FLUID MECHANICS D203
SAE SOLUTIONS TUTORIAL 9 – COMPRESSIBLE FLOW

SELF ASSESSMENT EXERCISE No. 4

1. Air discharges from a pipe into the atmosphere through an orifice. The stagnation pressure and temperature immediately upstream of the orifice is 10 bar and 287 K at all times.

Determine the diameter of the orifice which regulates the flow rate to 0.03 kg/s.
Determine the diameter of the orifice which regulates the flow rate to 0.0675 kg/s.

Atmospheric pressure is 1 bar, the flow is isentropic and the air should be treated as a perfect gas. The following formulae are given to you.

$T_o = T\{1 + M^2(\gamma-1)/2\}$ $\qquad$ $p_1/p_2 = (T_1/T_2)^{\gamma/(\gamma-1)}$

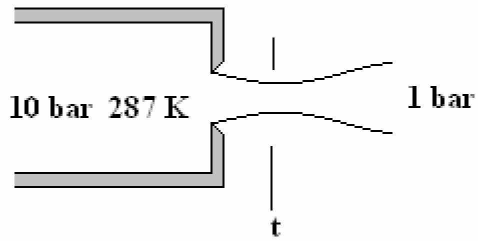
The relationship between areas for the flow of air through a convergent- divergent nozzle is given by $\qquad A/A^* = (1/M)\{(M^2 + 5)/6\}^3$

where A and A* are cross sectional areas at which the Mach Numbers are M and 1.0 respectively.

Determine the ratio of exit to throat areas of the nozzle when the Mach number is 2.44 at exit.

Confirm that an exit Mach number of 0.24 also gives the same area ratio.

Pressure ratio is 10/1 so it is clearly choked.



$$T_t = \frac{T_o}{1 + 0.2M^2} \qquad M = 1 \text{ at throat}$$

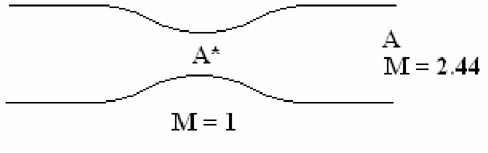
$$T_t = \frac{287}{1.2} = 239\text{ K}$$

$$\frac{p_o}{p_t} = \left(\frac{287}{239}\right)^{3.5} = 1.893 \qquad p_t = 10/1.893 = 5.28 \text{ bar}$$

$\rho_t = p/RT = 5.28 \times 10^5 /(287 \times 239) = 7.7$ kg/m^3 $\qquad$ $a = (\gamma R T)^{½} = 310$ m/s

m = 0.03 = ρ A a = 77 A x 310 $\quad$ A = 12.57 x 10^{-6} m^2 $\quad$ Diameter = √(4A/π) = 4 mm
m = 0.0675 = ρ A a = 77 A x 310 $\quad$ A = 28.278 x 10^{-6} m^2 $\quad$ Diameter = √(4A/π) = 6 mm

$$\frac{A}{A^*} = \frac{1}{M}\left(\frac{M^2 + 5}{6}\right)^3 = \frac{1}{2.44}\left(\frac{2.44^2 + 5}{6}\right)^3 = 2.49$$



With M = 0.24

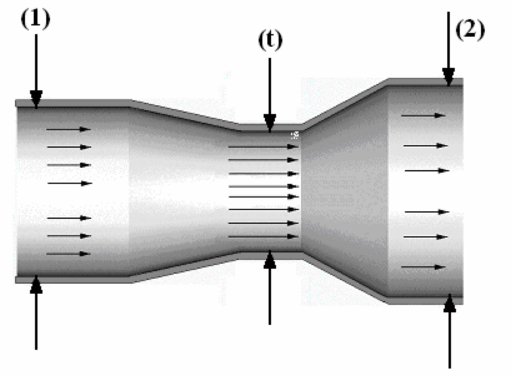
$$\frac{A}{A^*} = \frac{1}{2.4}\left(\frac{2.4^2 + 5}{6}\right)^3 = 2.49$$

Hence this is a correct solution and this is the theoretical result when m = 0.24 at inlet and 1.0 at the throat.

2. Air discharges from a vessel in which the stagnation temperature and pressure are 350 K and 1.3 bar into the atmosphere through a convergent-divergent nozzle. The throat area is 1 x 10^{-3} m^2. The exit area is 1.2 x 10^{-3} m^2. Assuming isentropic flow and no friction and starting with the equations

$a = (\gamma RT)^{1/2}$ $\quad C_p T_o = C_p T + v_2/2$ $\quad p\rho^{-\gamma} = \text{constant}$

Determine the mass flow rate through the nozzle , the pressure at the throat and the exit velocity.



$T_t/T_1 = (p_t/p_1)^{(\gamma-1)/\gamma}$ hence $T_t = 291.7$ K

$p_t = 0.528\, p_1 = 0.686$ bar if chocked.

$a_t = (\gamma R T_t)^{½} = 342.35$ m/s

$\rho_t = p_t/RT_t = 0.686 \times 10^5 /(287 \times 291.7) = 0.82$ kg/m^3

$m = \rho A a = 0.82 \times 1 \times 10^{-3} \times 342 = 0.28$ kg/s

$T_2/T_0 = \{1 + M^2(\gamma -1)/2\}$ hence $T_o = 350$ K

$T_2 = T_o (p_2/p_t)^{(\gamma-1)/\gamma} = 350(1.013/1.3)^{(\gamma-1)/\gamma} = 325.9$ K

$\rho_2 = p_2/RT_2 = 1.013 \times 10^5 /(287 \times 325.9) = 1.083$ kg/m^3

$c_2 = m/ (\rho_2 A_2) = 0.28/(1.083 \times 1.2 \times 10^{-3}) = 215.4$ m/s

3. Show that the velocity of sound in a perfect gas is given by $\quad a = (\gamma RT)^{½}$

Show that the relationship between stagnation pressure, pressure and Mach number for the isentropic flow of a perfect gas is $\quad p_o/p = \{1 + (\gamma-1)M^2/2\}^{\gamma/(\gamma-1)}$

It may be assumed that $ds = C_p\, d(l_n v) + C_v\, d(l_n p)$ where v is the specific volume.

A convergent - divergent nozzle is to be designed to produce a Mach number of 3 when the absolute pressure is 1 bar. Calculate the required supply pressure and the ratio of the throat and exit areas.

The solution for part 1 requires the derivations contained in the tutorial.

$$p_o = p\left(M^2\frac{\gamma-1}{2}+1\right)^{\frac{\gamma}{\gamma-1}} = 1 \times \left(3^2\frac{0.4}{2}+1\right)^{3.5} = 36.73\,\text{bar}$$

For chocked flow $p_t = 0.528\, p_o = 19.39$ bar

$c_t = 1\, M_t = (\gamma R T_t)^{½}$ $\quad c_e = 3\, M_e = 3(\gamma R T_e)^{½}$ $\quad m = \rho_t A_t c_t = \rho_e A_e c_e$

$$\frac{A_t}{A_e} = \frac{\rho_e c_e}{\rho_t c_t} \qquad \frac{\rho_e}{\rho_t} = \left(\frac{p_e}{p_t}\right)^{\frac{1}{\gamma}} \qquad \frac{A_t}{A_e} = \left(\frac{p_e}{p_t}\right)^{\frac{1}{\gamma}} \times 3 \times \frac{\sqrt{\gamma RT_e}}{\sqrt{\gamma RT_t}} \qquad \frac{T_e}{T_t} = \left(\frac{p_e}{p_t}\right)^{1-\frac{1}{\gamma}}$$

$$\frac{A_t}{A_e} = \left(\frac{p_e}{p_t}\right)^{\frac{1}{\gamma}} \times 3 \times \sqrt{\left(\frac{p_e}{p_t}\right)^{1-1/\gamma}} = 3\left(\frac{p_e}{p_t}\right)^{\frac{1+\gamma}{2\gamma}} = 3\left(\frac{1}{19.39}\right)^{2.4/2.8} = 0.236$$

SELF ASSESSMENT EXERCISE No. 5

1. An air storage vessel contains air at 6.5 bar and 15°C. Air is supplied from the vessel to a machine through a pipe 90 m long and 50 mm diameter. The flow rate is 2.25 m^3/min at the pipe inlet. The friction coefficient C_f is 0.005. Neglecting kinetic energy, calculate the pressure at the machine assuming isothermal flow.

m = pV/RT = 6.5 x 10^5 x 2.25/(60 x 288 x 287) = 0.295 kg/s

$(1-p_2^2/p_1^2) = (64\, m^2 RT\, C_f\, L)/(\pi^2 D^5 p_1^2)$

$$1-\left(\frac{p_2}{6.5}\right)^2 = \frac{64 \times 0.295^2 \times 287 \times 288 \times 0.005 \times 90}{\pi^2 \times 0.05^5 \times \left(6.5 \times 10^5\right)^2} = 0.1589$$

$$1-0.1589 = 0.841 = \left(\frac{p_2}{6.5}\right)^2$$

$p_2^2 - 35.5 \quad p_2 = 5.96$ bar

SELF ASSESSMENT EXERCISE No. 6

1. A natural gas pipeline is 1000 m long and 100 mm bore diameter. It carries 0.7 kg/s of gas at a constant temperature of 0° C. The viscosity is 10.3 x 10^{-6} N s/m^2 and the gas constant R = 519.6 J/kg K. The outlet pressure is 105 kPa. Calculate the inlet pressure. Using the Blazius formula to find f. (Answer 357 kPa.)

$$R_e = \frac{4m}{\pi\mu D} = \frac{4 \times 0.7}{\pi \times 10.3 \times 10^{-6} \times 0.1} = 865 \times 10^3$$

$$C_f = 0.079 R_e^{-0.25} = 0.00259$$

$$1-\left(\frac{p_2}{p_1}\right)^2 = \frac{64 \times 0.7^2 \times 519.6 \times 273 \times 0.00269 \times 1000}{\pi^2 \times 0.1^5 \times (p_1)^2}$$

$$1 = \frac{116.7 \times 10^9 + 11.025 \times 10^9}{(p_1)^2} \qquad p_1 = 357 \text{ kPa}$$

2. A pipeline is 20 km long and 500 mm bore diameter. 3 kg/s of natural gas must be pumped through it at a constant temperature of 20°C. The outlet pressure is 200 kPa. Calculate the inlet pressure using the same gas constants as Q.1.

$$R_e = \frac{4m}{\pi\mu D} = \frac{4 \times 3}{\pi \times 10.3 \times 10^{-6} \times 0.5} = 741693$$

$$C_f = 0.079 R_e^{-0.25} = 0.00269$$

$$1-\left(\frac{200 \times 10^3}{p_1}\right)^2 = \frac{64 \times 3^2 \times 519.6 \times 293 \times 0.00269 \times 20000}{\pi^2 \times 0.5^5 \times (p_1)^2} = \frac{15.296 \times 10^9}{p_1^2}$$

$p_1^2 = 15.296 \times 10^9 + 40 \times 10^9$

p_1= 235 kPa

3. Air flows at a mass flow rate of 9.0 kg/s isothermally at 300 K through a straight rough duct of constant cross sectional area of 1.5×10^{-3} m^2. At end A the pressure is 6.5 bar and at end B it is 8.5 bar. Determine

a. the velocities at each end. (Answers 794.8m/s and 607.7 m/s)
b. the force on the duct. (Answer 1 380 N)
c. the rate of heat transfer through the walls. (Answer 1.18 MJ)
d. the entropy change due to heat transfer. (Answer 3.935 KJ/k)
e. the total entropy change. (Answer 0.693 kJ/K)
It may be assumed that $ds = C_p\, dT/T + R\, dp/p$

$v_2 = mRT/p_2A = 9 \times 287 \times 300/(6.5 \times 10^5 \times 1.5 \times 10^{-3}) = 794.8$ m/s
$v_1 = mRT/p_1A = 9 \times 287 \times 300/(8.5 \times 10^5 \times 1.5 \times 10^{-3}) = 607.7$ m/s

$p_1A_1 + m\, v_1 = p_2A_2 + m\, v_2 + F$
$F = 1.5 \times 10^{-3}\, (2 \times 10^5) + 9\, (607.76 - 794.48) = 300 - 1680 = -1380$ N
The force to accelerate the gas is greater than the pressure force.
$\Phi + P = mc_p\Delta T + (m/2)(\, v_2^{\,2} - v_1^{\,2}\,)$ $\quad \Delta T = 0 \quad P = 0$
$\Phi = c_p\Delta T + (m/2)(\, v_2^{\,2} - v_1^{\,2}\,)$
$\Phi = 0 + (9/2)(\, 794.8^2 - 607.7^2\,) = 1.18$ MJ
$\Phi = \int T\, ds = T\, \Delta s \qquad \Delta s = \Phi/T = 1180/300 = 3.935$ kJ/k
$\Delta s = mR\ln(p_1/p_2) = 9 \times 287 \ln(8.5/6.5) = 693$ J/K

4. A gas flows along a pipe of diameter D at a rate of m kg/s.

Show that the pressure gradient is $-\dfrac{dp}{dL} = \dfrac{32C_f\, m^2RT}{\pi^2 p\, D^5}$

Methane gas is passed through a pipe 500 mm diameter and 40 km long at 13 kg/s. The supply pressure is 11 bar. The flow is isothermal at 15^{o}C. Given that the molecular mass is 16 kg/kmol and the friction coefficient C_f is 0.005 determine

a. the exit pressure.
b. the inlet and exit velocities.
c. the rate of heat transfer to the gas.
d. the entropy change resulting from the heat transfer.
e. the total entropy change calculated from the formula $ds = C_p \ln(T_2/T_1) - R \ln(p_2/p_1)$

The derivation is given in the tutorial.

$$R = \frac{R_o}{\tilde{N}} = \frac{8314.4}{16} = 520 \text{ J/kgK} \quad -\int_o^{p_2} p\,dp = \frac{32C_f\, m^2RT}{\pi^2 D^5}\int_0^L dL \quad -\left(\frac{p_2^2 - p_1^2}{2}\right) = \frac{32C_f\, m^2RTL}{\pi^2 D^5}$$

$$-\left(p_2^2 - p_1^2\right) = \frac{64C_f\, m^2RTL}{\pi^2 D^5} \qquad -\left[p_2^2 - \left(11 \times 10^5\right)^2\right] = \frac{64 \times 0.005 \times 13^2 \times 520 \times 288 \times 40000}{\pi^2 \times 0.5^5}$$

$$-\left[p_2^2 - \left(11 \times 10^5\right)^2\right] = 1.05 \times 10^{12} \qquad \left[\left(11 \times 10^5\right)^2\right] - 1.05 \times 10^{12} = p_2^2 \qquad p_2 = 3.99 \text{ bar}$$

$v_1 = mRT_1/p_1A_1 = 13 \times 520 \times 288/(11 \times 10^5 \times \pi \times 0.25^2) = 9.014$ m/s
$v_2 = mRT_2/p_2A_2 = 13 \times 520 \times 288/(3.99 \times 10^5 \times \pi \times 0.25^2) = 24.85$ m/s
$\Phi + P = mc_p\Delta T + (m/2)(\, v_2^{\,2} - v_1^{\,2}\,)$ $\quad \Delta T = 0 \quad P = 0$
$\Phi = c_p\Delta T + (m/2)(\, v_2^{\,2} - v_1^{\,2}\,)$
$\Phi = 0 + (13/2)(24.85^2 - 9.014^2\,) = 3.484$ kW
$\Delta s = \Phi/T = 3484/288 = 12.09$ J/k
$\Delta s = -R\ln(p_2/p_1) = -520 \ln(3.99/11) = 526$ J/K

SELF ASSESSMENT EXERCISE No. 7

1. Write down the equations representing the conservation of mass, energy and momentum across a normal shock wave.
Carbon dioxide gas enters a normal shock wave at 300 K and 1.5 bar with a velocity of 450 m/s. Calculate the pressure, temperature and velocity after the shock wave. The molecular mass is 44 kg/kmol and the adiabatic index is 1.3.

R = 8314/44 = 188.95 J/kg K $a_1 = \sqrt{(\gamma RT_1)} = \sqrt{(1.3 \times 188.95 \times 300)} = 271.5$ m/s
$M_1 = v_1/a_1 = 450/271.5 = 1.6577$

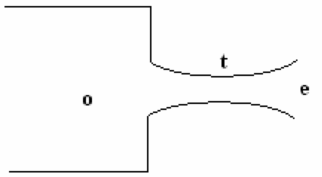
$$M_2^2 = \frac{M_1^2 + \frac{2}{\gamma - 1}}{\frac{2\gamma M_1^2}{\gamma - 1} - 1} = \frac{1.6577^2 + 2/0.3}{(2 \times 1.3 \times 1.657^2) - 1} = 0.4126 \qquad M_2 = 0.643$$

$$\frac{p_2}{p_1} = \frac{1 + \gamma M_1^2}{(1 + \gamma M_2^2)} \qquad \frac{p_2}{p_1} = \frac{1 + 1.3 \times 1.6577^2}{(1 + 1.3 \times 0.643^2)} = 2.976 \qquad p_2 = 1.5 \times 10^5 \times 2.976 = 446 \text{ kPa}$$

$$\frac{T_2}{T_1} = \frac{1 + (\gamma - 1)\frac{M_1^2}{2}}{1 + (\gamma - 1)\frac{M_2^2}{2}} \qquad \frac{T_2}{T_1} = \frac{1 + (0.3)\frac{1.6577^2}{2}}{1 + (0.3)\frac{0.643^2}{2}} = 1.329 \qquad T_2 = 300 \times 1.329 = 398.8 \text{ K}$$

$a_2 = \sqrt{(\gamma RT_2)} = \sqrt{(1.3 \times 188.95 \times 398.8)} = 313$ m/s v = $a_2 M_2$ = 201 m/s

2. Air discharges from a large container through a convergent - divergent nozzle into another large container at 1 bar. the exit mach number is 2.0. Determine the pressure in the container and at the throat.
When the pressure is increased in the outlet container to 6 bar, a normal shock wave occurs in the divergent section of the nozzle. Sketch the variation of pressure, stagnation pressure, stagnation temperature and Mach number through the nozzle.
Assume isentropic flow except through the shock. The following equations may be used.



Energy Balance from o to e o is the stagnation condition $u_o = 0$
$\gamma/(\gamma-1)RT_o + 0 = \gamma/(\gamma-1)RT_e + u_e^2/2$ $u_e = 2\sqrt{(\gamma RT_e)}$
$3.5\, RT_o + 0 = 3.5RT_e + 2\gamma RT_e$

$$\frac{p_o}{p_e} = \left(\frac{T_o}{T_e}\right)^{\frac{\gamma}{\gamma-1}} \quad \frac{T_o}{T_e} = \left(\frac{p_o}{p_e}\right)^{\frac{\gamma-1}{\gamma}} \quad 3.5RT_e = \left(\frac{p_o}{p_e}\right)^{\frac{\gamma-1}{\gamma}} = 3.5RT_e + 2\gamma\gamma R_e$$

$$\left(\frac{p_o}{p_e}\right)^{\frac{\gamma-1}{\gamma}} = \frac{(3.5 + 2.8)}{3.5} = 1.8 \quad p_e = 1 \text{ bar} \qquad p_o = 1.8^{(1/0.286)} = 7.82 \text{ bar}$$

The throat is chocked $\left(\frac{p_t}{p_o}\right) = \left(\frac{2}{\gamma - 1}\right)^{\frac{\gamma}{\gamma-1}} = 0.528$