FLUID MECHANICS
TUTORIAL No. 3

BOUNDARY LAYER THEORY

In order to complete this tutorial you should already have completed tutorial 1 and 2 in this series. This tutorial examines boundary layer theory in some depth.

When you have completed this tutorial, you should be able to do the following.

- Discuss the drag on bluff objects including long cylinders and spheres.
- Explain skin drag and form drag.
- Discuss the formation of wakes.
- Explain the concept of momentum thickness and displacement thickness.
- Solve problems involving laminar and turbulent boundary layers.

Throughout there are worked examples, assignments and typical exam questions. You should complete each assignment in order so that you progress from one level of knowledge to another.

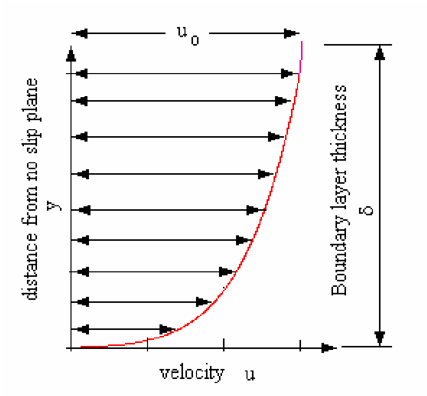
Let us start by examining how drag is created on objects.

1. DRAG

When a fluid flows around the outside of a body, it produces a force that tends to drag the body in the direction of the flow. The drag acting on a moving object such as a ship or an aeroplane must be overcome by the propulsion system. Drag takes two forms, skin friction drag and form drag.

1.1 SKIN FRICTION DRAG

Skin friction drag is due to the viscous shearing that takes place between the surface and the layer of fluid immediately above it. This occurs on surfaces of objects that are long in the direction of flow compared to their height. Such bodies are called ***STREAMLINED***. When a fluid flows over a solid surface, the layer next to the surface may become attached to it (it wets the surface). This is called the 'no slip condition'. The layers of fluid above the surface are moving so there must be shearing taking place between the layers of the fluid. The shear stress acting between the wall and the first moving layer next to it is called ***the wall shear stress*** and denoted τ_w.



The result is that the velocity of the fluid u increases with height y. The boundary layer thickness δ is taken as the distance required for the velocity to reach 99% of u_o. This layer is called the ***BOUNDARY LAYER*** and δ is the boundary layer thickness. Fig. 1.1 Shows how the velocity "u" varies with height "y" for a typical boundary layer.

Fig.1.1

In a pipe, this is the only form of drag and it results in a pressure and energy lost along the length. A thin flat plate is an example of a streamlined object. Consider a stream of fluid flowing with a uniform velocity u_o. When the stream is interrupted by the plate (fig. 1.2), the boundary layer forms on both sides. The diagram shows what happens on one side only.

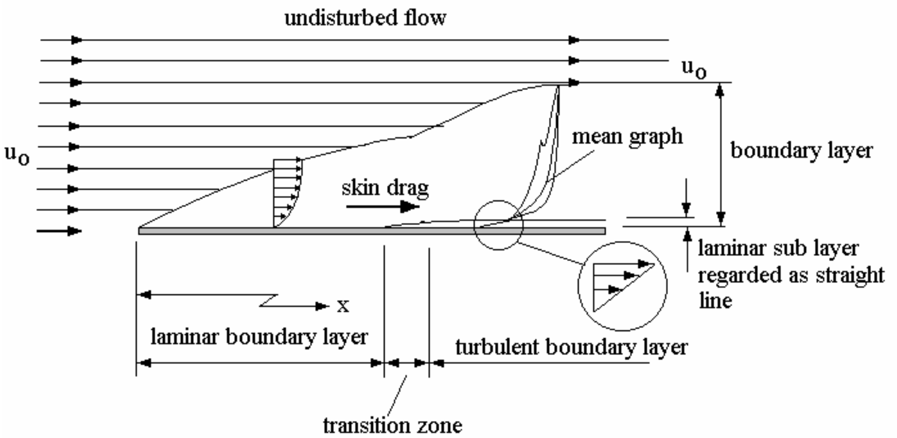


Fig. 1.2

The boundary layer thickness δ grows with distance from the leading edge. At some distance from the leading edge, it reaches a constant thickness. It is then called a ***FULLY DEVELOPED BOUNDARY LAYER***.

The Reynolds number for these cases is defined as: $(R_e)_x = \frac{\rho u_o x}{\mu}$

x is the distance from the leading edge. At low Reynolds numbers, the boundary layer may be laminar throughout the entire thickness. At higher Reynolds numbers, it is turbulent. This means that at some distance from the leading edge the flow within the boundary layer becomes turbulent. A turbulent boundary layer is very unsteady and the streamlines do not remain parallel. The boundary layer shape represents an average of the velocity at any height. There is a region between the laminar and turbulent section where transition takes place

The turbulent boundary layer exists on top of a thin laminar layer called the ***LAMINAR SUB LAYER***. The velocity gradient within this layer is linear as shown. A deeper analysis would reveal that for long surfaces, the boundary layer is turbulent over most of the length. Many equations have been developed to describe the shape of the laminar and turbulent boundary layers and these may be used to estimate the skin friction drag.

Note that for this ideal example, it is assumed that the velocity is the undisturbed velocity u_o everywhere outside the boundary layer and that there is no acceleration and hence no change in the static pressure acting on the surface. There is hence no drag force due to pressure changes.

CALCULATING SKIN DRAG

The skin drag is due to the wall shear stress τ_w and this acts on the surface area (wetted area). The drag force is hence: **R = τ_w x wetted area.** The dynamic pressure is the pressure resulting from the conversion of the kinetic energy of the stream into pressure and is defined by the expression $\frac{\rho u_o^2}{2}$ .The drag coefficient is defined as

$$C_{Df} = \frac{\text{Drag force}}{\text{dynamic pressure} \ \text{ x wetted area}}$$

$$C_{Df} = \frac{2R}{\rho u_0^2 \text{ x wetted area}} = \frac{2\tau_w}{\rho u_0^2}$$

Note that this is the same definition for the pipe friction coefficient C_f and it is in fact the same thing. It is used in the Darcy formula to calculate the pressure lost in pipes due to friction. For a smooth surface, it can be shown that $C_{Df} = 0.074\ (R_e)_x^{-1/5}$

$(Re)_x$ is the Reynolds number based on the length. $(R_e)_x = \frac{\rho u_o L}{\mu}$

WORKED EXAMPLE 1.1

Calculate the drag force on each side of a thin smooth plate 2 m long and 1 m wide with the length parallel to a flow of fluid moving at 30 m/s. The density of the fluid is 800 kg/m^3 and the dynamic viscosity is 8 cP.

SOLUTION

$$(R_e)_x = \frac{\rho u_o L}{\mu} = \frac{800 \text{ x } 30 \text{ x } 2}{0.008} = 6 \text{ x } 10^6$$

$$C_{Df} = 0.074 \text{ x } (6 \text{ x} 10^6)^{-\frac{1}{5}} = 0.00326$$

$$\text{Dynamic pressure} = \frac{\rho u_0^2}{2} = \frac{800 \text{ x } 30^2}{2} = 360 \text{ kPa}$$

$$\tau_w = C_{Df} \text{ x dynamic pressure} = 0.00326 \text{ x } 360 \text{ x } 10^3 = 1173.6 \text{ Pa}$$

$$R = \tau_w \text{ x Wetted Area} = 1173.6 \text{ x } 2 \text{ x } 1 = 2347.2 \text{ N}$$

On a small area the drag is dR = τ_w dA. If the body is not a thin plate and has an area inclined at an angle θ to the flow direction, the drag force in the direction of flow is τ_w dA cosθ.

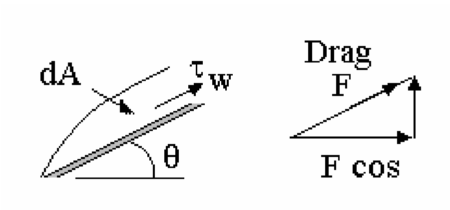


Fig.1.3

The drag force acting on the entire surface area is found by integrating over the entire area.

$$R = \oint \tau_w \cos\theta \, dA$$

Solving this equation requires more advanced studies concerning the boundary layer and students should refer to the classic textbooks on this subject.

SELF ASSESSMENT EXERCISE No. 1

1. A smooth thin plate 5 m long and 1 m wide is placed in an air stream moving at 3 m/s with its length parallel with the flow. Calculate the drag force on each side of the plate. The density of the air is 1.2 kg/m^3 and the kinematic viscosity is 1.6 x 10^{-5} m^2/s.
   (0.128 N)

2. A pipe bore diameter D and length L has fully developed laminar flow throughout the entire length with a centre line velocity u_o. Given that the drag coefficient is given as C_{Df} = 16/Re where Re $= \frac{\rho u_o D}{\mu}$, show that the drag force on the inside of the pipe is given as R=8$\pi\mu u_o$L and hence the pressure loss in the pipe due to skin friction is
   $p_L = 32\mu u_o L/D^2$

1.2 FORM DRAG and WAKES

Form or pressure drag applies to bodies that are tall in comparison to the length in the direction of flow. Such bodies are called ***BLUFF BODIES***.

Consider the case below that could for example, be the pier of a bridge in a river. The water speeds up around the leading edges and the boundary layer quickly breaks away from the surface. Water is sucked in from behind the pier in the opposite direction. The total effect is to produce eddy currents or whirl pools that are shed in the wake. There is a build up of positive pressure on the front and a negative pressure at the back. The pressure force resulting is the form drag. When the breakaway or separation point is at the front corner, the drag is almost entirely due to this effect but if the separation point moves along the side towards the back, then a boundary layer forms and skin friction drag is also produced. In reality, the drag is always a combination of skin friction and form drag. The degree of each depends upon the shape of the body.

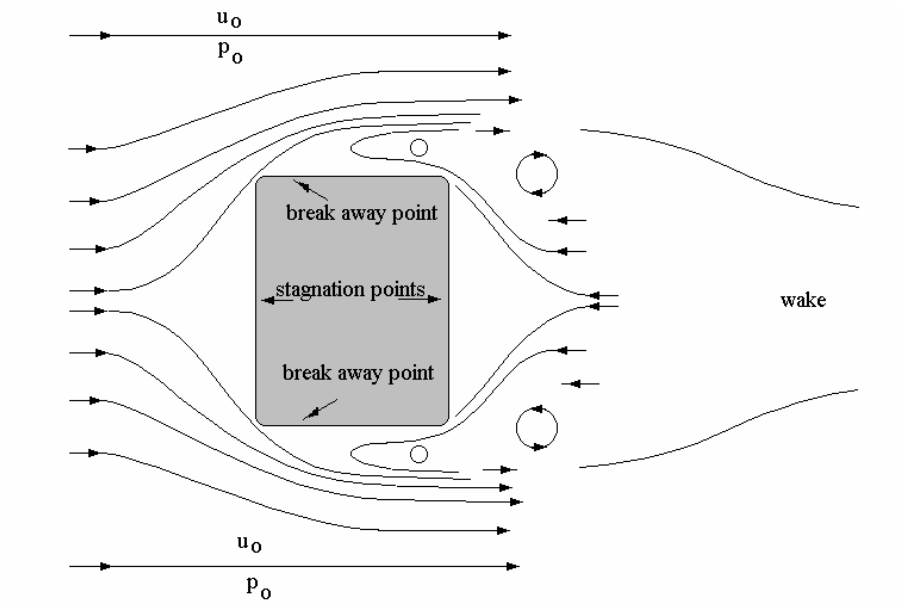


Fig.1.4

The next diagram typifies what happens when fluid flows around a bluff object. The fluid speeds up around the front edge. Remember that the closer the streamlines, the faster the velocity. The line representing the maximum velocity is shown but also remember that this is the maximum at any point and this maximum value also increases as the stream lines get closer together.

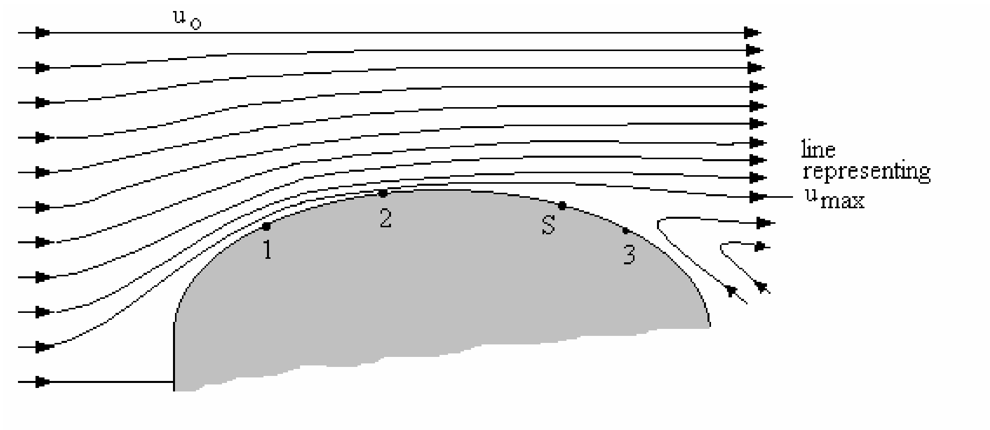


Fig.1.5

Two important effects affect the drag.

Outside the boundary layer, the velocity increases up to point 2 so the pressure acting on the surface goes down. The boundary layer thickness δ gets smaller until at point S it is reduced to zero and the flow separates from the surface. At point 3, the pressure is negative. ***This change in pressure is responsible for the form drag.***

Inside the boundary layer, the velocity is reduced from u_{max} to zero and skin friction drag results.

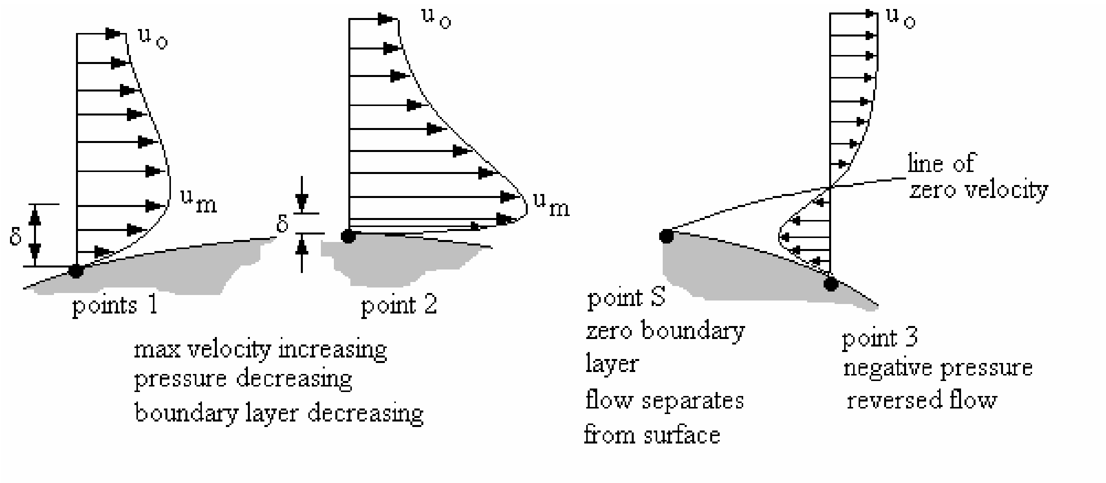


Fig.1.6

In problems involving liquids with a free surface, a negative pressure shows up as a drop in level and the pressure build up on the front shows as a rise in level. If the object is totally immersed, the pressure on the front rises and a vacuum is formed at the back. This results in a pressure force opposing movement (form drag). The swirling flow forms ***vortices*** and the wake is an area of great turbulence behind the object that takes some distance to settle down and revert to the normal flow condition.

Here is an outline of the mathematical approach needed to solve the form drag.

Form drag is due to pressure changes only. The drag coefficient due to pressure only is denoted C_{Dp} and defined as

$$C_{Dp} = \frac{\text{Drag force}}{\text{dynamic pressure x projected area}}$$

$$C_{Dp} = \frac{2R}{\rho u_0^2 \text{ x projected area}}$$

The projected area is the area of the outline of the shape projected at right angles to the flow. The pressure acting at any point on the surface is p. The force exerted by the pressure on a small surface area is p dA. If the surface is inclined at an angle θ to the general direction of flow, the force is p cosθ dA. The total force is found by integrating all over the surface.

$$R = \oint p \cos\theta \, dA$$

The pressure distribution over the surface is often expressed in the form of a pressure coefficient defined as follows.

$$C_p = \frac{2(p - p_o)}{\rho u_o^2}$$

p_o is the static pressure of the undisturbed fluid, u_o is the velocity of the undisturbed fluid and $\frac{\rho u_o^2}{2}$ is the dynamic pressure of the stream.

Consider any streamline that is affected by the surface. Applying Bernoulli between an undisturbed point and another point on the surface, we have the following.

$$p_0 + \frac{\rho u_o^2}{2} = p + \frac{\rho u^2}{2}$$

$$p - p_o = \frac{\rho}{2}(u_o^2 - u^2)$$

$$C_p = \frac{2(p - p_o)}{\rho u_o^2} = \frac{2\left(\frac{\rho}{2}(u_o^2 - u^2)\right)}{\rho u_o^2} = \frac{(u_o^2 - u^2)}{u_o^2} = 1 - \frac{u^2}{u_o^2}$$

In order to calculate the drag force, further knowledge about the velocity distribution over the object would be needed and students are again recommended to study the classic textbooks on this subject. The equation shows that if $u<u_o$ then the pressure is positive and if $u>u_o$ the pressure is negative.

1.3 TOTAL DRAG

It has been explained that a body usually experiences both skin friction drag and form drag. The total drag is the sum of both. This applies to aeroplanes and ships as well as bluff objects such as cylinders and spheres. The drag force on a body is very hard to predict by purely theoretical methods. Much of the data about drag forces is based on experimental data and the concept of a drag coefficient is widely used.

The ***DRAG COEFFICIENT*** is denoted $\mathbf{C_D}$ and is defined by the following expression.

$$C_D = \frac{\text{Resistance force}}{\text{Dynamic pressure x projected Area}}$$

$$C_D = \frac{2R}{\rho u_o^2 \text{ x projected Area}}$$

WORKED EXAMPLE 1.2

A cylinder 80 mm diameter and 200 mm long is placed in a stream of fluid flowing at 0.5 m/s. The axis of the cylinder is normal to the direction of flow. The density of the fluid is 800 kg/m^3. The drag force is measured and found to be 30 N.

Calculate the drag coefficient.

At a point on the surface the pressure is measured as 96 Pa above the ambient level.

Calculate the velocity at this point.

SOLUTION

Projected area = 0.08 x 0.2 =0.016 m^2
R = 30 N
u_o = 0.5 m/s
ρ = 800 kg/m^3
dynamic pressure = $\rho u_o^2/2$ = 800 x 0.5^2/2 = 100 Pa

$$C_D = \frac{\text{Resistance force}}{\text{Dynamic pressure x projected Area}} = \frac{30}{100 \text{ x } 0.016} = 18.75$$

$$p - p_o = \frac{\rho}{2}(u_o^2 - u^2)$$

$$96 = \frac{800}{2}(0.5^2 - u^2)$$

$$\frac{96 \text{ x } 2}{800} = (0.5^2 - u^2)$$

$$0.24 = 0.25 - u^2$$

$$u^2 = 0.01$$

$$u = 0.1 \text{ m/s}$$

1.4 APPLICATION TO A CYLINDER

The drag coefficient is defined as : $C_D = \dfrac{2R}{\rho u_o^2 \text{ x projected Area}}$ The projected Area is LD where L is the length and D the diameter. The drag around long cylinders is more predictable than for short cylinders and the following applies to long cylinders. Much research has been carried out into the relationship between drag and Reynolds number. $Re = \dfrac{\rho u_o d}{\mu}$ and d is the diameter of the cylinder. At very small velocities, (Re <0.5) the fluid sticks to the cylinder all the way round and never separates from the cylinder. This produces a streamline pattern similar to that of an ideal fluid. The drag coefficient is at its highest and is mainly due to skin friction. The pressure distribution shows that the dynamic pressure is achieved at the front stagnation point and vacuum equal to three dynamic pressures exists at the top and bottom where the velocity is at its greatest.

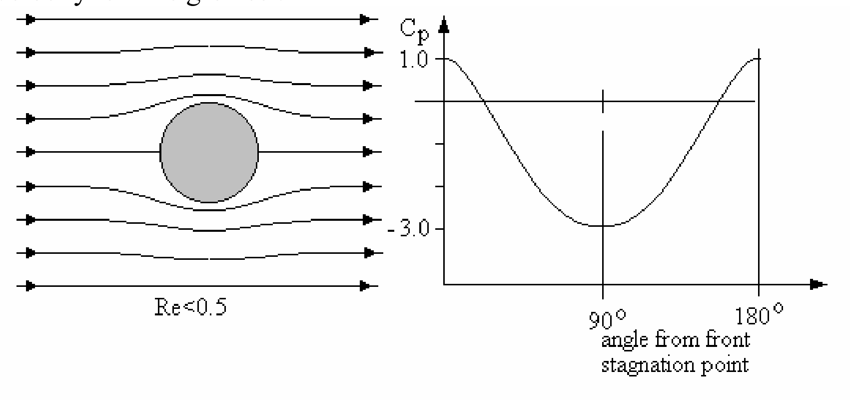


Fig.1.7

As the velocity increases the boundary layer breaks away and eddies are formed behind. The drag becomes increasingly due to the pressure build up at the front and pressure drop at the back.

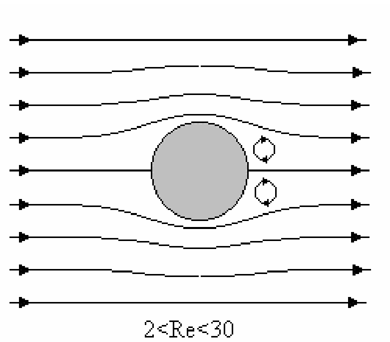


Fig.1.8

Further increases in the velocity cause the eddies to elongate and the drag coefficient becomes nearly constant. The pressure distribution shows that ambient pressure exists at the rear of the cylinder.

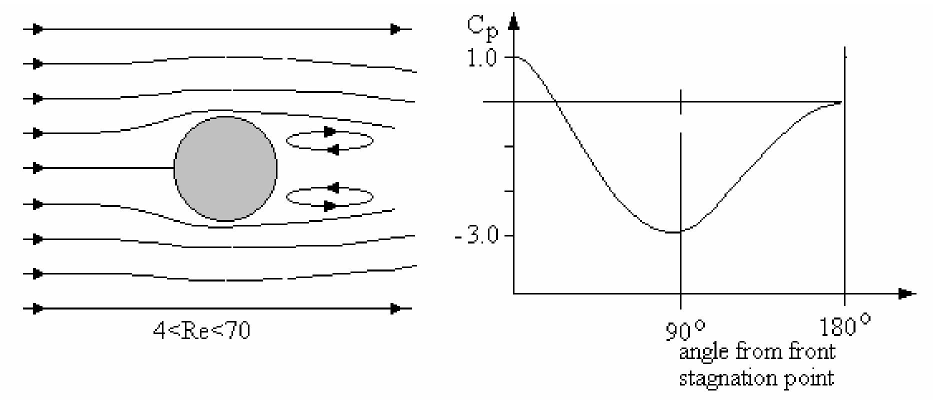


Fig.1.9

At a Reynolds number of around 90 the vortices break away alternatively from the top and bottom of the cylinder producing a vortex street in the wake. The pressure distribution shows a vacuum at the rear.

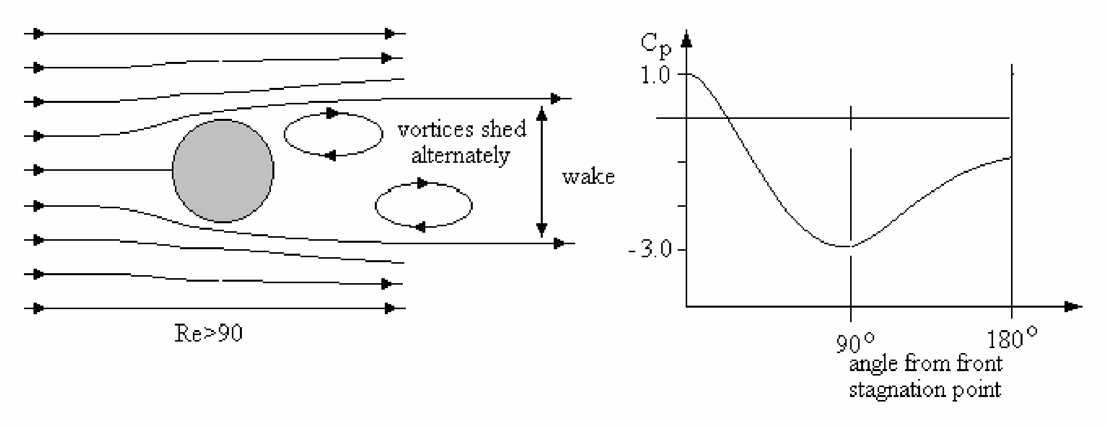


Fig.1.10

Up to a Reynolds number of about 2 x 10^5, the drag coefficient is constant with a value of approximately 1. The drag is now almost entirely due to pressure. Up to this velocity, the boundary layer has remained laminar but at higher velocities, flow within the boundary layer becomes turbulent. The point of separation moves back producing a narrow wake and a pronounced drop in the drag coefficient.

When the wake contains vortices shed alternately from the top and bottom, they produce alternating forces on the structure. If the structure resonates with the frequency of the vortex shedding, it may oscillate and produce catastrophic damage. This is a problem with tall chimneys and suspension bridges. The vortex shedding may produce audible sound.

Fig. 1.12 shows an approximate relationship between C_D and R_e for a cylinder and a sphere.

SELF ASSESSMENT EXERCISE No. 2

1. Calculate the drag force for a cylindrical chimney 0.9 m diameter and 50 m tall in a wind blowing at 30 m/s given that the drag coefficient is 0.8.
   The density of the air is 1.2 kg/m^3. (19.44 N)

2 Using the graph (fig.1.12) to find the drag coefficient, determine the drag force per metre length acting on an overhead power line 30 mm diameter when the wind blows at 8 m/s. The density of air may be taken as 1.25 kg/m^3 and the kinematic viscosity as 1.5×10^{-5} m^2/s. (1.8 N).

1.5 APPLICATION TO SPHERES

The relationship between drag and Reynolds number is roughly the same as for a cylinder but it is more predictable. The Reynolds number is $Re = \frac{\rho u_o d}{\mu}$ where d is the diameter of the sphere. The projected area of a sphere of diameter d is ¼ πd^2. In this case, the expression for the drag coefficient is as follows. $C_D = \frac{8R}{\rho u^2 \text{ x } \pi d^2}$.

At very small Reynolds numbers (less than 0.2) the flow stays attached to the sphere all the way around and this is called Stokes flow. The drag is at its highest in this region.

As the velocity increases, the boundary layer separates at the rear stagnation point and moves forward. A toroidal vortex is formed. For 0.2<Re<500 the flow is called Allen flow.

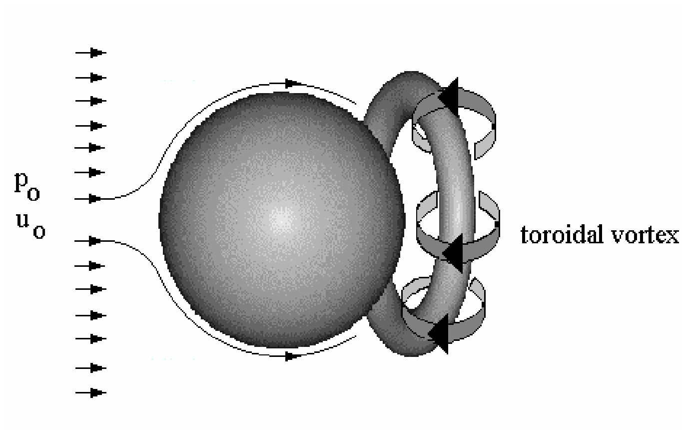


Fig.1.11

The breakaway or separation point reaches a stable position approximately 80° from the front stagnation point and this happens when R_e is about 1000. For 500<R_e the flow is called Newton flow. The drag coefficient remains constant at about 0.4. Depending on various factors, when R_e reaches 10^5 or larger, the boundary layer becomes totally turbulent and the separation point moves back again forming a smaller wake and a sudden drop in the drag coefficient to about 0.3. There are two empirical formulae in common use.

For $0.2 < R_e < 10^5$ $\qquad C_D = \frac{24}{R_e} + \frac{6}{1+\sqrt{R_e}} + 0.4$

For $R_e < 1000$ $$C_D = \frac{24}{R_e}\left[1 + 0.15\,\mathrm{Re}^{0.687}\right]$$

Fig. 1.12 shows this approximate relationship between C_D and Re.

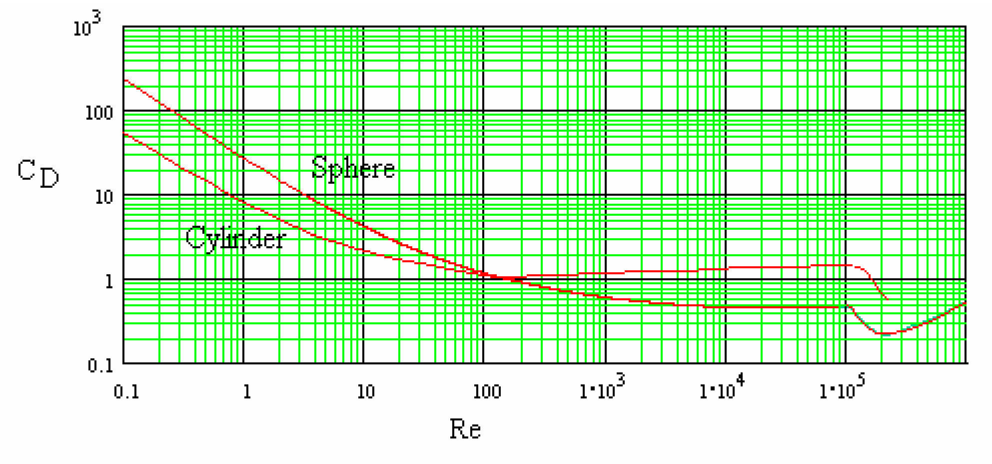


Fig.1.12

WORKED EXAMPLE 1.3

A sphere diameter 40 mm moves through a fluid of density 750 kg/m^3 and dynamic viscosity 50 cP with a velocity of 0.6 m/s. Note 1 cP = 0.001 Ns/m^2.

Calculate the drag on the sphere.

Use the graph to obtain the drag coefficient.

SOLUTION

$$\mathrm{Re} = \frac{\rho u d}{\mu} = \frac{750 \text{ x } 0.6 \text{ x } 0.04}{0.05} = 360$$

from the graph $C_D = 0.8$

$$C_D = \frac{2R}{\rho u^2 \text{ x projected Area}} \qquad \text{Projected area} = \pi\frac{d^2}{4} = \pi\frac{0.04^2}{4} = 1.2566 \text{ x } 10^{-3}\,\mathrm{m}^2$$

$$R = \frac{C_D \rho u^2 A}{2} = \frac{0.8 \text{ x } 750 \text{ x } 0.6^2 \text{ x } 1.2566 \text{ x } 10^{-3}}{2} = 0.136\ \mathrm{N}$$

1.6 TERMINAL VELOCITY

When a body falls under the action of gravity, a point is reached, where the drag force is equal and opposite to the force of gravity. When this condition is reached, the body stops accelerating and the terminal velocity reached. Small particles settling in a liquid are usually modelled as small spheres and the preceding work may be used to calculate the terminal velocity of small bodies settling in a liquid. A good application of this is the falling sphere viscometer described in earlier work.

For a body immersed in a liquid, the buoyant weight is W and this is equal to the viscous resistance R when the terminal velocity is reached.

R = W = volume x gravity x density difference $= \frac{\pi d^3 g(\rho_s - \rho_f)}{6}$

ρ_s = density of the sphere material
ρ_f = density of fluid
d = sphere diameter

STOKES' FLOW

For $R_e<0.2$ the flow is called Stokes flow and Stokes showed that $R = 3\pi d\mu u_t$
For a falling sphere viscometer, Stokes flow applies. Equating the drag force and the buoyant weight we get

$$3\pi d\mu u_t = \frac{\pi d^3 g(\rho_s - \rho_f)}{6}$$

$$\mu = \frac{d^2 g(\rho_s - \rho_f)}{18u_t}$$

The terminal velocity for Stokes flow is $u_t = \frac{d^2 g(\rho_s - \rho_f)}{18\mu}$

This formula assumes a fluid of infinite width but in a falling sphere viscometer, the liquid is squeezed between the sphere and the tube walls and additional viscous resistance is produced. The Faxen correction factor F is used to correct the result.

WORKED EXAMPLE 1.4

The terminal velocity of a steel sphere falling in a liquid is 0.03 m/s. The sphere is 1 mm diameter and the density of the steel is 7830 kg/m^3. The density of the liquid is 800 kg/m^3. Calculate the dynamic and kinematic viscosity of the liquid.

SOLUTION

Assuming Stokes' flow the viscosity is found from the following equation.

$$\mu = \frac{d^2 g(\rho_s - \rho_f)}{18u_t} = \frac{0.001^2 \text{ x } 9.81 \text{ x } (7830 - 800)}{18 \text{ x } 0.03} = 0.1277 \text{ Ns/m}^2 = 127.7 \text{ cP}$$

$$\nu = \frac{\mu}{\rho_s} = \frac{0.1277}{800} = 0.0001596 \text{ m}^2/\text{s} = 159.6 \text{ cSt}$$

Check the Reynolds number. $R_e = \frac{\rho_f u d}{\mu} = \frac{800 \text{ x } 0.03 \text{ x } 0.001}{0.0547} = 0.188$

As this is smaller than 0.2 the assumption of Stokes' flow is correct.

ALLEN FLOW

For $0.2 < R_e < 500$ the flow is called Allen flow and the following expression gives the empirical relationship between drag and Reynolds number. $\mathbf{C_D=18.5R_e^{-0.6}}$

Equating for C_D gives the following result. $C_D = \dfrac{8R}{\rho_f u_t^2 \ \pi d^2} = 18.5R_e^{-0.6}$

Substitute $R = \dfrac{\pi d^3 g(\rho_s - \rho_f)}{6}$

$$C_D = \frac{8dg(\rho_s - \rho_f)}{6\rho_f u_t^2} = 18.5R_e^{-0.6} = 18.5\left(\frac{\rho_f u_t d}{\mu}\right)^{-0.6}$$

$$\frac{8dg(\rho_s - \rho_f)}{6\rho_f u_t^2} = 18.5\left(\frac{\rho_f u_t d}{\mu}\right)^{-0.6}$$

From this equation the velocity u_t may be found.

NEWTON FLOW

For $500 < R_e < 10^5$ C_D takes on a constant value of 0.44.

Equating for C_D gives the following. $C_D = \dfrac{8R}{\rho_f u_t^2 \ \pi d^2} = 0.44$

Substitute $R = \dfrac{\pi d^3 g(\rho_s - \rho_f)}{6}$

$$\frac{8dg(\rho_s - \rho_f)}{6\rho_f u_t^2} = 0.44$$

$$u_t = \sqrt{\frac{29.73dg(\rho_s - \rho_f)}{\rho_f}}$$

When solving the terminal velocity, you should always check the value of the Reynolds number to see if the criterion used is valid.

WORKED EXAMPLE 1.5

Small glass spheres are suspended in an up wards flow of water moving with a mean velocity of 1 m/s. Calculate the diameter of the spheres. The density of glass is 2630 kg/m^3. The density of water is 1000 kg/m^3 and the dynamic viscosity is 1 cP.

SOLUTION

First, try the Newton flow equation. This is the easiest.

$$u_t = \sqrt{\frac{29.73d\,g(\rho_s - \rho_f)}{\rho_f}}$$

$$d = \frac{u_t^2 \rho_f}{29.73\,g(\rho_s - \rho_f)} = \frac{1^2 \text{ x } 1000}{29.73 \text{ x } 9.81 \text{ x } (2630 - 1000)} = 0.0021\,\text{m or } 2.1\,\text{mm}$$

Check the Reynolds number.

$$R_e = \frac{\rho_f u_t d}{\mu} = \frac{1000 \text{ x } 1 \text{ x } 0.0021}{0.001} = 2103$$

The assumption of Newton flow was correct so the answer is valid.

WORKED EXAMPLE 1.6

Repeat the last question but this time with a velocity of 0.05 m/s. Determine the type of flow that exists.

SOLUTION

If no assumptions are made, we should use the general formula $C_D = \frac{24}{R_e} + \frac{6}{1 + \sqrt{R_e}} + 0.4$

$$R_e = \frac{\rho_f u_t d}{\mu} = \frac{1000 \text{ x } 0.05 \text{ x } d}{0.001} = 50\,000d$$

$$C_D = \frac{24}{R_e} + \frac{6}{1 + \sqrt{R_e}} + 0.4$$

$$C_D = \frac{24}{50000d} + \frac{6}{1 + \sqrt{50000d}} + 0.4$$

$$C_D = 0.00048d^{-1} + \frac{6}{1 + 223.6d^{0.5}} + 0.4$$

$$C_D = \frac{8dg(\rho_s - \rho_f)}{6\rho_f u^2} = \frac{8d \text{ x } 9.81 \text{ x } (2630 - 1000)}{6 \text{ x } 1000 \text{ x } 0.05^2} = 8528.16d$$

$$8528.16d = 0.00048d^{-1} + \frac{6}{1 + 223.6d^{0.5}} + 0.4$$

This should be solved by any method known to you such as plotting two functions and finding the point of interception.

$$f1(d) = 8528.16d$$

$$f2(d) = 0.00048d^{-1} + \frac{6}{1 + 223.6d^{0.5}} + 0.4$$

The graph below gives an answer of d = 0.35 mm.

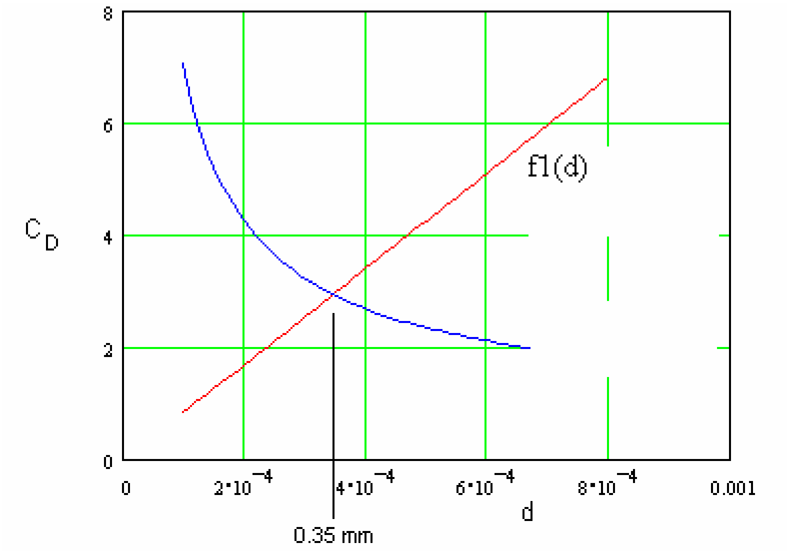


Fig. 1.13

Checking the Reynolds' number $R_e = \frac{\rho_f u_t d}{\mu} = \frac{1000 \text{ x } 0.05 \text{ x } 0.00035}{0.001} = 17.5$

This puts the flow in the Allen's flow section.

ANOTHER METHOD OF SOLUTION

It has been shown previously that the drag coefficient for a sphere is given by the formula $C_D = \frac{8R}{\pi d^2 \rho u^2}$. R is the drag force. One method of solving problems is to arrange the formula into the form $C_D R_e^2$ as follows.

$$C_D = \frac{8R}{\pi d^2 \rho_f u^2} x \frac{\rho_f \mu^2}{\rho_f \mu^2} = \frac{8R\rho_f}{\pi\mu^2} x \frac{\mu^2}{\rho_f^2 u^2 d^2} = \frac{8R\rho_f}{\pi\mu^2} x \frac{1}{R_e^2}$$

$$C_D R_e^2 = \frac{8R\rho_f}{\pi\mu^2}$$

If the sphere is falling and has reached its terminal velocity, R = buoyant weight.

$$R = \frac{\pi d^3 g(\rho_s - \rho_f)}{6}$$

$$C_D R_e^2 = \frac{\pi d^3 g(\rho_s - \rho_f)8\rho_f}{6\pi\mu^2}$$

$$C_D R_e^2 = \frac{4d^3 g\rho_f(\rho_s - \rho_f)}{3\mu^2}..............(A)$$

The drag coefficient for a sphere is related to the Reynolds number as described previously. There are two equations commonly used for this relationship as follows.

$$C_D = \frac{24}{R_e} + \frac{6}{1+\sqrt{R_e}} + 0.4.................(B)$$

and

$$C_D = \frac{24}{R_e}\left[1 + 0.15R_e^{0.687}\right]................(C)$$

Either B or C may be used in the solution of problems. The general method is to solve $R_eC_D^2$ from equation A. Next compose a table of values of R_e, C_D, and $R_eC_D^2$. Plot $R_eC_D^2$ vertically and Re horizontally. Find the value of R_e that gives the required value of $R_eC_D^2$. From this the velocity may be deduced.

WORKED EXAMPLE 1.7

A sphere 1.5 mm diameter falls in water. The density of the sphere is 2500 kg/m^3. The density and dynamic viscosity of water is 997 kg/m^3 and 0.89 x 10^{-3} Ns/m^2 respectively. The drag coefficient is given by the formula $C_D = \frac{24}{R_e}\left[1 + 0.15 R_e^{0.687}\right]$. Determine the terminal velocity.

SOLUTION

$$C_D R_e^2 = \frac{4d^3 g\rho_f\left(\rho_s - \rho_f\right)}{3\mu^2} = \frac{4(0.0015)^3 x 9.81 x 997\left(2500 - 997\right)}{3(0.89 x 10^{-3})^2} = 83513$$

Next compile a table using the formula $C_D = \frac{24}{R_e}\left[1 + 0.15\,\mathrm{Re}^{0.687}\right]$.

R_e	0.1	1	10	100	1000
C_D	24.7	27.6	4.15	1.09	0.44
$C_D R_e^2$	2.47	27.6	415	109	438288

We are looking for a value of $C_D R_e^2$ = 83513 and it is apparent that this occurs when R_e is between 100 and 1000. By plotting or by narrowing down the figure by trial and error we find that the correct value of R_e is 356.

$R_e = 356 = \rho_f u d/\mu$

356 = 997 x u x 0.0015/0.89 x 10^{-3}

u = 0.212 m/s and this is the terminal velocity.

SELF ASSESSMENT EXERCISE No. 3

1. a. Explain the term Stokes flow and terminal velocity.

   b. Show that the terminal velocity of a spherical particle with Stokes flow is given by the formula $u = d^2g(\rho_s - \rho_f)/18\mu$

   Go on to show that $C_D=24/R_e$

2. Calculate the largest diameter sphere that can be lifted upwards by a vertical flow of water moving at 1 m/s. The sphere is made of glass with a density of 2630 kg/m^3. The water has a density of 998 kg/m^3 and a dynamic viscosity of 1 cP. (20.7 mm)

3. Using the same data for the sphere and water as in Q2, calculate the diameter of the largest sphere that can be lifted upwards by a vertical flow of water moving at 0.5 m/s. (5.95 mm).

4. Using the graph (fig. 1.12) to obtain the drag coefficient of a sphere, determine the drag on a totally immersed sphere 0.2 m diameter moving at 0.3 m/s in sea water. The density of the water is 1025 kg/m^3 and the dynamic viscosity is 1.05 x 10^{-3} Ns/m^2. (0.639 N).

5. A glass sphere of diameter 1.5 mm and density 2 500 kg/m^3 is allowed to fall through water under the action of gravity. The density of the water is 1000 kg/m^3 and the dynamic viscosity is 1 cP.

   Calculate the terminal velocity assuming the drag coefficient is
   $C_D = 24\, R_e^{-1}\,(1+ 0.15R_e^{0.687})$ (Ans. 0.215 m/s

6. Similar to part of Q1 1990

   A glass sphere of density 2 690 kg/m^3 falls freely through water. Find the terminal velocity for a 4 mm diameter sphere and a 0.4 mm diameter sphere.

   The drag coefficient is $C_D = 8F/\{\pi d^2 \rho u^2\}$
   This coefficient is related to the Reynolds number as shown.

R_e	15	20	25	30	35
C_D	3.14	2.61	2.33	2.04	1.87

   The density and viscosity of the water is 997 kg/m^3 and 0.89 x 10^{-3} N s/m^2.

   Answer 0.45 m/s and 0.06625 m/s.

7. Similar to part of Q4 1988.

   A glass sphere of diameter 1.5 mm and density 2 500 kg/m^3 is allowed to fall through water under the action of gravity. Find the terminal velocity assuming the drag coefficient is $C_D = 24\, R_e^{-1}(1+ 0.15R_e^{0.687})$
   (Ans. 0.215 m/s)

8. Similar to Q1 1986

The force F on a sphere of diameter d moving at velocity u_m in a fluid is given by
$F = C_D \{\pi d^2 \rho u_m^2\}/8$

For Reynolds numbers less than 1 000, C_D is given by $C_D = 24\ R_e^{-1}(1 + 0.15R_e^{0.687})$

Estimate the terminal velocity of a glass sphere 1 mm diameter and density 2 650 kg/m^3 in water of density 997 kg/m^3 and viscosity 0.89 x 10^{-3} N s/m^2.

Answer 0.15 m/s

2. BOUNDARY LAYERS

In this tutorial we will look at the shape of various types of boundary layers. We will look at the mathematical equations for the shape of the boundary layer and use them to solve problems.

You may recall that the definition of a BL is the thickness of that layer next to a surface in which the velocity grows from zero to a maximum value (or so close to a maximum as to be of no practical difference). This thickness is usually given the symbol δ (small delta).

The boundary layer, once established may have a constant thickness but, for example, when a flow meets the leading edge of a surface, the boundary layer will grow as shown (fig.2.1).

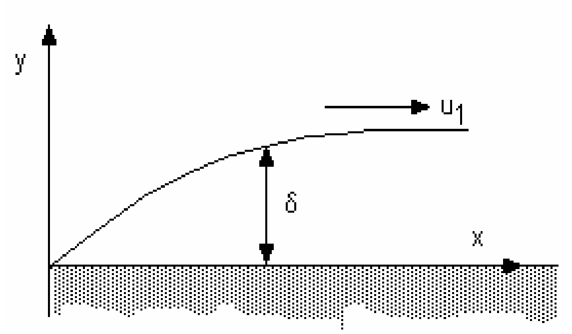


Fig.2.1

When the flow enters a pipe the BL builds up from all around the entrance and a cross section shows the layer meets at the centre (fig.2.2)

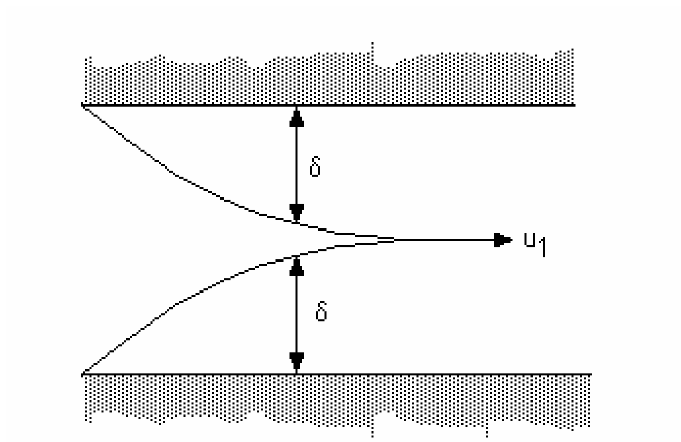


Fig.2.2

The symbol u_1 is used to designate the maximum velocity in the fully developed layer. The fully developed layer may be laminar or turbulent depending on the Reynolds' Number.

The velocity profile for a typical case is shown on fig.2.3.

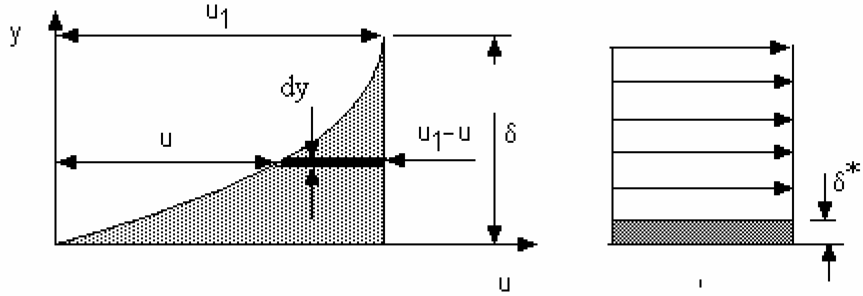


Fig.2.3

The shear stress between any two horizontal layers is τ. For a Newtonian Fluid the relationship between shear stress, dynamic viscosity (μ) and rate of shear strain (du/dy) is

$$\tau = \mu \, du/dy$$

At the wall the shear stress is called the WALL SHEAR STRESS, τ_o and occurs at y = 0. Note that the gradient du/dy is the rate of shear strain and it is steeper for turbulent flow than for laminar flow giving a greater shear resistance.

The solution of problems is simplified by the concepts of DISPLACEMENT THICKNESS AND MOMENTUM THICKNESS which we will now examine.

2.1. DISPLACEMENT THICKNESS δ*

The flow rate within a boundary layer is less than that for a uniform flow of velocity u_1. The reduction in flow is equal to the area under the curve in fig.2.3. If we had a uniform flow equal to that in the boundary layer, the surface would have to be displaced a distance δ* in order to produce the reduction. This distance is called the displacement thickness and it is given by :

$$\text{flow redution} = \int_0^\delta [u_1 - u]dy = u_1\delta^*$$

If this is equivalent to a flow of velocity u_1 in a layer δ* thick then :

$$\delta^* = \int_0^\delta \left[1 - \frac{u}{u_1}\right]dy$$

2.2. MOMENTUM THICKNESS θ

The momentum in a flow with a BL present is less than that in a uniform flow of the same thickness. The momentum in a uniform layer at velocity u_1 and height h is $\rho h u_1^2$. When a BL exists this is reduced by $\rho u_1^2\theta$. Where θ is the thickness of the uniform layer that contains the equivalent to the reduction. Using the same reasoning as before we get :

$$\theta = \int_0^\delta \left[\frac{u}{u_1}\right]\left[1 - \frac{u}{u_1}\right]dy$$

2.3. BOUNDARY LAYER LAWS

The velocity at any distance y above a surface is a function of the wall shear stress, the dynamic viscosity and the density.

$$u = \phi(y, \tau_o, \rho, \mu)$$

If you are familiar with the method of dimensional analysis you may wish to show for yourself that :

$$u(\rho/\tau_o)^{1/2} = \phi\{y(\tau_o)^{1/2}\}$$

Generally the law governing the growth of a BL is of the form $u = \phi(y)$ and the limits must be that $u = 0$ at the wall and $u = u_1$ in the fully developed flow. There are many ways in which this is expressed according to the Reynolds' Number for the flow. The important boundary conditions that are used in the formulation of boundary layer laws are:

1. The velocity is zero at the wall ($u = 0$ @ $y = 0$).

2. The velocity is a maximum at the top of the layer ($u = u_1$ @ $y = \delta$).

3. The gradient of the b.l. is zero at the top of the layer ($du/dy = 0$ @ $y = \delta$).

4. The gradient is constant at the wall ($du/dy = C$ @ $y = 0$).

5. Following from (4) $d^2u/dy^2 = 0$ @ $y = 0$).

Let us start by considering LAMINAR BOUNDARY LAYERS.

2.3.1 LAMINAR BOUNDARY LAYERS

One of the laws which seem to work for laminar flow is $u = u_1 \sin(\pi y/2\delta)$

WORKED EXAMPLE No.2.1

Find the displacement thickness δ^* for a Laminar BL modelled by the equation $u = u_1 \sin(\pi y/2\delta)$

$$\delta^* = \int_0^\delta \left[1 - \frac{u}{u_1}\right] dy = \int_0^\delta \left[1 - \sin\left\{\frac{\pi y}{2\delta}\right\}\right]$$

$$\delta^* = \left[y + \frac{2\delta}{\pi}\cos\left\{\frac{\pi y}{2\delta}\right\}\right]_0^\delta = \{\delta + 0\} - \left\{0 + \frac{2\delta}{\pi}\right\} = 0.364\delta$$

Another way of expressing the shape of the laminar BL is with a power law. The next example is typical of that used in the examination.

WORKED EXAMPLE No.2.2

The velocity distribution inside a laminar BL over a flat plate is described by the cubic law :

$$u/u_1 = a_0 + a_1 y + a_2 y^2 + a_3 y^3$$

Show that the momentum thickness is $39\delta/280$

SOLUTION

At $y = 0$, $u = 0$ so it follows that $a_0 = 0$

$d^2u/dy^2 = 0$ @ $y = 0$ so $a_2 = 0$. Show for yourself that this is so.

The law is reduced to $u/u_1 = a_1 y + a_3 y^3$
at $y = \delta$, $u = u_1$ so $1 = a_1\delta + 3a_3\delta^2$
hence $a_1 = (1-a_3\delta^3)/\delta$

Now differentiate and $du/dy = u_1(a_1+3a_3y^2)$
at $y = \delta$, du/dy is zero so $0 = a_1 + 3a_3\delta^2$ so $a_1 = -3a_3\delta^2$

Hence by equating $a_1 = 3/2\delta$ and $a_3 = -1/2\delta^3$

Now we can write the velocity distribution as $u/u_1 = 3y/2\delta - (y/\delta)^3/2$

and $du/dy = u_1\{3/2\delta + 3y^2/2\delta^3\}$

If we give the term y/δ the symbol η we may rewrite the equation as:

$$u/u_1 = 3\eta/2 - \eta^3/2$$

The momentum thickness θ is given by :

$$\theta = \int_0^\delta \left[\frac{u}{u_1}\right]\left[1 - \frac{u}{u_1}\right] dy \quad \text{but dy} = \delta \, \mathrm{d}\eta$$

$$\theta = \int_0^1 \left\{\frac{3\eta}{2} - \frac{\eta^3}{2}\right\}\left\{1 - \frac{3\eta}{2} - \frac{\eta^3}{2}\right\} d\eta$$

Integrating gives :

$$\theta = \delta\left[\frac{3\eta^2}{4} - \frac{\eta^4}{8} - \frac{9\eta^3}{12} - \frac{\eta^7}{28} + \frac{3\eta^5}{10}\right]$$

between the limits $\eta=0$ and $\eta=1$ this evaluates to

$$\theta = 39\delta/280$$

WORKED EXAMPLE No.2.3

Show that $\delta/x=4.64R_e^{-0.5}$ for the same case as before.

SOLUTION

We must first go back to the basic relationship. From the previous page

$$du/dy = u_1\{3/2\delta + 3y^2/2\delta^3\}$$

At the wall where y=0 the shear stress is

$$\tau_o=\mu du/dy = \mu u_1\{3/2\delta + 3y^2/2\delta^3\} =(\mu u_1/\delta)\ \delta\ [(3/2\delta) +3y^2/2\delta^3]$$

Putting $y/\delta= \eta$ we get :

$$\tau_o = (\mu u_1/\delta)\ \delta\ [(3/2\delta) +3\delta^2/2\delta]$$

$$\tau_o = (\mu u_1/\delta)\ [(3/2) +3\delta^2/2]$$

at the wall $\eta=0$ $\quad \tau_o = (\mu u_1/\delta)\ (3/2)$……….......(2.1)

The friction coefficient C_f is always defined as

$$C_f = \tau_o/(\rho\ u_1^2/2)\text{.............(2.2)}$$

It has been shown elsewhere that $C_f = 2d\theta/dx$. The student should search out this information from test books.

Putting $\theta = 39\delta/280$ (from the last example) then

$$C_f = 2d\theta/dx = (2x39/280)\ d\delta/dx \quad \text{..............(2.3)}$$

equating (2.2) and (2.3) gives

$$\tau_o=(\ \rho u_1^2)(39/280)d\delta/dx \text{ ....(2.4)}$$

equating (2.1) and (2.4) gives

$$(\rho u_1^2)(39/280)d\delta/dx = (\mu u/\delta)(3/2)$$

hence $\quad (3 \times 280)/(2 \times 39)(\mu dx)/\rho u) = \delta d\delta$

Integrating $\quad 10.77(\mu x/\ \rho u_1) = \delta^2/2 + C$

Since $\delta = 0$ at $x = 0$ (the leading edge of the plate) then $C=0$

hence $\quad \delta= \{21.54\mu x/\rho u_1\}^{½}$

dividing both sides by x gives $\quad \delta/x = 4.64(\mu/\rho u_1 x)^{-½} = 4.64R_e^{-½}$

NB $\quad R_{ex}= \ \rho u_1 x/\mu$ and is based on length from the leading edge.

SELF ASSESSMENT EXERCISE No. 4

1. The BL over a plate is described by $u/u_1=\sin(\pi y/2\delta)$. Show that the momentum thickness is 0.137δ.

2. The velocity profile in a laminar boundary layer on a flat plate is to be modelled by the cubic expression $u/u_1=a_0+a_1y + a_2y^2+a^3y^3$
where u is the velocity a distance y from the wall and u_1 is the main stream velocity.
Explain why a_0 and a_2 are zero and evaluate the constants a_1 and a_3 in terms of the boundary layer thickness δ.

   Define the momentum thickness θ and show that it equals $39\delta/280$
Hence evaluate the constant A in the expression
$\delta/x = A\,(R_{ex})^{-0.5}$
where x is the distance from the leading edge of the plate. It may be assumed without proof that the friction factor $C_f = 2\,d\theta/dx$

3. (a) The velocity profile in a laminar boundary layer is sometimes expressed in the form
$$u/u_1=a_0+a_1(y/\delta)+a_2(y/\delta)^2+a_3\,(y/\delta)^3+a_4(y/\delta)^4$$
where u_1 is the velocity outside the boundary layer and δ is the boundary layer thickness. Evaluate the coefficients a_0 to a_4 for the case when the pressure gradient along the surface is zero.

   (b) Assuming a velocity profile $u/u_1=2(y/\delta) - (y/\delta)^2$ obtain an expression for the mass and momentum fluxes within the boundary layer and hence determine the displacement and momentum thickness.

4. When a fluid flows over a flat surface and the flow is laminar, the boundary layer profile may be represented by the equation

$$u/u_1= 2(\eta) - (\eta)^2 \quad \text{where } \eta = y/\delta$$

   y is the height within the layer and δ is the thickness of the layer. u is the velocity within the layer and u_1 is the velocity of the main stream.

   Show that this distribution satisfies the boundary conditions for the layer.

   Show that the thickness of the layer varies with distance (x) from the leading edge by the equation $\delta=5.48x(R_{ex})^{-0.5}$

   It may be assumed that $\tau_o = \rho u_1^2\,d\theta/dx$

5. Define the terms *displacement thickness* δ^* and *momentum thickness* θ.
Find the ratio of these quantities to the boundary layer thickness δ if the velocity profile within the boundary layer is given by

$$u/u_1=\sin(\pi y/2\delta)$$

Show, by means of a momentum balance, that the variation of the boundary layer thickness δ with distance (x) from the leading edge is given by $\delta = 4.8(R_{e_x})^{-0.5}$

It may be assumed that $\tau_o = \rho u_1^2\, d\theta/dx$

Estimate the boundary layer thickness at the trailing edge of a plane surface of length 0.1 m when air at STP is flowing parallel to it with a free stream velocity u_1 of 0.8 m/s. It may be assumed without proof that the friction factor C_f is given by
$C_f = 2\, d\theta/dx$

N.B. standard data $\mu = 1.71 \times 10^{-5}$ N s/m^2. $\rho = 1.29$ kg/m^3.

6. In a laminar flow of a fluid over a flat plate with zero pressure gradient an approximation to the velocity profile is

$$u/u_1=(3/2)(\eta) - (1/2)(\eta)^3$$

$\eta = y/\delta$ and u is the velocity at a distance y from the plate and u_1 is the mainstream velocity. δ is the boundary layer thickness.

Discuss whether this profile satisfies appropriate boundary conditions.

Show that the local skin-friction coefficient C_f is related to the Reynolds' number (Re_x) based on distance x from the leading edge by the expression
$C_f = A\,(Re_x)^{-0.5}$
and evaluate the constant A.

It may be assumed without proof that $C_f = 2\, d\theta/dx$

and that θ is the integral of $(u/u_1)(1 - u/u_1)dy$ between the limits 0 and δ

2.3.2 TURBULENT BOUNDARY LAYERS

When a fluid flows at high velocities, the boundary layer becomes turbulent and the gradient at the wall becomes smaller so the wall shear stress is larger and the drag created on the surface increases.

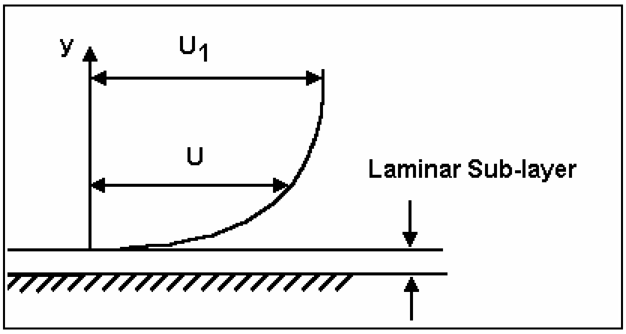


Fig. 2.4

Prandtl found that a law which fits the turbulent case well for Reynolds' numbers below 10^7 is:

$$u = u_1(y/\delta)^{1/7}$$

This is called the $1/7^{th}$ law.

The gradient of the B.L. is $du/dy = u_1\delta^{1/7}y^{-6/7}/7$

This indicates that at the wall where y=0, the gradient is infinite (horizontal). This is obviously incorrect and is explained by the existence of a laminar sub-layer next to the wall. In this layer the velocity grows very quickly from zero and merges with the turbulent layer. The gradient is the same for both at the interface of laminar and turbulent flow. The drag on the surface is due to the wall shear stress in the laminar sub-layer.

WORKED EXAMPLE No.2.4

Show that the mean velocity in a pipe with fully developed turbulent flow is 49/60 of the maximum velocity. Assume the 1/7th law.

For a pipe, the B.L. extends to the centre so δ = radius = R. Consider an elementary ring of flow.

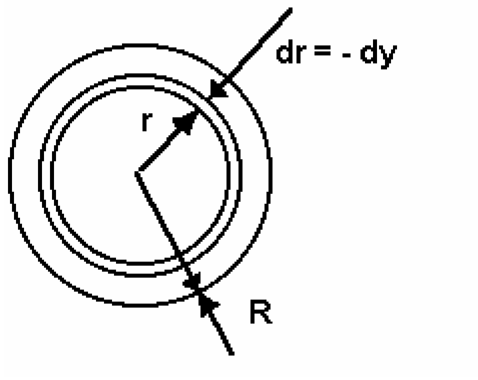


Fig.2.5

The velocity through the ring is u.
The volume flow rate through the ring is $2\pi r u dr$
The volume flow rate in the pipe is $Q = 2\pi\int r u dr$

Since $\delta = R$ then $u = u_1(y/R)^{1/7}$

also $r = R-y$

$$Q = 2\pi\int (R-y)udr = 2\pi\int u_1 R^{-1/7} (R-y)y^{1/7}dy$$

$$Q = 2\pi u_1 R^{-1/7}[Ry^{1/7} - y^{8/7}]$$

$$Q = 2\pi u_1 R^{-1/7}[(7/8)Ry^{8/7} - (7/15)y^{15/7}]$$

$$Q = (49/60)\pi R^2 u_1.$$

The mean velocity is defined by $u_m = Q/\pi R^2$

hence $u_m = (49/60)u_1$

2.4 FRICTION COEFFICIENT AND BOUNDARY LAYERS

Earlier it was explained that the friction coefficient C_f is the ratio of the wall shear stress to the dynamic pressure so :

$$C_f = 2\tau_o/(\rho u_m^2) \text{ ....(2.4.1)}$$

For smooth walled pipes, Blazius determined that $C_f = 0.079R_e^{-0.25}$ ....(2.4.2)

Equating (2.4.1) and (2.4.2) gives : $2\tau_o/(\rho u_m^2) = 0.079R_e^{-0.25}$

Note that u_m is the mean velocity and u_1 is the maximum velocity.

Research shows that $u_m = 0.8u_1$

Also Note that $R_e = \rho u_1 D/\mu$ and $D = 2\delta$.

Hence $\tau_o = 0.02125\rho u_1^2(\mu/\rho\delta u_1)^{0.25}$.............(2.4.3)

2.5 FORCE BALANCE IN THE BOUNDARY LAYER

The student should refer to textbooks for finer details of the following work.

Consider again the growth of the B.L. as the fluid comes onto a flat surface. A stream line for the flow is not parallel to the B.L. Now consider a control volume A, B, C, D.

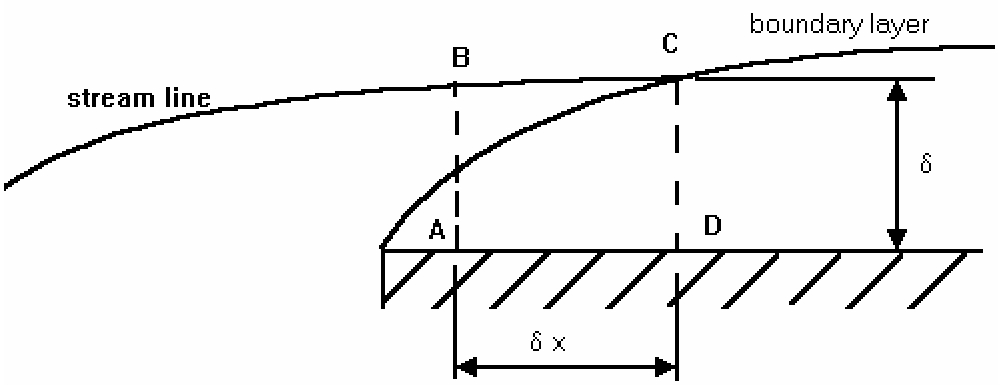


Fig.2.6

Balancing pressure force and shear force at the surface with momentum changes gives :

$$\tau_o = \rho\left(\frac{\delta}{\delta x}\right)\int_0^\delta [u - u_1]u\,dy + \rho\left(\frac{\delta u}{\delta x}\right)\int_0^\delta [u - u_1]u\,dy \text{...........(2.4.4)}$$

Using equations (2.4.2), (2.4.3) and (2.4.4) gives $(4/5)\delta^{5/4} = 0.231\{\mu/\rho u\}^{1/5}x = R_e^{-1/5}$

The shear force on the surface is $F_s = \tau_o$ x surface area

The surface skin friction coefficient is $C_f = 2F_s/(\rho u_1^2) = 0.072R_e^{-1/5}$

Experiments have shown that a more accurate figure is : $C_f = 0.074R_e^{-1/5}$

SELF ASSESSMENT EXERCISE No. 5

1. Under what circumstances is the velocity profile in a pipe adequately represented by the 1/7 th power law $u/u_1=(y/R)^{1/7}$ where u is the velocity at distance y from the wall, R is the pipe radius and u_1 is the centre-line velocity ?

   The table shows the measured velocity profile in a pipe radius 30 mm. Show that these data satisfy the 1/7 th power law and hence evaluate

   (i) the centre-line velocity

   (ii) the mean velocity u_m

   (iii) the distance from the wall at which the velocity equals u_m.

1.0	2.0	5.0	10.0	15.0	20.0	y (mm)
1.54	1.70	1.94	2.14	2.26	2.36	u (m/s)

2. (a) Discuss the limitations of the 1/7th power law $u/u_1=(y/R)^{1/7}$ for the velocity profile in a circular pipe of radius R, indicating the range of Reynolds numbers for which this law is applicable.

   (b) Show that the mean velocity is given by $49u_1/60$.

   (c) Water flows at a volumetric flow rate of 1.1 x 10^{-3} m^3/s in a tube of diameter 25 mm. Calculate the centre-line velocity and the distance from the wall at which the velocity is equal to the mean velocity.

   (d) Assuming that $C_f=0.079(Re)^{-0.25}$ evaluate the wall shear stress and hence estimate the laminar sub-layer thickness.

   μ = 0.89 x 10^{-3} N s/m^2. ρ=998 kg/m^3.