direction of \( \mathbf{A} \) and bend the other fingures to the direction of \( \mathbf{B} \). The thumb then points the direction of \( \mathbf{C} \).

Let’s illustrate this with the following case. Consider the force acting at an angle \( \theta \) to a torque wrench as shown. A turning moment (Torque) is produced about the centre.

The component of the force acting normal to the radius is \( \mathbf{F} \sin \theta \).

It follows that \( \mathbf{T} = \mathbf{F} \mathbf{R} \sin \theta \). In this case the direction is into the page.

Since \( \mathbf{F} \) and \( \mathbf{R} \) are vectors, the result is obtained by using the cross product \( \mathbf{F} \times \mathbf{R} = \mathbf{F} \mathbf{R} \sin \theta \)

In general \( \mathbf{A} \times \mathbf{B} = \mathbf{A} \mathbf{B} \sin \theta \mathbf{n} \)

\( \theta \) is the angle between them and \( \mathbf{n} \) is the unit vector normal to the plane made by them and this makes the result a vector. B is counter clockwise of A on that plane.
6. **ANGULAR QUANTITIES**

When \( \theta = 90^\circ \) the vector product represents any angular quantity such as angle, angular velocity, angular acceleration, torque and angular momentum. All these are vector quantities and can be represented as a vector. The rule for determining the direction of the vector is that if you view the plane so that the direction of rotation is clockwise, the vector points away from you. This is known as the corkscrew rule because when a corkscrew is used you rotate clockwise and it moves into the cork. The vector direction is summed up in the diagram.

![Diagram](image)

**Figure 6**

**WORKED EXAMPLE No. 3**

Given the vectors \( \mathbf{A} = (3, 1, 2) \) and \( \mathbf{B} = (2, -2, 3) \) determine \( \mathbf{A} \times \mathbf{B} \)

**SOLUTION**

\[ \mathbf{A} \times \mathbf{B} = A \mathbf{B} \sin \theta \hat{n} \]

First find the magnitudes. \( A = \sqrt{3^2 + 1^2 + 2^2} = 3.742 \) \( B = \sqrt{2^2 + (-2)^2 + 3^2} = 4.123 \)

Next find the angle between them by using the scalar product method.

Scalar product

\[ \mathbf{A} \cdot \mathbf{B} = (3)(2) + (1)(-2) + (2)(3) = 10 \]

\[ 10 = A \mathbf{B} \cos (\theta) \]

\[ \theta = \cos^{-1}(10/(3.742)(4.123)) = 49.6^\circ \]

\[ \mathbf{A} \times \mathbf{B} = A \mathbf{B} \sin (49.6^\circ) \hat{n} = (3.742)(4.123) \sin 49.6^\circ = 11.747 \hat{n} \]

\( \hat{n} \) is the unit vector in the direction normal to the plane of \( \mathbf{A} \mathbf{B} \)
7. PROPERTIES OF CROSS PRODUCTS

Since the direction of the cross product is determined by the angle, then \( \mathbf{A} \times \mathbf{B} \) is equal and opposite to \( \mathbf{B} \times \mathbf{A} \) so:

\[
\mathbf{A} \times \mathbf{B} = - \mathbf{B} \times \mathbf{A}
\]

If either vector \( \mathbf{A} \) or \( \mathbf{B} \) is multiplied by a scalar \( \lambda \), the cross product \( \mathbf{C} \) will also be multiplied by \( \lambda \) so:

\[
\lambda \mathbf{A} \times \mathbf{B} = \mathbf{A} \times \lambda \mathbf{B} = \lambda (\mathbf{A} \times \mathbf{B})
\]

The distributive law can be proved to be

\[
\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}
\]

8. CARTESIAN REPRESENTATION

Consider the cross product of two identical parallel vectors \( \mathbf{A} \). \( \mathbf{A} \times \mathbf{A} = \mathbf{A} \mathbf{A} \sin(0) = 0 \)

If we have unit vectors \( \mathbf{i}, \mathbf{j}, \mathbf{k} \), then \( \mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0 \)

Consider the cross product of two identical vectors at 90° to each other. \( \mathbf{A} \times \mathbf{A} = \mathbf{A} \mathbf{A} \sin(90) = \mathbf{A}^2 \)

If we have unit vectors then

\[
\begin{align*}
\mathbf{i} \times \mathbf{i} &= \mathbf{j} \\
\mathbf{j} \times \mathbf{j} &= \mathbf{k} \\
\mathbf{k} \times \mathbf{k} &= \mathbf{i}
\end{align*}
\]

The unit vector indicates direction so the Cartesian coordinates used for \( \mathbf{A} \) and \( \mathbf{B} \) will be as follows.

\[
\mathbf{A} = i a_1 + j a_2 + k a_3 \quad \mathbf{B} = i b_1 + j b_2 + k b_3
\]

\[
\mathbf{A} \times \mathbf{B} = (i a_1 + j a_2 + k a_3) \times (i b_1 + j b_2 + k b_3)
\]

By application of the property rules we can produce the result

\[
\begin{align*}
\mathbf{A} \times \mathbf{B} &= \{a_1 b_1(i \times i) + a_1 b_2(i \times j) + a_1 b_3(i \times k)\} + \{a_2 b_1(j \times i) + a_2 b_2(j \times j) + a_2 b_3(j \times k)\} + ... \\
\end{align*}
\]

\[
\begin{align*}
\mathbf{A} \times \mathbf{B} &= \{a_3 b_1(k \times i) + a_3 b_2(k \times j) + a_3 b_3(k \times k)\}
\end{align*}
\]

\[
\begin{align*}
\mathbf{A} \times \mathbf{B} &= \{a_1 b_2(i \times j) + a_1 b_3(i \times k)\} + \{a_2 b_1(j \times i) + a_2 b_2(j \times j) + a_2 b_3(j \times k)\} + ... \\
\end{align*}
\]

\[
\begin{align*}
\mathbf{A} \times \mathbf{B} &= \{a_3 b_1(k \times i) + a_3 b_2(k \times j) + a_3 b_3(k \times k)\}
\end{align*}
\]

\[
\begin{align*}
\mathbf{A} \times \mathbf{B} &= \{a_1 b_2(k \times a_1 b_3(\cdot j))\} + \{a_2 b_1(\cdot k) + a_2 b_3(i)\} + \{a_3 b_1(j) + a_3 b_2(-i)\}
\end{align*}
\]

\[
\begin{align*}
\mathbf{A} \times \mathbf{B} &= a_1 b_2(k) + a_1 b_3(-j) + a_2 b_1(-k) + a_2 b_3(i) + a_3 b_1(j) + a_3 b_2(-i)
\end{align*}
\]

\[
\begin{align*}
\mathbf{A} \times \mathbf{B} &= i(a_2 b_3 - a_1 b_2) + j(a_3 b_1 - a_1 b_3) + k (a_1 b_2 - a_2 b_1)
\end{align*}
\]

If you have studies matrices, you will recognise this law happens to coincide with the determinant for a 3 x 3 matrix.

\[
\begin{vmatrix}
i & j & k \\
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3
\end{vmatrix}
\]
WORKED EXAMPLE No. 4

Find the resulting vector when \( \mathbf{A} = (3, 1, 2) \) and \( \mathbf{B} = (2, -2, 3) \) are crossed. Express the result in Cartesian and Spherical coordinates.

SOLUTION

\[
\mathbf{A} \times \mathbf{B} = \mathbf{i}(a_2 b_3 - a_3 b_2) + \mathbf{j}(a_3 b_1 - a_1 b_3) + \mathbf{k}(a_1 b_2 - a_2 b_1)
\]

\[
\mathbf{A} \times \mathbf{B} = \mathbf{i} \{(1)(3) - (2)(-2)} + \mathbf{j} \{(2)(2) - (3)(3)} + \mathbf{k} \{(3)(-2) - (1)(2)}
\]

\[
\mathbf{C} = \mathbf{A} \times \mathbf{B} = 7 \mathbf{i} - 5 \mathbf{j} - 8 \mathbf{k}
\]

The resulting vector is 7, -5, -8

The magnitude is 

\[
C = \sqrt{(7)^2 + (-5)^2 + (-8)^2} = 11.747
\]

\[
\theta = \tan^{-1}\left[\frac{c_2}{\sqrt{c_1^2 + c_3^2}}\right] = \tan^{-1}\left[\frac{-5}{\sqrt{7^2 + (-8)^2}}\right] = -25.2^\circ
\]

\[
\phi = \tan^{-1}\left[\frac{c_3}{c_1}\right] = \tan^{-1}\left[\frac{-8}{7}\right] = -48.8^\circ
\]

SELF ASSESSMENT EXERCISE No. 2

Given \( \mathbf{A} \) and \( \mathbf{B} \) in Cartesian coordinates determine the cross product and express the resulting vector in Cartesian and Spherical coordinates.

1. \( \mathbf{A} = (1, 2, 3) \) \( \mathbf{B} = (3, 2, 1) \)

   Answers \( \mathbf{A} \times \mathbf{B} = 9.798 \) n, Cartesian coordinates \( (-4, 8, -4) \) Spherical \( (9.798, 54.7^\circ \) and \( 45^\circ) \)

2. \( \mathbf{A} = (-2, -7, 4) \) \( \mathbf{B} = (5, -2, 5) \)

   Answers \( \mathbf{A} \times \mathbf{B} = 56.125 \) n, Cartesian \( (-27,30, 39) \) and spherical \( (56.1, 32.3^\circ \) and \( -55.3^\circ) \)
9. **THE TRIPLE SCALAR PRODUCT**

The triple SCALAR product is produced by three vectors. It is a scalar produced by the combination \((A \times B) \cdot C\).

Studying the geometry of three vectors it may be shown that the triple product produces the volume of a parallelepiped.

![Diagram showing volume of parallelepiped](image)

First consider again the cross product of \(A\) and \(B\) and let them lay on the \(x – z\) plane as shown. The area of the parallelogram is \(A_w = A \cdot B \sin \theta\).

Since \(A \times B = A \cdot B \sin \theta \cdot j\) it follows that \(A \times B = \) is the volume of a parallelogram extruded one unit in the \(y\) direction.

If we now examine vector \(C\) drawn as shown for convenience, it has a height normal to the \(x – z\) plane of \(h = j \cdot C \cos \phi\).

The parallelepiped has a volume \(A \cdot w \cdot h = A \cdot B \sin \theta \cdot h\).

The volume is hence \(A \cdot B \sin \theta \cdot j \cdot C \cos \phi = (A \times B) \cdot C\).

Since any of the flat faces of the parallelepiped could have been chosen as the reference plane, it must follow that:

\[(A \times B) \cdot C = (A \times C) \cdot B = (B \times C) \cdot A = (A \times B) = B \cdot (A \times C) = A \cdot (B \times C)\]

In other words you swap any two vectors.

The three vectors may be expressed as a matrix of the Cartesian coordinates as shown.

It can further be shown that the determinant gives the triple product.

\[A \cdot (B \times C) = a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1\]

### WORKED EXAMPLE No. 5

Given three vectors \(A\) (2, 1, 0), \(B\) (2, -1, 1), and \(C\) (0, 1, 1) find \(A \cdot (B \times C)\) and \(B \cdot (C \times A)\)

**SOLUTION**

\[A \cdot (B \times C) = a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1\]

\[A \cdot (B \times C) = (2)(-1)(1) – (2)(1)(1) – (1)(2)(1) + (1)(1)(0) + (0)(-1)(1) = -6\]

To solve \(B \cdot (C \times A)\) we change the above pattern by simply changing a to b, b to c and c to a.

\[B \cdot (C \times A) = b_1c_2a_3 - b_1c_3a_2 - b_2c_1a_3 + b_2c_3a_1 + b_3c_1a_2 - b_3c_2a_1\]

\[B \cdot (C \times A) = 0 - 2 \cdot 0 - 2 + 0 - 2 = -6\]

As expected the result is the same.
SELF ASSESSMENT EXERCISE No. 3

1. Find the volume of a parallelepiped formed by the three vectors $A(4, 0, -1)$ $B(1, 0, -4)$ and $C(2, 6, -4)$ all projected from the origin. (Answer 90 units)

2. Given the three vectors $A(-2, 2, -1)$ $B(-3, 1, 5)$ and $C(5, 3, 4)$ all projected from the origin, evaluate $A \bullet (B \times C)$ and $B \bullet (C \times A)$ and show that they are the same. (Answer 110 units).

10. THE TRIPLE VECTOR PRODUCT

The triple product of three vectors $A$, $B$ and $C$ is defined as $(A \times B) \times C$ and this is a vector.

We already know that $A \times B = i(a_2 b_3 - a_3 b_2) + j(a_3 b_1 - a_1 b_3) + k(a_1 b_2 - a_2 b_1)$

This is a vector with coordinates $(a_2 b_3 - a_3 b_2)$, $(a_3 b_1 - a_1 b_3)$ and $(a_1 b_2 - a_2 b_1)$

Let this vector be designated $R$ with coordinates:
$r_1 = (a_2 b_3 - a_3 b_2)$, $r_2 = (a_3 b_1 - a_1 b_3)$ and $r_3 = (a_1 b_2 - a_2 b_1)$

Now deduce $R \times C$

$R \times C = i(r_2 c_3 - r_3 c_2) - j(r_1 c_3 - r_3 c_1) + k(r_1 c_2 - r_2 c_1)$

Substitute for $r_1$, $r_2$ and $r$

$R \times C = (A \times B) \times C = i\{(a_3 b_1 - a_1 b_3) c_2 - (a_1 b_2 - a_2 b_1) c_1\} - j\{(a_2 b_3 - a_3 b_2) c_3 - (a_1 b_2 - a_2 b_1) c_1\} + k\{(a_2 b_3 - a_3 b_2) c_2 - r(a_3 b_1 - a_1 b_3) c_1\}$

Examine the first coordinate of the vector $i\{(a_3 b_1 c_3 - a_1 b_3 c_3) - (a_1 b_2 c_2 - a_2 b_1 c_2)\}$

With manipulation it can be shown that this is the result of $(A \bullet C) b_1 - (B \bullet C) a_1$

Repeating the process for the other two components it can be shown that the three coordinates of the vector are:

$$(A \times B) \times C = \{(A \bullet C) b_1 - (B \bullet C) a_1\}, \{(A \bullet C) b_2 - (B \bullet C) a_2\}, \{(A \bullet C) b_3 - (B \bullet C) a_3\}$$

This further simplifies to

$$(A \times B) \times C = (A \bullet C) B - (B \bullet C) A$$

We can go on to show that $(B \times C) \times A = (A \bullet C) B - (A \bullet B) C$
WORKED EXAMPLE No. 6

Given three vectors \( \mathbf{A} (3, -2, 1) \), \( \mathbf{B} (-1, 3, 4) \), and \( \mathbf{C} (2, 1, -3) \) find \( (\mathbf{A} \times \mathbf{B}) \times \mathbf{C} \) and show that the same result is obtained from \( (\mathbf{A} \cdot \mathbf{C}) \mathbf{B} - (\mathbf{B} \cdot \mathbf{C}) \mathbf{A} \)

SOLUTION

\[
(\mathbf{A} \times \mathbf{B}) = i(a_2 b_3 - a_3 b_2) + j(a_3 b_1 - a_1 b_3) + k(a_1 b_2 - a_2 b_1)
\]

\[
(\mathbf{A} \times \mathbf{B}) = i[(-2)(4) - (1)(3)] + j[(1)( -1) - (3)(4)] + k [(3)(3) – (-2)(-1)]
\]

\[
(\mathbf{A} \times \mathbf{B}) = i[(-8) – (3)] + j[(-11)] + k [(7)]
\]

Let this be a vector \( \mathbf{R} \)

\[
\mathbf{R} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = \begin{pmatrix} -11 \\ -13 \\ 7 \end{pmatrix}
\]

Now deduce \( \mathbf{R} \times \mathbf{C} \)

\[
\mathbf{R} \times \mathbf{C} = i(r_2 c_3 - r_3 c_2) – j(r_1 c_3 - r_3 c_1) + k(r_1 c_2 - r_2 c_1)
\]

\[
\mathbf{R} \times \mathbf{C} = i[(-13)(-3) – (7)(1)] – j[(-11)(-3) – (7)(2)] + k[(-11)(1) – (-13)(2)]
\]

\[
\mathbf{R} \times \mathbf{C} = i[(39) – (7)] – j[(33) – (14)] + k[(-11) –(-26)]
\]

\[
\mathbf{R} \times \mathbf{C} = 32i –19j + 15k
\]

\( (\mathbf{A} \times \mathbf{B}) \times \mathbf{C} \) is a vector 32i –19j + 15k

Now evaluate \( (\mathbf{A} \cdot \mathbf{C}) \mathbf{B} - (\mathbf{B} \cdot \mathbf{C}) \mathbf{A} \)

\[
(\mathbf{A} \cdot \mathbf{C}) = a_1 c_1 + a_2 c_2 + a_3 c_3 = (3)(2) + (-2)(1) + (1)(-3)
\]

\[
(\mathbf{A} \cdot \mathbf{C}) = 6 - 2 -3 = 1
\]

\( (\mathbf{A} \cdot \mathbf{C}) \mathbf{B} \) is a vector 1(-1i, 3j, 4k) or simply (-i, 3j, 4k)

\[
(\mathbf{B} \cdot \mathbf{C}) = b_1 c_1 + b_2 c_2 + b_3 c_3 = (-1)(2) + (3)(1) + (4)(-3)
\]

\[
(\mathbf{B} \cdot \mathbf{C}) = -2 + 3 -12 = -11
\]

\( (\mathbf{B} \cdot \mathbf{C}) \mathbf{A} \) is a vector -11(3i, -2j, 1k) or (-33i, 22j, -11k)

\[
(\mathbf{A} \cdot \mathbf{C}) \mathbf{B} - (\mathbf{B} \cdot \mathbf{C}) \mathbf{A} \text{ is a vector found by subtracting the coordinates to give}
\]

\[
(\mathbf{A} \cdot \mathbf{C}) \mathbf{B} - (\mathbf{B} \cdot \mathbf{C}) \mathbf{A} = (-i, 3j, 4k) - (-33i, 22j, -11k) = (32i, -19j, 15k)
\]

SELF ASSESSMENT EXERCISE No. 4

1. Given the three vectors \( \mathbf{A}(2, -3, 3) \), \( \mathbf{B}(-3, 2, 2) \) and \( \mathbf{C}(1, 2, -2) \) find the vector \( (\mathbf{A} \times \mathbf{B}) \times \mathbf{C} \) and show that the same result is obtained from \( (\mathbf{A} \cdot \mathbf{C}) \mathbf{B} - (\mathbf{B} \cdot \mathbf{C}) \mathbf{A} \)

\( 36, -29, -11 \) in both cases

2. Given the three vectors \( \mathbf{A}(2, -3, 3) \), \( \mathbf{B}(-3, 2, 2) \) and \( \mathbf{C}(1, 2, -2) \) find the vector \( \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \) and show that the same result is obtained from \( (\mathbf{A} \cdot \mathbf{C}) \mathbf{B} - (\mathbf{A} \cdot \mathbf{B}) \mathbf{C} \)

\( 36, -8, -32 \) in both cases

3. Given the three vectors \( \mathbf{A}(-5, -3, 1) \), \( \mathbf{B}(3, 2, -4) \) and \( \mathbf{C}(2, -3, 10) \) find the vector \( (\mathbf{A} \times \mathbf{B}) \times \mathbf{C} \) and show that the same result is obtained from \( (\mathbf{A} \cdot \mathbf{C}) \mathbf{B} - (\mathbf{B} \cdot \mathbf{C}) \mathbf{A} \)

\( -173, -102, 4 \) in both cases