APPLIED FLUID MECHANICS

TUTORIAL No.6

DIMENSIONAL ANALYSIS

When you have completed this tutorial you should be able to do the following.

- Explain the basic system of dimensions.
- Find the relationship between variables affecting a phenomenon.
- Define and use dimensionless numbers.
- Solve problems by the use of model tests.
- Solve typical exam questions.

1. BASIC DIMENSIONS

All quantities used in engineering can be reduced to six basic dimensions. These are the dimensions of

Mass	M
Length	L
Time	T
Temperature	θ
Electric Current	I
Luminous Intensity	J

The last two are not used in fluid mechanics and temperature is only used sometimes.

All engineering quantities can be defined in terms of the four basic dimensions M,L,T and θ. We could use the S.I. units of kilogrammes, metres, seconds and Kelvins, or any other system of units, but if we stick to M,L,T and θ we free ourselves of any constraints to a particular system of measurements.

Let's now explain the above with an example. Consider the quantity *density*. The S.I. units are kg/m^3 and the imperial units are lb/in^3. In our system the units would be $Mass/Length^3$ or M/L^3. It will be easier in the work ahead if we revert to the inverse indice notation and write it as ML^{-3}.

Other engineering quantities need a little more thought when writing out the basic MLTθ dimensions. The most important of these is the unit of force or the Newton in the S.I. system. Engineers have opted to define force as that which is needed to accelerate a mass such that 1 N is needed to accelerate 1 kg at 1 m/s^2. From this we find that the Newton is a derived unit equal to 1 kg m/s^2. In our system the dimensions of force become MLT^{-2}. This must be considered when writing down the dimensions of anything containing force.

Another unit that produces problems is that of angle. Angle is a ratio of two sides of a triangle and so has no units nor dimensions at all. This also applies to revolutions which are angular measurements. Strain is also a ratio and has no units nor dimensions. Angle and strain are in fact examples of dimensionless quantities that will be considered in detail later.

WORKED EXAMPLE No. 1

Write down the basic dimensions of pressure p.

SOLUTION

Pressure is defined as p = Force/Area

The S.I. unit of pressure is the Pascal which is the name for $1N/m^2$.
Since force is MLT^{-2} and area is L^2 then the basic dimensions of pressure are

$$ML^{-1}T^{-2}$$

When solving problems it is useful to use a notation to indicate the MLT dimensions of a quantity and in this case we would write

$$[p] = ML^{-1}T^{-2}$$

WORKED EXAMPLE No.2

Deduce the basic dimensions of dynamic viscosity.

SOLUTION

Dynamic viscosity was defined in an earlier tutorial from the formula $\tau = \mu du/dy$

τ is the shear stress, du/dy is the velocity gradient and μ is the dynamic viscosity. From this we have $\mu = \tau\, dy/du$

Shear stress is force/area.
The basic dimensions of force are MLT^{-2}
The basic dimensions of area are L^2.
The basic dimensions of shear stress are $ML^{-1}T^{-2}$.
The basic dimensions of distance y are L.
The basic dimensions of velocity v are LT^{-1}.
It follows that the basic dimension of dy/du (a differential coefficient) is T.
The basic dimensions of dynamic viscosity are hence $(ML^{-1}T^{-2})(T) = ML^{-1}T^{-1}$.

$[\mu] = ML^{-1}T^{-1}$.

2. LIST OF QUANTITIES AND DIMENSIONS FOR REFERENCE.

AREA	$(\text{LENGTH})^2$	L^2
VOLUME	$(\text{LENGTH})^3$	L^3
VELOCITY	LENGTH/TIME	LT^{-1}
ACCELERATION	$\text{LENGTH}/(\text{TIME}^2)$	LT^{-2}
ROTATIONAL SPEED	REVOLUTIONS/TIME	T^{-1}
FREQUENCY	CYCLES/TIME	T^{-1}
ANGULAR VELOCITY	ANGLE/TIME	T^{-1}
ANGULAR ACCELERATION	$\text{ANGLE}/(\text{TIME})^2$	T^{-2}
FORCE	MASS X ACCELERATION	MLT^{-2}
ENERGY	FORCE X DISTANCE	ML^2T^{-2}
POWER	ENERGY/TIME	ML^2T^{-3}
DENSITY	MASS/VOLUME	ML^{-3}
DYNAMIC VISCOSITY	STRESS/VELOCITY GRADIENT	$ML^{-1}T^{-1}$
KINEMATIC VISCOSITY	DYN. VISCOSITY/DENSITY	L^2T^{-1}
PRESSURE	FORCE/AREA	$ML^{-1}T^{-2}$
SPECIFIC HEAT CAPACITY	ENERGY/(MASS X TEMP)	$L^2T^{-2}\theta^{-1}$
TORQUE	FORCE X LENGTH	ML^2T^{-2}
BULK MODULUS	PRESSURE/STRAIN	$ML^{-1}T^{-2}$

3. HOMOGENEOUS EQUATIONS

All equations must be homogeneous. Consider the equation $F = 3 + T/R$
F is force, T is torque and R is radius. Rearranging we have $3 = F - T/R$

Examine the units. F is Newton. T is Newton metre and R is metre.

hence $3 = F\ (N) - T/R\ (N\ m)/m)$
$3 = F(N) - T/R\ (N)$

It follows that the number 3 must represent 3 Newton. It also follows that the unit of F and T/R must both be Newton. If this was not so, the equation would be nonsense. In other words all the components of an equation that add together must have the same units. You cannot add dissimilar quantities. For example you cannot say that 5 apples + 6 pears = 11 plums. This is clearly nonsense. When all parts of an equation that add together have the same dimensions, then the equation is homogeneous.

WORKED EXAMPLE No.3

Show that the equation Power = Force x velocity is homogeneous in both S.I. units and basic dimensions.

SOLUTION

The equation to be checked is $P = F\ v$

The S.I. Unit of power (P) is the Watt. The Watt is a Joule per second. A Joule is a Newton metre of energy. Hence a Watt is 1 N m/s.

The S.I. unit of force (F) is the Newton and of velocity (v) is the metre/second.

The units of F v are hence N m/s.

It follows that both sides of the equation have S.I. units of N m/s so the equation is homogeneous.

Writing out the MLT dimensions of each term we have

$$[P] = ML^2T^{-3}$$
$$[v] = LT^{-1}$$
$$[F] = MLT^{-2}$$

Substituting into the equation we have $ML^2T^{-3} = MLT^{-2}\ LT^{-1} = ML^2T^{-3}$

Hence the equation is homogeneous.

4. **INDECIAL EQUATIONS**

When a phenomenon occurs, such as a swinging pendulum as shown in figure 3.44we observe the variables that effect each other. In this case we observe that the frequency, (f) of the pendulum is affected by the length (l) and the value of gravity (g). We may say that frequency is a function of l and g. In equation form this is as follows.

$$f = \phi(l,g)$$ where ϕ is the function sign.

When we remove the function sign we must put in a constant because there is an unknown number and we must allocate unknown indices to l and g because we do know not what if any they are. The equation is written as follows.

$$f = C\,l^a\,g^b$$

C is a constant and has no units. a and b are unknown indices.

This form of relating variables is called an indecial equation. The important point here is that because we know the units or dimensions of all the variables, we can solve the unknown indices.

WORKED EXAMPLE No.4

Solve the relationship between f, l and g for the simple pendulum.

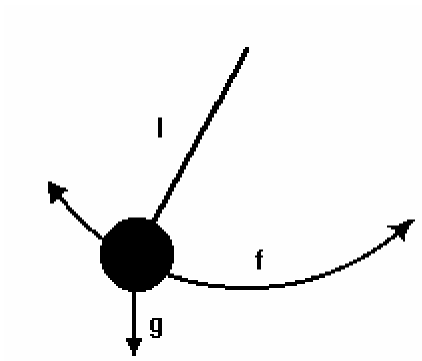

Fig.1

SOLUTION

First write down the indecial form of the equation (covered overleaf).

$f = C\, l^a g^b$

Next write down the basic dimensions of all the variables.

$[f] = T^{-1}$

$[l] = L^1$

$[g] = LT^{-2}$

Next substitute the dimensions in place of the variables.

$$T^{-1} = (L^1)^a (LT^{-2})^b$$

Next tidy up the equation. $T^{-1} = L^{1a} L^b T^{-2b}$

Since the equation must be homogeneous then the power of each dimension must be the same on the left and right side of the equation. If a dimension does not appear at all then it is implied that it exists to the power of zero. We may write them in until we get use to it. The equation is written as follows.

$$M^0 L^0 T^{-1} = L^{1a} LT^{-2b} M^0$$

Next we equate powers of each dimension. First equate powers of Time.

$T^{-1} = T^{-2b}$

$-1 = -2b$

$b = 1/2$

Next equate powers of Length.

$L^0 = L^{1a} L^b$

$0 = 1a + b$ hence $a = -b = -1/2$

$M^0 = M^0$ yields nothing in this case.

Now substitute the values of a and b back into the original equation and we have the following.

$f = C\, l^{-½}\, g^{½}$

$f = C\, (g/l)^{½}$

The frequency of a pendulum may be derived from basic mechanics and shown to be

$f = (1/2\pi)(g/l)^{½}$

If we did not know how to find C = (½ π) from basic mechanics, then we know that if we conducted an experiment and measured the values f for various values of l and g, we could find C by plotting a graph of f against $(g/l)^{½}$. This is the importance of dimensional analysis to fluid mechanics. We are able to determine the basic relationships and then conduct experiments and determine the remaining unknown constants. We are able to plot graphs because we know what to plot against what.

SELF ASSESSMENT EXERCISE No.1

1. It is observed that the velocity 'v' of a liquid leaving a nozzle depends upon the pressure drop 'p' and the density 'ρ'. Show that the relationship between them is of the form

$$v = C\left(\frac{p}{\rho}\right)^{\frac{1}{2}}$$

2. It is observed that the speed of a sound in 'a' in a liquid depends upon the density 'ρ' and the bulk modulus 'K'. Show that the relationship between them is

$$a = C\left(\frac{K}{\rho}\right)^{\frac{1}{2}}$$

3. It is observed that the frequency of oscillation of a guitar string 'f' depends upon the mass 'm', the length 'l' and tension 'F'. Show that the relationship between them is

$$f = C\left(\frac{F}{ml}\right)^{\frac{1}{2}}$$

5. DIMENSIONLESS NUMBERS

We will now consider cases where the number of unknown indices to be solved, exceed the number of equations to solve them. This leads into the use of dimensionless numbers.

Consider that typically a problem uses only the three dimensions M, L and T. This will yield 3 simultaneous equations in the solution. If the number of variables in the equation gives 4 indices say a, b, c and d, then one of them cannot be resolved and the others may only be found in terms of it.

In general there are n unknown indices and m variables. There will be m-n unknown indices. This is best shown through a worked example.

WORKED EXAMPLE No. 5

The pressure drop per unit length 'p' due to friction in a pipe depends upon the diameter 'D' , the mean velocity 'v' , the density 'ρ' and the dynamic viscosity 'μ'. Find the relationship between these variables.

SOLUTION

$p = \text{function}(D\ v\ \rho\ \mu) = K\ D^a\ v^b\ \rho^c\ \mu^d$

p is pressure per metre

$[p] = ML^{-2}T^{-2}$

$[D] = L$

$[v] = LT^{-1}$

$[\rho] = ML^{-3}$

$[\mu] = ML^{-1}T^{-1}$

$$ML^{-2}T^{-2} = L^a (LT^{-1})^b (ML^{-3})^c (ML^{-1}T^{-1})^d$$

$$ML^{-2}T^{-2} = L^{a+b-3c-d} M^{c+d} T^{-b-d}$$

The problem is now deciding which index not to solve. The best way is to use experience gained from doing problems. Viscosity is the quantity that causes viscous friction so the index associated with it (d) is the one to identify. We will resolve a, b and c in terms of d.

TIME $-2 = -b - d$ hence $b = 2 - d$ is as far as we can resolve

MASS $1 = c + d$ hence $c = 1 - d$

LENGTH $-2 = a + b - 3c - d$

$-2 = a + (2 - d) - 3(1-d) - d$ hence $a = -1 - d$

Next put these back into the original formula.

$$p = K\ D^{-1-d}\ v^{2-d}\ \rho^{1-d}\ \mu^{d}$$

Next group the quantities with same power together as follows :

$$p = K\{\rho v^2 D^{-1}\}\ \{\mu\ \rho^{-1} v^{-1} D^{-1}\}^d$$

Remember that p was pressure drop per unit length so the pressure loss over a length L is

$$P = K\ L\{\rho\ v^2 D^{-1}\}\ \{\mu\ \rho^{-1} v^{-1} D^{-1}\}^d$$

We have two unknown constants K and d. The usefulness of dimensional analysis is that it tells us the form of the equation so we can deduce how to present experimental data. With suitable experiments we could now find K and d.

Note that this equation matches up with Poiseuille's equation which gives the relationship as :

$$p = 32\ \mu\ L\ v\ D^{-2}$$

It may be deduced that K = 32 and d = 1 (laminar flow only)

The term $\{\rho v D \mu^{-1}\}$ has no units. If you check it out all the units will cancel. This is a ***DIMENSIONLESS NUMBER***, and it is named after Reynolds.

Reynolds Number is denoted R_e. The whole equation can be put into a dimensionless form as follows.

$$\{p\ \rho^{-1}\ L^{-1}\ v^{-2} D^{1}\} = K\ \{\mu\ \rho^{-1} v^{-1} D^{-1}\}^d$$

$$\{p\ \rho^{-1}\ L^{-1}\ v^{-2} D^{1}\} = \text{function}\ (R_e)$$

This is a dimensionless equation. The term $\{p\rho^{-1}\ L^{-1} v^{-2} D^{1}\}$ is also a dimensionless number.

Let us now examine another similar problem.

WORKED EXAMPLE No.6

Consider a sphere moving through an viscous fluid completely submerged. The resistance to motion R depends upon the diameter D, the velocity v, the density ρ and the dynamic viscosity μ. Find the equation that relates the variables.

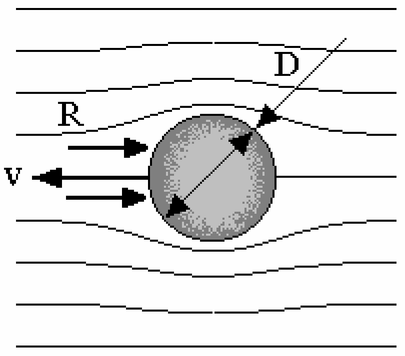

Fig.2

SOLUTION

$R = \text{function}(D\ v\ \rho\ \mu) = K\ D^a\ v^b\ \rho^c\ \mu^d$

First write out the MLT dimensions.

$[R] = ML^1T^{-2}$

$[D] = L \qquad ML^1T^{-2} = L^a (LT^{-1})^b (ML^{-3})^c (ML^{-1}T^{-1})^d$

$[v] = LT^{-1} \qquad ML^1T^{-2} = L^{a+b-3c-d}\ M^{c+d}\ T^{-b-d}$

$[\rho] = ML^{-3}$

$[\mu] = ML^{-1}T^{-1}$

Viscosity is the quantity which causes viscous friction so the index associated with it (d) is the one to identify. We will resolve a,b and c in terms of d as before.

TIME $-2 = -b - d$ hence $b = 2 - d$ is as far as we can resolve b

MASS $1 = c + d$ hence $c = 1 - d$

LENGTH $1 = a + b - 3c - d$

$1 = a + (2 - d) - 3(1 - d) - d$ hence $a = 2 - d$

Next put these back into the original formula.

$$R = K\ D^{2-d}\ v^{2-d}\ \rho^{1-d}\ \mu^d$$

Next group the quantities with same power together as follows :

$$R = K\{\rho v^2 D^2\}\ \{\mu\ \rho^{-1} v^{-1} D^{-1}\}^d$$
$$R\{\rho v^2 D^2\}^{-1} = K\ \{\mu\ \rho^{-1} v^{-1} D^{-1}\}^d$$

The term $\{\rho v D \mu^{-1}\}$ is the Reynolds Number R_e and the term $R\{\rho v^2 D^2\}^{-1}$ is called the Newton Number N_e. Hence the relationship between the variables may be written as follows.

$$R\{\rho v^2 D^2\}^{-1} = \text{function } \{\rho v D\ \mu^{-1}\}$$
$$N_e = \text{function } (R_e)$$

Once the basic relationship between the variables has been determined, experiments can be conducted to find the parameters in the equation. For the case of the sphere in an incompressible fluid we have shown that

$$N_e = \text{function } (R_e)$$

Or put another way $\quad N_e = K\ (R_e)^n$

where K is a constant of proportionality and n is an unknown index (equivalent to -d in the earlier lines). In logarithmic form the equation is

$$\log(N_e) = \log(K) + n \log(R_e)$$

This is a straight line graph from which log K and n are taken. Without dimensional analysis we would not have known how to present the information and plot it. The procedure now would be to conduct an experiment and plot log(Ne) against log(Re). From the graph we would then determine K and n.

6. BUCKINGHAM'S Π (Pi) THEORY

Many people prefer to find the dimensionless numbers by intuitive methods. Buckingham's theory is based on the knowledge that if there are m basic dimensions and n variables, then there are m - n dimensionless numbers. Consider worked example No.6 again. We had the basic equation

$$R = \text{function } (D\ v\ \rho\ \mu)$$

There are 5 quantities and there will be 3 basic dimensions ML and T. This means that there will be 2 dimensionless numbers Π_1 and Π_2. These numbers are found by choosing two prime quantities (R and μ).

Π_1 is the group formed between μ and D v ρ
Π_2 is the group formed between R and D v ρ

First taking μ. Experience tells us that this will be the Reynolds number but suppose we don't know this.

The dimensions of μ are $ML^{-1}T^{-1}$
The dimensions of D v ρ must be arranged to be the same.

$$\mu = \Pi_1\ D^a v^b\ \rho^c$$

$$M^1L^{-1}T^{-1} = \Pi_1\ (L)^a\ (LT^{-1})^b\ (ML^{-3})^c$$

Time	**-1 = -b**	**b = 1**
Mass		**c = 1**
Length	-1 = a + b -3c	
	-1 = a + 1 – 3	a=1

$$\mu = \Pi_1\ D^1\ v^1\ \rho^1 \qquad \Pi_1 = \frac{\mu}{Dv\rho}$$

The second number must be formed by combining R with ρ,v and D

$$R = \Pi_2\ D^a\ v^b\ \rho^c$$

$$MLT^{-2} = \Pi_2\ (L)^a\ (LT^{-1})^b (ML^{-3})^c$$

Time	**-2 = -b**	**b = 2**
Mass		**c = 1**
Length	**1 = a + b -3c**	
	1 = a + 2 – 3	**a = 2**

$$R = \Pi_2\ D^2\ v^2\ \rho^1 \qquad \Pi_2 = \frac{R}{\rho v^2 D^2}$$

The dimensionless equation is $\mathbf{\Pi_2 = f(\Pi_1)}$

WORKED EXAMPLE No.7

The resistance to motion 'R' for a sphere of diameter 'D' moving at constant velocity 'v' through a compressible fluid is dependant upon the density 'ρ' and the bulk modulus 'K'. The resistance is primarily due to the compression of the fluid in front of the sphere. Show that the dimensionless relationship between these quantities is N_e = function (M_a)

SOLUTION

$R = \text{function}(D\ v\ \rho\ K) = C\ D^a\ v^b\ \rho^c\ K^d$

There are 3 dimensions and 5 quantities so there will be 5 –3 = 2 dimensionless numbers. Identify that the one dimensionless group will be formed with R and the other with K.

Π_1 is the group formed between K and D v ρ

Π_2 is the group formed between R and D v ρ

$K = \Pi_2 D^a\ v^b \rho^c$

$[K] = ML^{-1}\ T^{-2}$

$[D] = L$

$[v] = LT^{-1}$

$[\rho] = ML^{-3}$

$ML^{-1}T^{-2} = L^a (LT^{-1})^b\ (ML^{-3})^c$

$ML^{-1}T^{-2} = L^{a+b-3c}\ M^c\ T^{-b}$

Time -2 = -b **b = 2**

Mass **c = 1**

Length -1 = a + b -3c

-1 = a + 2 – 3 **a = 0**

$K = \Pi_2\ D^o\ v^2\ \rho^1$

$$\Pi_2 = \frac{K}{\rho v^2}$$

$R = \Pi_1\ D^a\ v^b \rho^c$

$[R] = MLT^{-2}$

$[D] = L$

$[v] = LT^{-1}$

$[\rho] = ML^{-3}$

$MLT^{-2} = L^a (LT^{-1})^b\ (ML^{-3})^c$

$ML^1T^{-2} = L^{a+b-3c}\ M^c\ T^{-b}$

Time -2 = -b **b = 2**

Mass **c = 1**

Length **1 = a + b -3c**

1 = a + 2 – 3 **a = 2**

$R = \Pi_1\ D^2\ v^2\ \rho^1$

$$\Pi_1 = \frac{R}{\rho v^2 D^2}$$

It was shown earlier that the speed of sound in an elastic medium is given by the following formula. $a = (k/\rho)^{½}$

It follows that $(k/\rho) = a^2$ and so $\Pi_2 = (a/v)^2$

The ratio v/a is called the Mach number (Ma) so $(Ma)^{-2}$

Π_1 is the Newton Number Ne.

The equation may be written as $\Pi_1 = \phi\Pi_2\ N_e$ or $Ne = \phi(M_a)$

SELF ASSESSMENT EXERCISE No.2

1. The resistance to motion 'R' for a sphere of diameter 'D' moving at constant velocity 'v' on the surface of a liquid is due to the density 'ρ' and the surface waves produced by the acceleration of gravity 'g'. Show that the dimensionless equation linking these quantities is N_e = function(F_r)

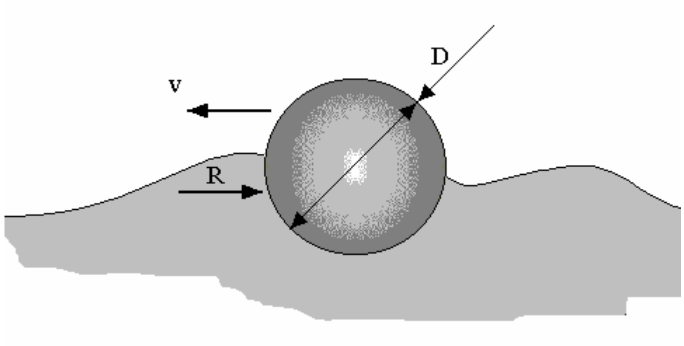

Fig. 3

F_r is the Froude number and is given by $F_r = \sqrt{\frac{v^2}{gD}}$

Here is a useful tip. It is the power of g that cannot be found.

2. The Torque 'T' required to rotate a disc in a viscous fluid depends upon the diameter 'D' , the speed of rotation 'N' the density 'ρ' and the dynamic viscosity 'μ'. Show that the dimensionless equation linking these quantities is :

$$\{T\,D^{-5}\,N^{-2}\,\rho^{-1}\} = \text{function}\ \{\rho\,N\,D^{2}\,\mu^{-1}\}$$

MORE DIFFICULT PROBLEMS

The problems so far seen have one unknown index in the solution. When there are two (or more) unknown indexes, the procedure is the same as before. A group of quantities must be formed for each unknown index left in the penultimate part of the solution.

SELF ASSESSMENT EXERCISE No.3

1. The resistance to motion 'R' of a sphere travelling through a fluid which is both viscous and compressible, depends upon the diameter 'D' , the velocity 'v' , the density 'ρ' , the dynamic viscosity 'μ' and the bulk modulus 'K'. Show that the complete relationship between these quantities is :

$$N_e = \text{function}\{R_e\}\{M_a\}$$

where

$$N_e = R\ \rho^{-1}\ v^{-2}\ D^{-2}$$

$$R_e = \rho\ v\ D\ \mu^{-1}$$

$$M_a = v/a \quad \text{and} \quad a = (k/\rho)^{0.5}$$

7 MODEL TESTING

When we test a model in order to predict the performance of the real thing, the results are only valid when the forces acting on the model are in the same ratio to each other as they are on the real thing. When this occurs we have DYNAMIC SIMILARITY.

It will be shown that in order to have dynamic similarity, the model must also be a true scale model, in other words we must have GEOMETRIC SIMILARITY.

7.1 DYNAMIC SIMILARITY

We have already seen that certain dimensionless numbers occur in problems of fluid mechanics. Each of these is associated with a particular kind of force.

The Newton Number N_e is associated with total resistance.

The Reynolds Number R_e is associated with viscous resistance.

The Mach Number M_a is associated with compression wave resistance.

The Froude Number F_r is associated with surface wave resistance.

There are others and all dimensionless numbers can take various forms. In order to obtain dynamic similarity, these dimensionless numbers must have the same values on the model and the real thing. Consider for example the resistance to motion of a sphere due to viscosity and compressibility of the fluid. The dimensionless equation is:

$$N_e = \phi(R_e)(M_a)$$

To ensure that the viscous, compression and resistance forces are in the same ratio to each other on the model and on the object, then the three numbers must be the same on both. This is often difficult or impossible to obtain when there are more than three numbers for reasons which will become apparent.

7.2 GEOMETRIC SIMILARITY

In much of the forgoing work, the work has been about a sphere of diameter D so that only one actual length dimension was needed to define both the shape and size of the object. If we tried the same analysis for a submarine or an aeroplane, we should include all the linear dimensions necessary to define the shape and this would be enormous. Consider the following problem that needs two linear dimensions and it is the one we looked at previously in a slightly different way.

The pressure drop p in a pipe depends upon the diameter D, the length l, the density ρ and the viscosity μ. Dimensional analysis shows that :

$$\frac{p}{\rho v^2} = \phi\left(\frac{l}{D}\right)\left(\frac{\rho v D}{\mu}\right)$$

$p/(\rho v^2)$ is a form of the Newton number and $(\rho vD/\mu)$ is a form of the Reynolds number. It could have been arranged for Reynolds number to include l instead of D.

Because we needed two linear dimensions D and l, we now have another dimensionless number (l/D) that is the ratio of the two. In a model test this must be made the same as for the object and if the ratio is the same then geometric similarity exists.

If many such linear dimensions exist in a problem, then many dimensionless numbers will be created which are all the possible ratios of any one with all the others. To avoid all this work, we usually just assume a characteristic length. This is valid when geometric similarity exists as will become apparent.

We may express our equation as :

$$Ne = \phi\left(\frac{l}{d}\right)(\mathrm{Re})$$

Removing the function sign gives :

$$Ne = K\left(\frac{l}{d}\right)(\mathrm{Re})^n$$ where K is the constant of proportionality.

If we make the value of R_e the same on the model and the real object and if we have geometric similarity , then since the function is the same for both (K and n) then it follows that the Newton number must be the same also. In other words since

$$Ne_{object} = K\left(\frac{l}{d}\right)(\mathrm{Re})^n{}_{object} = Ne_{\mathrm{mod}el} = K\left(\frac{l}{d}\right)(\mathrm{Re})^n{}_{\mathrm{mod}el}$$

Then $\{Ne\}_{object} = \{Ne\}_{model}$

From this the resisting force may be predicted. Note that if we had many linear dimensions and many ratios like l/D, then they would also cancel so it is not necessary to include them, just a characteristic length. Let us finish this problem now as a worked example.

WORKED EXAMPLE No.8

The pipe in the previous analysis is 200 m long and 0.5 m diameter and must carry water with a mean velocity of 0.2 m/s. In order to predict the pressure drop, a model is made to a scale of 1/10. Calculate the velocity at which water must flow in the model in order to obtain dynamic similarity.

SOLUTION

For this section we must obtain dynamic similarity by equating the Reynolds numbers. Hence :

$$(\rho vD/\mu)_{model} = (\rho vD/\mu)_{object}$$

The density and viscosity will be the same in both since the same water is used so

$$(vD)_{model} = (vD)_{object}$$

$$v_{model} \times D/10 = 2 \times D \quad \text{hence } v_{model} = 2 \text{ m/s}$$

When the model is tested at the velocity, the pressure drop is found to be 100kPa. Predict the pressure drop in the real pipe.

Since R_e is now the same and l/D is the same for both cases then the Newton number is the same so

$$p/(\rho v^2)_{model} = p/(\rho v^2)_{pipe}$$

Again density and viscosity cancel so we have

$$100/2^2 = p/0.2^2$$

p = 1 kPa on the full size pipe.

WORKED EXAMPLE No.9a

The resistance to motion R of a hydrofoil depends upon the characteristic length l, the velocity v, the density ρ and the acceleration of gravity g.

It may be shown that $N_e = f(F_r)$ where $Ne = R/(\rho v^2 l^2)$ and $F_r = v/(gl)^{1/2}$

In order to predict the resistance of a hydrofoil, a model is made to a scale of 1/20. The actual hydrofoil must move at 0.8 m/s over water. Calculate the velocity of the model that gives dynamic similarity on the same water.

SOLUTION

For dynamic similarity the Froude numbers must be made the same.

$v/(gl)^{1/2}{}_{model} = v/(gl)^{1/2}(hydrofoil)$

$v/(l)^{1/2}{}_{model} = v/(l)^{1/2}(hydrofoil)$

$v_{model} \times (20/l)^{1/2} = 0.8/l^{1/2}$

$v_{model} = 0.8/20^{1/2}$

$v_{model} = 0.179$ m/s

WORKED EXAMPLE No.9b

The model is tested at 0.179 m/s and the resistance to motion was found to be 2.2N. Predict the resistance of the hydrofoil at 0.8 m/s.

SOLUTION

Since the Froude number is the same and the function is the same then the Newton number must be the same for both.

$R/(\rho v^2 l^2)_{model} = R/(\rho v^2 l^2)_{hydrofoil}$

Since the density is the same then $\{2.2 \times 20^2/(0.179\ l)^2\} = \{R/(vl)^2$

R = 17 570 N

SELF ASSESSMENT EXERCISE No.4

1. (a) The viscous torque produced on a disc rotating in a liquid depends upon the characteristic dimension D, the speed of rotation N, the density ρ and the dynamic viscosity μ. Show that :

$$\{T/(\rho N^2 D^5)\} = f(\rho N D^2/\mu)$$

(b) In order to predict the torque on a disc 0.5 m diameter which rotates in oil at 200 rev/min, a model is made to a scale of 1/5. The model is rotated in water. Calculate the speed of rotation for the model which produces dynamic similarity.

For the oil the density is 750 kg/m^3 and the dynamic viscosity is 0.2 Ns/m^2.

For water the density is 1000 kg/m^3 and the dynamic viscosity is 0.001 Ns/m^2.

(The answer is 18.75 rev/min)

(c) When the model is tested at 18.75 rev/min the torque was 0.02 Nm. Predict the torque on the full size disc at 200 rev/min. (Ans 5 333 N)

2. The resistance to motion of a submarine due to viscous resistance is given by :

$\{R/(\rho v^2 D^2)\} = f(\rho v D/\mu)$ where D is the characteristic dimension.

The submarine moves at 8 m/s through sea water. In order to predict its resistance, a model is made to a scale of 1/100 and tested in fresh water. Determine the velocity at which the model should be tested. (690.7 m/s)

The density of sea water is 1036 kg/m^3

The density of fresh water is 1000 kg/m^3

The viscosity of sea water is 0.0012 N s/m^2.

The viscosity of fresh water is 0.001 N s/m^2.

When run at the calculated speed, the model resistance was 200 N. Predict the resistance of the submarine. (278 N).

3. The resistance of an aeroplane is due to, viscosity and compressibility of the fluid. Show that:

$$\{R/(\rho v^2 D^2)\} = f(M_a)(R_e)$$

An aeroplane is to fly at an altitude of 30 km at Mach 2.0. A model is to be made to a suitable scale and tested at a suitable velocity at ground level. Determine the velocity of the model that gives dynamic similarity for the Mach number and then using this velocity determine the scale which makes dynamic similarity in the Reynolds number. (680.6 m/s and 1/61.86)

The properties of air are

sea level	a= 340.3 m/s	μ= 1.7897 x 10^{-5}	ρ = 1.225 kg/m^3
30 km	a= 301.7 m/s	μ= 1.4745 x 10^{-5}	ρ = 0.0184 kg/m^3

When built and tested at the correct speed, the resistance of the model was 50 N. Predict the resistance of the aeroplane. (2 259 N).

4. The force on a body of length 3 m placed in an air stream at 1 bar and moving at 60 m/s is to be found by testing a scale model. The model is 0.3 m long and placed in high pressure air moving at 30 m/s. Assuming the same temperature and viscosity, determine the air pressure which produced dynamic similarity.

The force on the model is found to be 500 N. Predict the force on the actual body. (Ans. 20 bar and 10 kN).

5. Show by dimensional analysis that the velocity profile near the wall of a pipe containing turbulent flow is of the form $u^+ = f(y^+)$

where $u^+ = u(\rho/\tau_o)^{1/2}$ and $y^+ = y(\rho\tau_o)^{1/2}/\mu$

When water flows through a smooth walled pipe 60 mm bore diameter at 0.8 m/s, the velocity profile is $u^+ = 2.5\ln(y^+) + 5.5$

Find the velocity 10 mm from the wall.
The friction coefficient is $C_f = 0.079\, R_e^{-0.25}$.

Answer 0.85 m/s