FLUID MECHANICS

TUTORIAL No.5

POTENTIAL FLOW

In this tutorial you will study the flow of ideal fluids. On completion you should be able to do the following.

- Define the stream function.
- Define the velocity potential.
- Understand the flow of an ideal fluid around a long cylinder.
- Understand the main points concerning vortices.

An ideal fluid has no viscosity (inviscid) and is incompressible. No such fluid exists but these assumptions make it possible to produce models for the flow of fluids in and around solid boundaries such as long cylinders. In particular, the concepts of POTENTIAL FLOW and STREAM FUNCTION give us useful mathematical models to study these phenomena.

1 STREAM FUNCTION

Consider the streamlines representing a 2 dimensional flow of a perfect fluid.

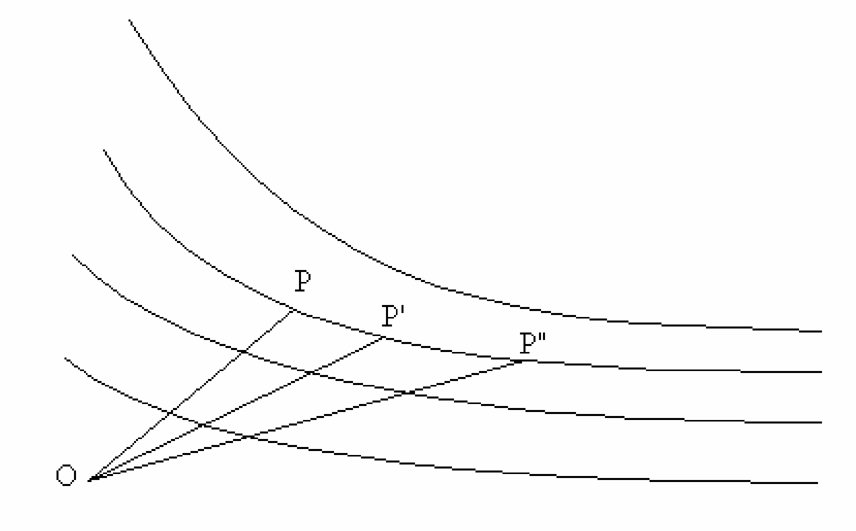


Figure 1

Flux is defined as the volume flow rate per metre depth normal to the page. The stream function is defined as the flux across the line O -P. The symbol used is ψ (psi). Since there is no flow rate normal to a stream line, then it follows that the stream function is the same between O and any point P, P' or P" on the same stream line. In other words, the stream line represents a constant value of the stream function.

It is easier to understand ψ in terms of small changes. Consider a short line of length ds perpendicular to a stream line. Let the velocity across this line have a mean value of v'. The flux crossing this line is hence v'ds and this is the small change in the stream function $d\psi$. It follows that $d\psi = v'ds$

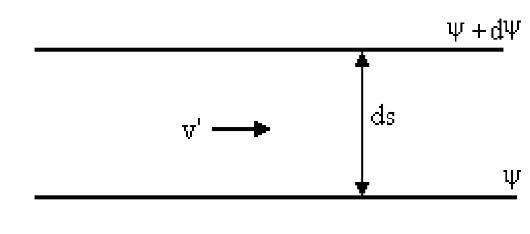


Figure 2

In this analysis, the stream function is positive when it crosses the line in an anti-clockwise direction (right to left on the diagram). This is quite arbitrary with some publications using clockwise as positive, others using anti-clockwise.

The stream function may be expressed with Cartesian or polar co-ordinates. The convention for velocity is that we use v for velocity in the y direction and u for velocity in the x direction. Consider a small flux entering a triangular area as shown. The fluid is incompressible so the volume per unit depth entering the area must be equal to that leaving. It follows that for a flux in the direction shown

$$d\psi + u\,dy = v dx \quad \text{and } d\psi = v'ds$$

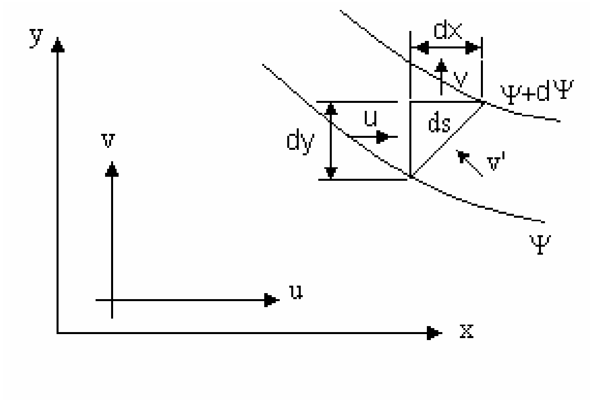


Figure 3

If the stream line is horizontal v' is velocity u and ds is dy hence $u = -d\psi/dy$

If the stream is vertical then v' is v and ds is dx hence $v = d\psi/dx$

When polar co-ordinates are used the flow directions are radial and tangential.

If the flow is radial and θ = 0, then v' becomes v_R and ds is r dθ.

$$v_R = -\frac{dy}{rdq}$$

If the flow is tangential and θ = 90° then v' becomes v_T and ds is dr hence

$$v_T = \frac{dy}{dr}$$

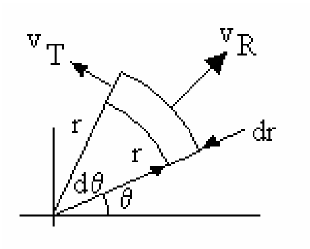


Figure 4

The sign convention agrees with the stream function being positive in a direction from right to left.

2 VELOCITY POTENTIAL

The velocity potential has a symbol φ. It is best explained as follows.

Consider a line along which the velocity v' varies. Over a short length ds the velocity potential varies by dφ. Hence dφ = v'ds or v' = dφ/ds. The velocity potential may be thought of as the product of velocity and length in the same direction. It follows that

$$\phi = \int v' \, ds$$

Some text books use a sign convention opposite to this and again this is arbitrary.

If the line is horizontal v' is velocity u and ds is dx hence $u = \frac{df}{dx}$

If the line is vertical then v' is v and ds is dy hence $v = \frac{df}{dy}$

If the flow is radial then v' is v_R and ds is dr hence $v_R = \frac{df}{dr}$

If the flow is tangential then v' is v_T and ds is r dθ hence $v_T = \frac{df}{rdq}$

The sign convention is positive for increasing radius and positive for anti- clockwise rotation.

Since v' is zero perpendicular to a stream line it follows that lines of constant φ run perpendicular to the stream lines. If these lines are superimposed on a flow we have a flow net.

Consider a flow with lines of constant ψ and φ as shown.

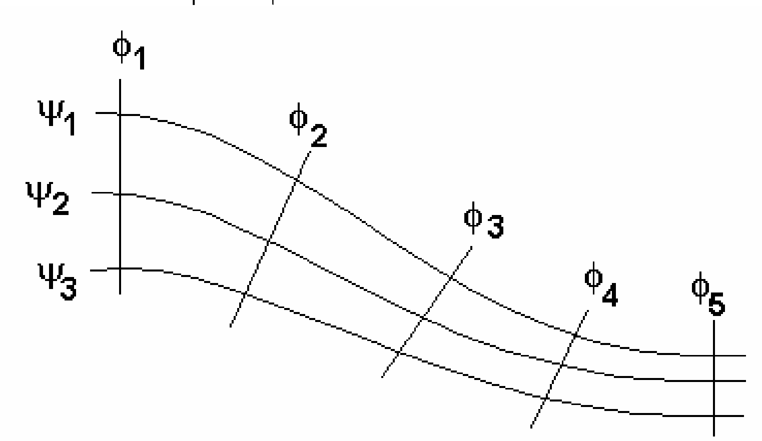


Figure 5

When we compare the velocity equations in terms of the stream function and the velocity potential we find :

$$u = -\frac{d\Psi}{dy} = \frac{df}{dx} \ldots\ldots\ldots\ldots(1)$$

$$v = \frac{d\Psi}{dx} = \frac{df}{dy} \ldots\ldots\ldots\ldots(2)$$

$$v_R = -\frac{d\Psi}{rdq} = \frac{df}{dr} \ldots\ldots\ldots(3)$$

$$v_T = \frac{d\Psi}{dr} = \frac{df}{rdq} \ldots\ldots\ldots\ldots(4)$$

3 UNIFORM FLOW

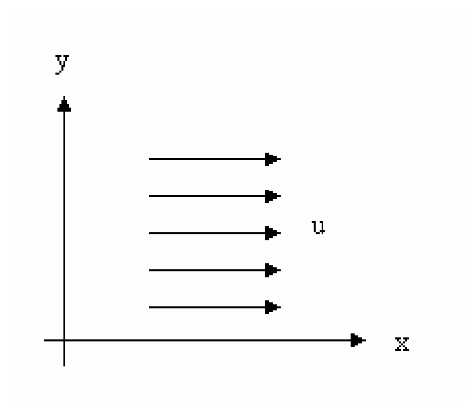


Figure 6

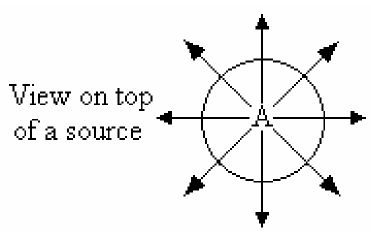
If the flow has a constant velocity u in the x direction and a uniform depth of 1 m then, the stream function is obtained from equation 1 and is $\psi = -uy$

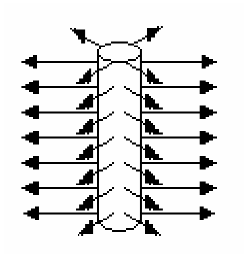
From equation 3,

$$\frac{df}{dr} = -\frac{d(-uy)}{rdq} = -\frac{d(-ur\sin q)}{rdq} = u\cos q$$

4. SOURCE AND SINK

A line source is a single point 1m deep from which fluid appears and flows away radially. A line sink is a single point 1m deep at which flow disappears. The flow rate through any circle centred on the source or sink must be the same at all radii. All radii are stream lines.





STREAM FUNCTION

Consider a source at point A with a flow emerging 1 m deep at a rate of Q m^3/s. At radius r the radial velocity is Q/area = $Q/2\pi r = v_R$. Flux outwards is taken as positive. Some texts use the opposite sign convention.

At radius r the stream function is defined as $d\psi = v_R ds$

ds is a tiny arc. $d\psi = (Q/2\pi r)\, ds$

Note that text books and examiners often use m for the strength of the source and this has the same meaning as Q. A sink is the exact opposite of a source.

$d\psi = -(Q/2\pi r)\,ds$ for a sink.

The arc subtends an angle $d\theta$ and $ds = r\,d\theta$

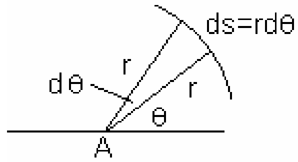


Figure 8

$d\Psi = \frac{Q}{2\pi r} r d\theta = \frac{Q}{2\pi} d\theta$ for a source.

$d\Psi = -\frac{Q}{2\pi r} r d\theta = -\frac{Q}{2\pi} d\theta$ for a sink

For a finite angle θ these become

$\Psi = \frac{Q}{2\pi}\theta$ for a source.

$\Psi = -\frac{Q}{2\pi}\theta$ for a sink.

VELOCITY POTENTIAL

Now consider a length in the radial direction.

$$ds = dr$$

At radius r the velocity potential is defined as

$$d\phi = v_R\,dr$$

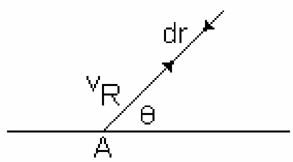


Figure 9

This becomes

$d\phi = \frac{Q}{2\pi r} dr$ for a source

$d\phi = -\frac{Q}{2\pi r} dr$ for a sink.

To find the expression for a length of one radius, we integrate with respect to r.

$\phi = \frac{Q}{2\pi} \ln r$ for a source

$\phi = -\frac{Q}{2\pi} \ln r$ for a sink.

From the preceding it may be deduced that the streamline are radial lines and the lines of constant ϕ are concentric circles.

5 DOUBLET

A doublet is formed when an equal source and a sink are brought close together. Consider a source and sink of equal strength placed at A and B respectively. The stream function for point P relative to A and B are respectively

$\Psi_B = \frac{Q}{2p} q_2$ for the source

$\Psi_A = -\frac{Q}{2p} q_1$ for the sink

$$\Psi_P = \Psi_B + \Psi_A = \frac{Q}{2p}(q_2 - q_1)$$

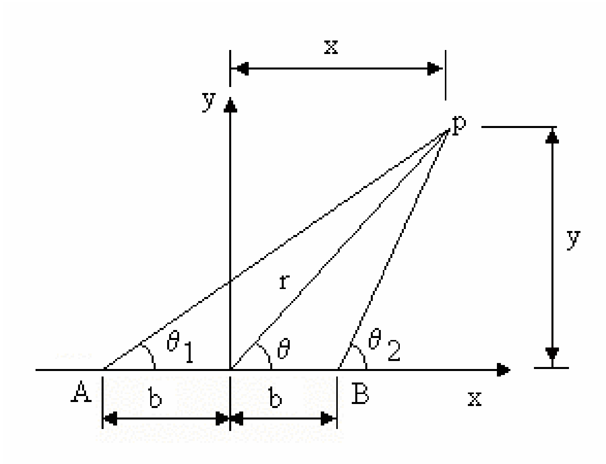


Figure 10

Referring to the diagram

$$\tan\theta_1 = \frac{y}{x+b}, \tan\theta_2 = \frac{y}{x-b}$$

$$\tan(\theta_2 - \theta_1) = \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_2 \tan\theta_1}$$

$$\tan(\theta_2 - \theta_1) = \frac{y(x-b) - y(x+b)}{1 + \frac{y^2}{x^2 - b^2}}$$

$$\tan(\theta_2 - \theta_1) = \frac{2by}{x^2 - b^2 + y^2}$$

As $b \to 0$, $b^2 \to 0$ and the tan of the angle becomes the same as he angle itself in radians.

$$(\theta_2 - \theta_1) = \frac{2by}{x^2 + y^2}$$

$$\Psi_p = \frac{Q}{2\pi}\left[\frac{2by}{x^2 + y^2}\right]$$

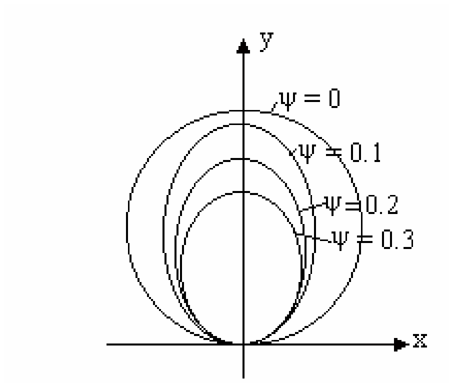
When the source and sink are brought close together we have ***DOUBLET*** but b remains finite.

Let $B = (Qb/\pi)$ $\quad \Psi = \frac{By}{x^2 + y^2}$

Since $y = r\sin\theta$ and $x^2 + y^2 = r^2$ then

$$\Psi = \frac{Br\sin q}{r^2} = \frac{B\sin q}{r}$$

$\psi = 0$ is the streamline across which there is no flux and this is a circle so it can be used to represent a cylinder.



e 11

6. COMBINATION OF UNIFORM FLOW AND SOURCE OR SINK

For this development, consider the case for the source at the origin of the x – y co-ordinates with a uniform flow of velocity u from left to right. The development for a sink in a uniform flow follows the same principles. The uniform flow encounters the flux from the source producing a pattern as shown. At large values of x the flow has become uniform again with velocity u. The flux from the source is Q. this divides equally to the top and bottom. At point s there is a stagnation point where the radial velocity from the source is equal and opposite of the uniform velocity u.

The radial velocity is $Q/2\pi r$. Equating to u we have $\mathbf{r = Q/2\pi u}$ and this is the distance from the origin to the stagnation point.

For uniform flow $\Psi_1 = -uy$ For the source $\Psi_2 = Q\theta/2\pi$. The combined value is $\Psi = \mathbf{-uy + Q\theta/2\pi}$

The flux between the origin and the stagnation point S is half the flow from the source. Hence, the flux is Q/2 and the angle θ is π radian (180°). The dividing streamline emanating from S is the zero streamline $\Psi = 0$. Since no flux crosses this streamline, the dividing streamline could be a solid boundary. When the flow is uniform, we have:

$\Psi = 0 = -uy + Q\theta/2\pi = -uy + Q\pi/2\pi$ $\qquad$ $y = -uy + Q/2$ $\qquad$ $y = Q/2u$

y is the distance from the x axis to the zero stream line where the flow is uniform (at large values of x). The thickness of the uniform stream emerging from the source is t = 2y.

Hence $\qquad$ **t =Q/u**.

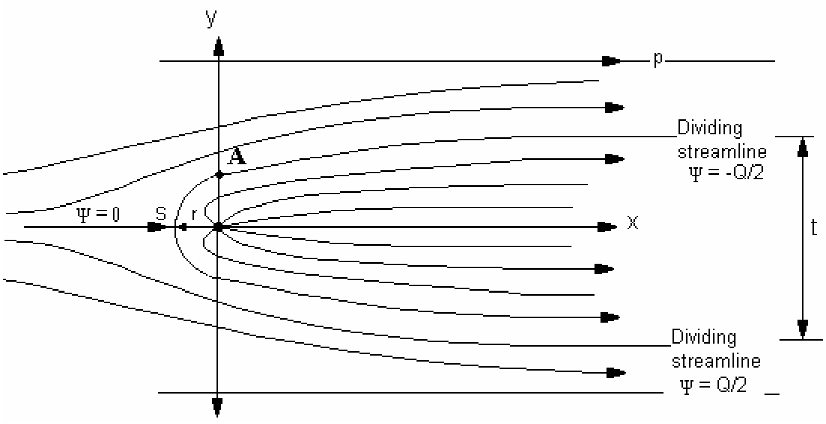


Figure 12

PRESSURE

Consider points S and A. At S there is a pressure p_s and no velocity. At point A there is a velocity v_A and pressure p_A. Applying Bernoulli between these points, we have:

$p_s = p_A + \rho v_A^2/2$

$p_s - p_A = \rho v_A^2/2$

To solve the pressure difference we need to know the velocity. At point A we can solve this as follows. The velocity is the resultant velocity of the uniform flow u and the radial velocity from the source v_R.

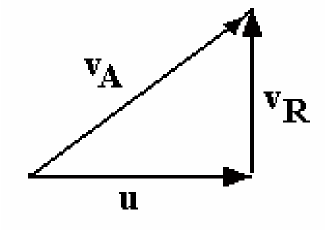


Figure 13

$v_A{}^2 = u^2 + v_R{}^2$

The stream line at point A is $\Psi = 0$ hence $0 = -uy + Q\theta/2\pi$ hence $y = Q\theta/2\pi u$

At this point $\theta = \pi/2$ (90^o) so $y = Q/4u$

This is the distance to point A along the y axis.

$v_R = Q/2\pi r$. The radius at point A is Q/4u hence $v_R = 2u/\pi$

$v_A{}^2 = u^2 + (2u/\pi)^2 = u^2 \{1 + 4/\pi^2\}$

$p_s - p_A = \rho\, v_A{}^2/2 = (\rho u^2/2)\{1 + 4/\pi^2\}$

WORKED EXAMPLE No.1

A uniform flow of fluid with a density of 800 kg/m^3 is from left to right with a velocity u = 2 m/s. It is combined with a source of strength Q = 8 m^2/s at the origin. Calculate:

1. The distance to the stagnation point.
2. The width of the flow stream emanating from the source when it has reached a uniform state.
3. The pressure difference between the stagnation point and the point where the zero streamline crosses the y axis.

SOLUTION

From the preceding work

Distance to stagnation point = $Q/2\pi u = 8/(2\pi \times 2) = 2/\pi$ metres

$t = Q/u = 8/2 = 4$ m

$p_s - p_A = (\rho u^2/2)\{1 + 4/\pi^2\}$

$p_s - p_A = (800 \times 2^2/2)\{1 + 4/\pi^2\}$

$p_s - p_A = 1600(1.405) = 2248\ N/m^2$

If a sink is placed at the origin, the flow pattern is like this.

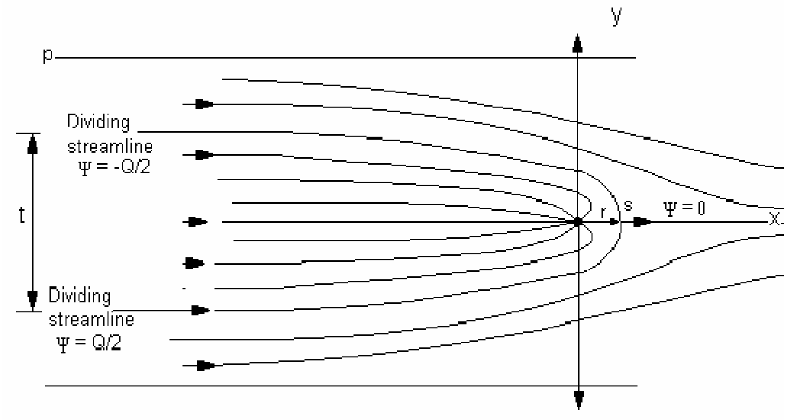


Figure 14

The analysis is similar and yields the same result.

7. FLOW AROUND A LONG CYLINDER

When an ideal fluid flows around a long cylinder, the stream lines and velocity potentials form the same pattern as a doublet placed in a constant uniform flow. It follows that we may use a doublet to study the flow pattern around a cylinder. The result of combining a doublet with a uniform flow at velocity u is shown below.

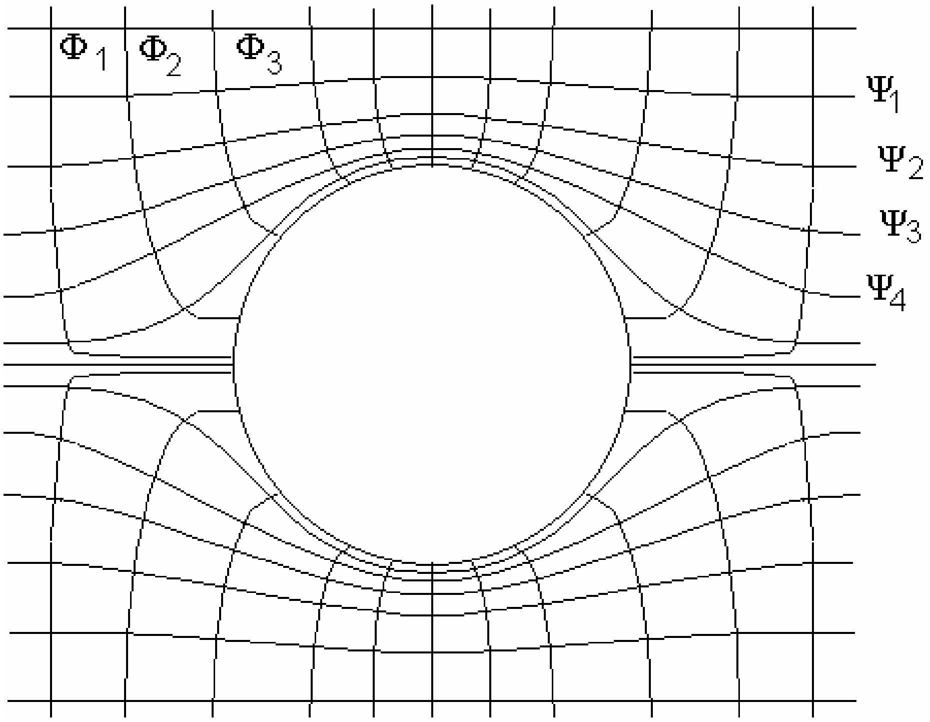


Figure 15

Consider a doublet at the origin with a uniform flow from left to right. The stream function for point p is obtained by adding the functions for a doublet and a uniform flow.

For a doublet is $\psi = B\sin\theta/r$

For a uniform flow $\psi = -uy$.

For the combined flow is $\psi = B\sin\theta/r - uy$

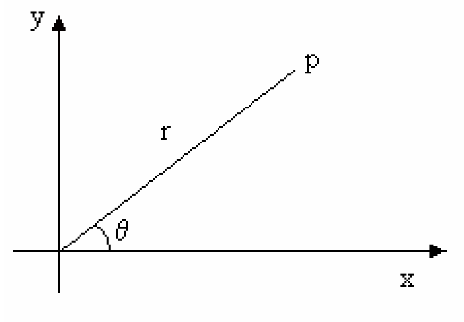


Figure 16

Where B = (Qb/π) From the diagram we have y = r sinθ and substituting this into the stream function gives

$$\Psi = \frac{B\sin\theta}{r} - ur\sin\theta = \left(\frac{B}{r} - ur\right)\sin\theta$$

$$\frac{d\Psi}{dr} = \left(\frac{-B}{r^2} - u\right)\sin\theta \qquad (5)$$

$$\Psi = \left(\frac{B}{r} - ur\right)\cos\theta \qquad (6)$$

The equation is usually given in the form $\Psi = \left(\frac{B}{r} - Ar\right)\cos\theta$ where A = u

The stream functions may be converted into velocity potentials by use of equations 3 and 5 or 4 and 6 as follows.

Equation 4

$$v_T = \frac{d\Psi}{dr} = \frac{d\phi}{r\,d\theta}$$

$$\frac{d\Psi}{dr} = \frac{d\phi}{r\,d\theta}$$

$$r\frac{d\Psi}{dr} = \frac{d\phi}{d\theta}$$

$$r\left\{-\frac{B}{r^2} - A\right\}\sin\theta = \frac{d\phi}{d\theta}$$

$$\left\{-\frac{B}{r} - Ar\right\}\sin\theta = \frac{d\phi}{d\theta}$$

$$\left\{\frac{B}{r} + Ar\right\}\cos\theta = \phi$$

Equation 3

$$v_R = -\frac{d\Psi}{r\,d\theta} = \frac{d\phi}{dr}$$

$$-\frac{d\Psi}{d\theta} = r\frac{d\phi}{dr}$$

$$-\left(\frac{B}{r} - Ar\right)dr\cos\theta = r\frac{d\phi}{dr}$$

$$-\left(\frac{B}{r^2} + A\right)dr\cos\theta = d\phi$$

$$\left(\frac{B}{r} + Ar\right)\cos\theta = \phi$$

At any given point in the flow with co-ordinates r,θ the velocity has a radial and tangential component. The true velocity v_θ is the vector sum of both which, being at a right angle to each other, is found from Pythagoras as

$$v_\theta = \sqrt{v_R^2 + v_T^2}$$

From equation 3 and 4 we can show that

$$v_R = -u\left\{\frac{R^2}{r^2} - 1\right\}\cos\theta ....(7)$$

$$v_T = -u\left\{\frac{R^2}{r^2} + 1\right\}\sin\theta ....(8)$$

R is the radius of the cylinder. From these equations we may find the true velocity at any point in the flow.

<u>WORKED EXAMPLE No. 2</u>

The velocity potential for an ideal fluid flowing around a long cylinder is given by

$$\left\{\frac{B}{r} + Ar\right\}\cos\theta = \phi$$

The cylinder has a radius R and is placed in a uniform flow of velocity u, which affects the velocity near to the cylinder. Determine the constants A and B and determine where the maximum velocity occurs.

<u>SOLUTION</u>

The values of the constants depend upon the quadrant selected to solve the boundary conditions. This is because the sign of the tangential velocity and radial velocity are different in each quadrant. Which ever one is used, the final result is the same. Let us select the quadrant from 90° to 180°.

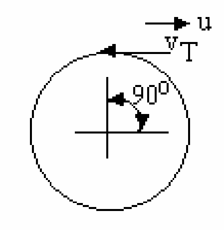
At a large distance from the cylinder and at the 90° position the velocity is from left to right so at this point v_T = -u. From equation 4 we have

$$v_T = \frac{d\phi}{r d\theta}$$

$$\phi = \left\{\frac{B}{r} + Ar\right\}\cos\theta$$

$$v_T = -\frac{1}{r}\left\{\frac{B}{r} + Ar\right\}\sin\theta$$

$$v_T = -\left\{\frac{B}{r^2} + A\right\}\sin\theta$$



Putting r = infinity and θ = 90^o and remembering that +v_T is anticlockwise +u is left to right, we have

$$v_T = -u = -\left\{\frac{B}{r^2} + A\right\}\sin\theta = -\{0 + A\}x1$$

Hence v_T = -A = -u so A = u as expected from earlier work.

At angle 180^o with r = R, the velocity is only radial in directions and is zero because it is arrested.

From equation 3 we have $$v_R = \frac{df}{dr} = \left(-\frac{B}{r^2} + A\right)\cos q$$

Putting r = R and v_R = 0 and θ = 180 we have

$$0 = \left(-\frac{B}{R^2} + A\right)(-1)$$

$$0 = \left(\frac{B}{R^2} - A\right)$$

Put A = u $\quad 0 = \frac{B}{R^2} - u \qquad B = uR^2$

Substituting for B = uR^2 and A = u we have

$$f = \left\{\frac{B}{r} + Ar\right\}\cos q = \left\{\frac{uR^2}{r} + ur\right\}\cos q$$

At the surface of the cylinder r = R the velocity potential is

$$f = \{uR + uR\}\cos q = 2uR\cos q$$

The tangential velocity on the surface of the cylinder is then

$$v_T = \frac{df}{rdq} = -\left\{\frac{B}{r^2} + A\right\}\sin q$$

$$v_T = -\left\{\frac{uR^2}{r^2} + u\right\}\sin q$$

$$v_T = -2u\sin q$$

This is a maximum at θ = ± 90^o where the streamlines are closest together so the maximum velocity is 2u on the top and bottom of the cylinder.

WORKED EXAMPLE No.3

The potential for flow around a cylinder of radius a is given by

$$\phi = ux\left[1 + \frac{a^2}{x^2 + y^2}\right]$$

where x and y are the Cartesian co-ordinates with the origin at the middle. Derive an expression for the stream function ψ.

SOLUTION

First convert from Cartesian to polar co-ordinates.

$x^2 + y^2 = r^2$ $\qquad$ $x = r\cos\theta$

$$\phi = ur\cos\theta\left[1 + \frac{a^2}{r^2}\right]$$

$$v_T = \frac{d\Psi}{dr} = \frac{d\phi}{r\,d\theta}$$

$$r\frac{d\Psi}{dr} = \frac{d\phi}{d\theta} = -u\left[r + \frac{a^2}{r}\right]\sin\theta$$

$$\frac{d\Psi}{dr} = -u\left[1 + \frac{a^2}{r^2}\right]\sin\theta$$

$$\Psi = -u\left[r - \frac{a^2}{r}\right]\sin\theta$$

$$\Psi = ur\left[\frac{a^2}{r^2} - 1\right]\sin\theta$$

Now change back to Cartesian co-ordinates

$$\Psi = ur\left[\frac{a^2}{x^2 + y^2} - 1\right]\sin\theta = uy\left[\frac{a^2}{x^2 + y^2} - 1\right]$$

8. PRESSURE DISTRIBUTION AROUND

The velocity of the main stream flow is u and the pressure is p'. When it flows over the surface of the cylinder the pressure is p because of the change in velocity. The pressure change is p - p'.

The dynamic pressure for a stream is defined as $\rho u^2/2$

The pressure distribution is usually shown in the dimensionless form

$$2(p - p')/(\rho u^2)$$

For an infinitely long cylinder placed in a stream of mean velocity u we apply Bernoulli's equation between a point well away from the stream and a point on the surface. At the surface the velocity is entirely tangential so :

$$p' + \rho u^2/2 = p + \rho v_T^2/2$$

From the work previous this becomes

$$p' + \rho u^2/2 = p + \rho(2u \sin\theta)^2/2$$

$$p - p' = \rho u^2/2 - (\rho/2)(4u^2 \sin^2\theta) = (\rho u^2/2)(1 - 4\sin^2\theta)$$

$$(p - p')/(\rho u^2/2) = 1 - 4\sin^2\theta$$

If this function is plotted against angle we find that the distribution has a maximum value of 1.0 at the front and back, and a minimum value of -3 at the sides.

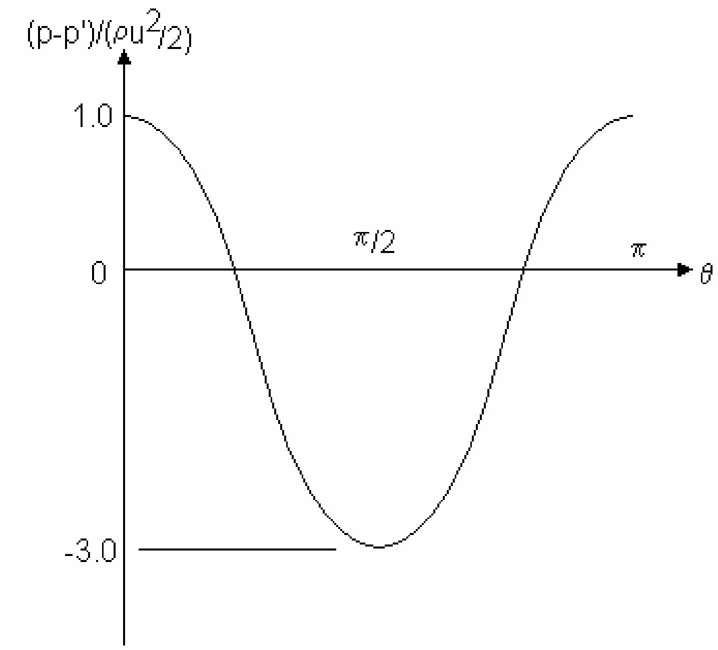


Figure 17

9. THE FLOW OF REAL INCOMPRESSIBLE FLUIDS AROUND A CYLINDER

This is covered in detail in tutorial 3. When the fluid is real, it has viscosity and where it flows over a surface a boundary layer is formed. Remember a boundary layer is the thickness of the layer in which the velocity grows from zero at the surface to a maximum in the main stream.

When the fluid flows around a cylinder, the tangential velocity reaches a theoretical maximum on the top edge. This means the velocity increases around the leading edge. The flow may be laminar or turbulent depending on conditions. If it remains laminar, then the boundary layer gets thinner as shown below. A point may be reached where the layer thickness is reduced to zero and then it actually becomes reversed with eddies forming as shown. At this point the boundary layer separates from the surface and a wake is formed.

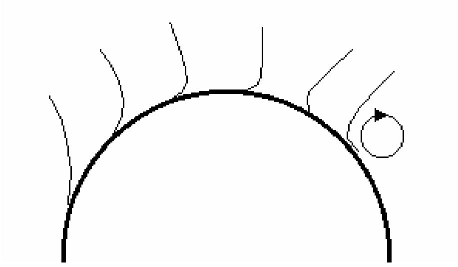


Figure 18

Research shows that the drag coefficient reduces with increased stream velocity and then remains constant when the boundary layer achieves separation. If the mainstream velocity is further increased, turbulent flow sets in around the cylinder and this produces a marked drop in the drag. This is shown below on the graph of C_D against Reynolds's number. The point where the sudden drop occurs is at a critical value of Reynolds's number of 5×10^5.

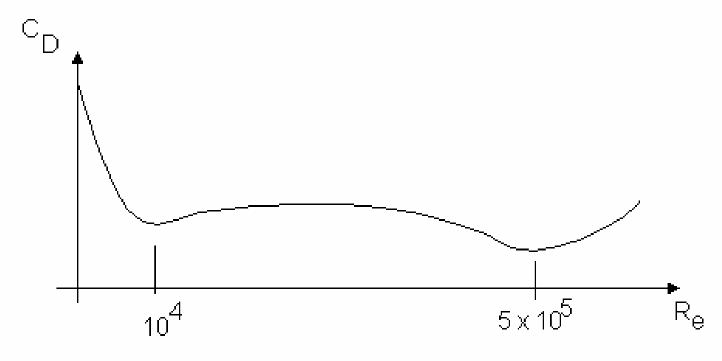


Figure 19

The drag coefficient is defined as : $C_D = \text{Drag Force}/\rho A(u^2/2)$

where A is the area normal to the flow in cases such as this.

The student should read up details of boundary layer formation, wakes and separation as this work is only a brief description of what occurs.

WORKED EXAMPLE No.4

Water flows around a cylinder 80 mm radius. At large distances from the cylinder, the velocity is 7.5 m/s in the x direction and the pressure is 1 bar. Find the velocity and pressure at the point x = -90 mm and y = 20 mm. The velocity and stream functions are as given in the last example.

SOLUTION

$v_R = u[R^2/r^2 - 1]\cos\theta$
$v_T = u[1 + R^2/r^2]\sin\theta$
changing the co-ordinates into angle we have
$\theta = \tan^{-1}(y/x) = 167.5^o$

R = 0.08 m u = 7.5 m/s
$r = (x^2 + y^2)^{1/2} = 92.19$ mm

$v_R = 7.5[0.08^2/0.092^2 - 1]\cos 167.5^o$
$v_R = 1.785$ m/s
$v_T = 7.5[1 + 0.08^2/0.092^2]\sin 167.5^o$
$v_T = 2.85$ m/s

The true velocity is the vector sum of these two so
$v = (1.785^2 + 2.85^2)^{1/2} = 3.363$ m/s

Applying Bernoulli between the mainstream flow and this point we have

$1 \times 10^5/\rho g + 7.5^2/2g = p/\rho g + 3.363^2/2g$

p = 122.47 kPa

SELF ASSESSMENT EXCERCISE No.1

1.a. Show that the potential function $\phi = A(r + B/r)\cos\theta$ represents the flow of an ideal fluid around a long cylinder. Evaluate the constants A and B if the cylinder is 40 mm radius and the velocity of the main flow is 3 m/s. (A = 3 m/s and B = 0.0016)

b. Obtain expressions for the tangential and radial velocities and hence the stream function ψ.

c. Evaluate the largest velocity in the directions parallel and perpendicular to the flow direction. (6 m/s for tangential velocity)

d. A small neutrally buoyant particle is released into the stream at r = 100 mm and $\theta = 150^o$. Determine the distance at the closest approach to the cylinder. (66.18 mm)

2.a. Show that the potential function $\phi = (Ar + B/r)\cos\theta$ gives the flow of an ideal fluid around a cylinder. Determine the constants A and B if the velocity of the main stream is u and the cylinder is radius R.

b. Find the pressure distribution around the cylinder expressed in the form $(p - p')/(\rho u^2/2)$ as a function of angle.

c. Sketch the relationship derived above and compare it with the actual pressure profiles that occur up to a Reynolds number of 5×10^5.

3. Show that in the region y>0 the potential function $\phi = a \ln[x^2 + (y-c)^2] + a \ln[x^2 + (y+c)^2]$ gives the 2 dimensional flow pattern associated with a source distance c above a solid flat plane at y=0.

b. Obtain expressions for the velocity adjacent to the plane at y = 0. Find the pressure distribution along this plane.

c. Derive an expression for the stream function ϕ.

4. A uniform flow has a sink placed in it at the origin of the Cartesian co-ordinates. The stream function of the uniform flow and sink are $\psi_1 = Uy$ and $\psi_2 = B\theta$
Write out the combined stream function in Cartesian co-ordinates.

Given U=0.001 m/s and B= -0.04 $m^3/3$ per m thickness, derive the velocity potential.

Determine the width of the flux into the sink at a large distance upstream.
(Ans. 80π m)

10. VORTICES

10.1 CIRCULATION

Consider a stream line that forms a closed loop. The velocity of the streamline at any point is tangential to the radius of curvature R. the radius is rotating at angular velocity ω. Now consider a small length of that streamline ds.

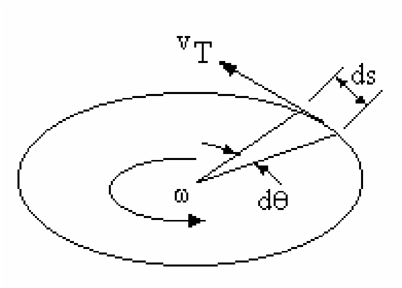


Figure 20

The circulation is defined as $K = \int v_T\, ds$ and the integration is around the entire loop.

Substituting $v_T = \omega R$ $\quad ds = R\, d\theta$

$K = \int \omega R^2\, d\theta$ The limits are 0 and 2π

$K = 2\pi\omega R^2$

In terms of v_T $\quad K = 2\pi v_T R$

10.2 VORTICITY

Vorticity is defined as $G = \int v_T\, ds/A$ where A is the area of the rotating element.

The area of the element shown in the diagram is a small sector of arc ds and angle $d\theta$.

$$dA = \frac{dq}{2p} pR^2 = R^2 \frac{dq}{2}$$

$$A = \int R^2 \frac{dq}{2}$$

$$G = \frac{\int wR^2 dq}{\int R^2 \frac{dq}{2}} = 2w \quad \text{at any point.}$$

It should be remembered in this simplistic approach that ω may vary with angle.

10.3 VORTICES

Consider a cylindrical mass rotating about a vertical axis. The streamlines form concentric circles. The angular velocity of the streamlines are the same at all radii for a forced vortex , but varies with radius for a free vortex.

Consider a small annular element between two streamlines. The streamlines are so close that the circumference of each is the same and length $2\pi r$. Let the depth be dh, a small part of the actual depth.

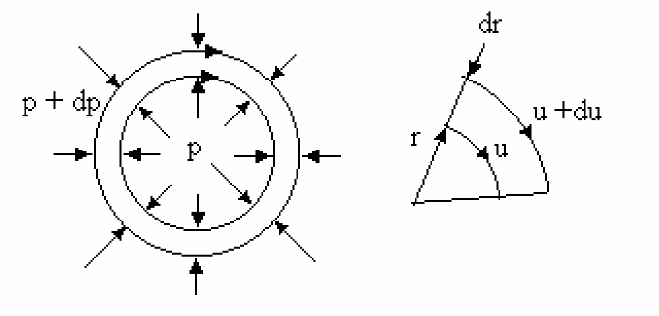


Figure 21

The velocity of the outer streamline is u + du and the inner streamline is u. The pressure at the inner streamline is p and at the outer streamline is p + dp.

The mass of the element is $\rho 2\pi r\ dh\ dr$

The centrifugal force acting on the mass is $\rho 2\pi r\ dh\ dr\ u^2/r$

It must be the centrifugal force acting on the element that gives rise to the change in pressure dp. It follows that

$dp\ 2\pi r\ dh = \rho 2\pi r\ dh\ dr\ u^2/r$ and $dp/\rho = u^2\ dr/r$

Changing pressure into head $dp = \rho g\ dh$ so $dh/dr = u^2/gr$

The kinetic head at the inner streamline is $u^2/2g$

Differentiating w.r.t. radius we get $u\ du/(g\ dr)$

The total energy may be represented as a Head H where H = Total Energy/mg

The rate of change of energy head with radius is dH/dr. It follows that this must be the sum of the rate of change of pressure and kinetic heads so

$dH/dr = u^2/gr + u\ du/(g\ dr)$

10.3.1 FREE VORTEX

A free vortex is one with no energy added nor removed so dH/dr = 0. It is also irrotational which means that although the streamlines are circle and the individual molecules orbit the axis of the vortex, they do not spin. This may be demonstrated practically with a vorticity meter that is a float with a cross on it. The cross can be seen to orbit the axis but not spin as shown.

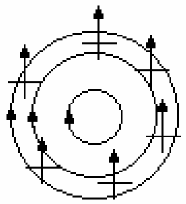


Figure 22

Since the total head H is the same at all radii it follows the dH/dr = 0. The equation reduces to

$$u/r + du/dr = 0$$
$$dr/r + du/v = 0$$

Integrating $\ln u + \ln r = \text{Constant}$

$$\ln(ur) = \text{constant}$$
$$\mathbf{ur = C}$$

Note that a vortex is positive for anti-clockwise rotation. C is the strength of the free vortex with units of m^2/s

10.3.2 STREAM FUNCTION FOR A FREE VORTEX

The tangential velocity was shown to be linked to the stream function by

$$d\Psi = v_T\, dr$$

Substituting $v_T = C/r$

$$d\Psi = C\, dr/r$$

Suppose the vortex has an inner radius of a and an outer radius of R.

$$\Psi = C\int dr/r = C\ln(R/a)$$

10.3.3 VELOCITY POTENTIAL FOR A FREE VORTEX

The velocity potential was defined in the equation $d\phi = v_T\, r\, d\theta$

Substituting $v_T = C/r$ and integrating.

$$\phi = \int (C/r) r\, d\theta$$

Over the limits 0 to θ we have

$$\phi = C\theta$$

10.3.4 SURFACE PROFILE OF A FREE VORTEX

It was shown earlier that $dh/dr = u^2/gr$ where h is the depth. Substituting u=C/r we get

$$dh/dr = C^2/gr^3$$
$$dh = C^2 gr^{-3}\ dr/g$$

Integrating between a small radius r and large radius R we get

$$\mathbf{h_2 - h_1 = (C^2/2g)(1/R^2 - 1/r^2)}$$

Plotting h against r produces a shape like this.

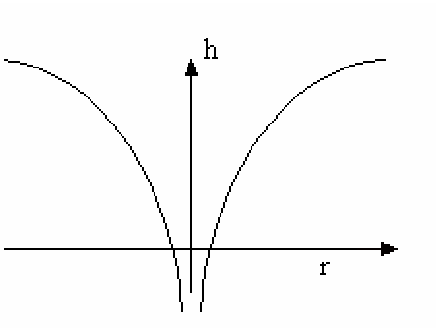


Figure 23

10.3.5 FORCED VORTEX

A forced vortex is one in which the whole cylindrical mass rotates at one angular velocity ω. It was shown earlier that $dH/dr = u^2/gr + u\ du/(g\ dr)$ where h is the depth. Substituting $u = \omega r$ and noting $du/dr = \omega$ we have

$$dH/dr = (\omega r)^2/gr + \omega^2 r/g$$
$$dH/dr = 2\omega^2 r/g$$

Integrating without limits yields

$$H = \omega^2 r^2/g + A$$

H was also given by

$$H = h + u^2/2g = h + \omega^2 r^2/2g$$

Equating we have

$$h = \omega^2 r^2/2g + A$$

At radius r $h_1 = \omega^2 r^2/2g + A$

At radius R $h_2 = \omega^2 R^2/2g + A$

$$\mathbf{h_2 - h_1 = (w^2/2g)(R^2 - r^2)}$$

This produces a parabolic surface profile like this.

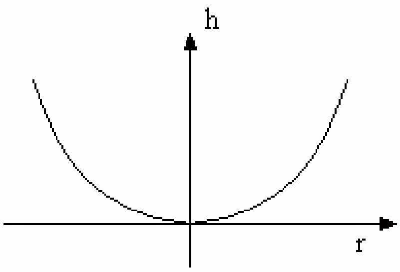


Figure 24

WORKED EXAMPLE No.5

A free vortex of strength C is placed in a uniform flow of velocity u. Derive the stream function and velocity potential for the combined flow.

SOLUTION

The derivation of the stream function and velocity potential for a free vortex is given previously as

$\Psi = C \ln(r/a)$ and $\phi = C\theta$

The corresponding functions for a uniform flow are

$$\Psi = -uy = -ur\sin\theta \text{ and } \phi = ur\cos\theta$$

Combining the functions we get

$$\Psi = C\ln(r/a) - ur\sin\theta$$
$$\phi = C\theta + ur\cos\theta$$

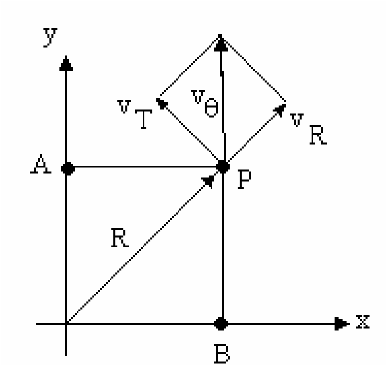
WORKED EXAMPLE No.6

The strength of a free vortex is 2 m^2/s and it is placed in a uniform flow of 3 m/s in the x direction. Calculate the pressure difference between the main stream and a point at x= 0.5 and y=0.5. The density of the fluid is 997 kg/m^3.

SOLUTION

The velocity of the combined flow at this point is v_θ. This the vector sum of the radial and tangential velocities so

$$v_\theta = \{v_T^2 + v_R^2\}^{1/2}$$



C = 2 u = 3

$$v_R = d\phi /dr = u \cos\theta$$
$$v_T = d\Psi /dr = C/r - u \sin\theta$$

At point A $\theta = 90^o$ R = 0.5 $v_R = 0$ hence $v_T = 7$ m/s and $v_{\theta A} = 7$m/s

At point B $\theta = 0^o$ R = 0.5 $v_R = 3$ m/s hence $v_T = 4$ m/s and $v_{\theta B} = 5$ m/s
The mainstream pressure is p and the velocity is u.

Apply Bernoulli between the main stream and point A .

$$p + \rho u^2/2 = p_A + \rho v_{\theta A}^2/2$$

Apply Bernoulli between the main stream and point B .

$$p + \rho u^2/2 = p_B + \rho v_{\theta B}^2/2$$

The pressure difference is then

$$p_A - p_B = (\rho/2)\{v_{\theta B}^2 - v_{\theta A}^2\} = -11\ 964 \text{ Pascal}$$

WORKED EXAMPLE 7

A rectangular channel 1 m deep carries 2 m^3/s of water around a 90^o bend with an inner radius of 2 m and outer radius of 4 m. Treating the around the bend as part of a free vortex, determine the difference in levels between the inner and outer edge.

SOLUTION

Free vortex ur = C m^2/s
ψ = C ln (R/r)

ψ = Flux = Flow/depth = 2 m^2/s and this must be the same across any radial line on the bend.

Putting r = 2 m and R = 4 m

C ln (4/2) = 2 hence C = 2.885

ψ = 2.885 ln (R/r)

The surface profile of a free vortex is $h_2 - h_1 = (C^2/2g)(1/r_1^2 - 1/r_2^2)$

Let the inside level of the bend be 0 so h_2 is the change in level over the bend.

$h_2 = \{(2.885)^2/2g\}(1/2^2 - 1/4^2) = 0.08$ m

SELF ASSESSMENT EXERCISE No.2

1. Define the following terms.

   Stream function.
   Velocity potential function.
   Streamline
   Stream tube
   Circulation
   Vorticity.

2. A free vortex of with circulation $K = 2\pi v_T R$ is placed in a uniform flow of velocity u.

   Derive the stream function and velocity potential for the combined flow.

   The circulation is 7 m^2/s and it is placed in a uniform flow of 3 m/s in the x direction. Calculate the pressure difference between a point at x= 0.5 and y=0.5.

   The density of the fluid is 1000 kg/m^3.

   (Ans. 6695 Pascal)