FLUID MECHANICS D203
SAE SOLUTIONS TUTORIAL 6 – DIMENSIONAL ANALYSIS

SELF ASSESSMENT EXERCISE No. 1

1. It is observed that the velocity 'v' of a liquid leaving a nozzle depends upon the pressure drop 'p' and the density 'ρ'.

Show that the relationship between them is of the form $v = C\left(\frac{p}{\rho}\right)^{\frac{1}{2}}$

$v = C\{p^a \rho^b\}$ $[v] = LT^{-1}$ $[p] = ML^{-1}T^{-2}$ $[\rho] = ML^{-3}$

$M^0L^1T^{-1} = (ML^{-1}T^{-2})^a(ML^{-3})^b$

(T) $-1 = -2a$ $a = ½$
(M) $0 = a + b$ $b = -½$

$v = C\{p^{1/2}\rho^{-1/2}\}$ $v = C\left(\frac{p}{\rho}\right)^{\frac{1}{2}}$

2. It is observed that the speed of a sound in 'a' in a liquid depends upon the density 'ρ' and the bulk modulus 'K'.

Show that the relationship between them is $a = C\left(\frac{K}{\rho}\right)^{\frac{1}{2}}$

$a = C\{K^a \rho^b\}$ $[a] = LT^{-1}$ $[K] = ML^{-1}T^{-2}$ $[\rho] = ML^{-3}$

$M^0L^1T^{-1} = (ML^{-1}T^{-2})^a(ML^{-3})^b$

(T) $-1 = -2a$ $a = ½$
(M) $0 = a + b$ $b = -½$

$v = C\{K^{1/2}\rho^{-1/2}\}$ $v = C\left(\frac{K}{\rho}\right)^{\frac{1}{2}}$

3. It is observed that the frequency of oscillation of a guitar string 'f' depends upon the mass 'm', the length 'l' and tension 'F'.

Show that the relationship between them is $f = C\left(\frac{F}{ml}\right)^{\frac{1}{2}}$

$f = C\{F^a m^b l^c\}$

$[f] = T^{-1}$ $[F] = MLT^{-2}$ $[m] = M$ $[l] = L$

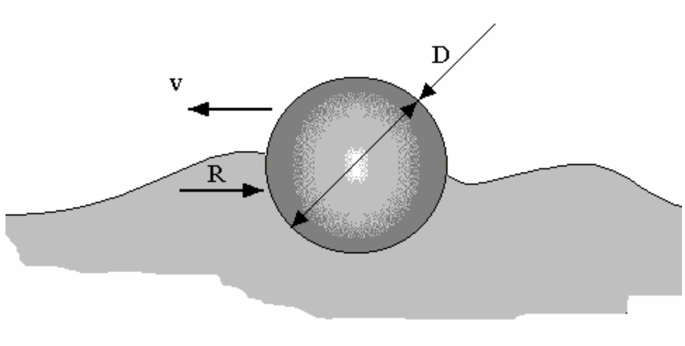
$T^{-1} = (MLT^{-2})^a(M)^b(L)^c$

(T) $-1 = -2a$ $a = ½$
(M) $0 = a + b$ $b = -½$
(L) $0 = a + c$ $c = -½$

$f = C\{F^{1/2}m^{-1/2}l^{-1/2}\}$ $f = C\left(\frac{F}{ml}\right)^{\frac{1}{2}}$

SELF ASSESSMENT EXERCISE No.2

1. The resistance to motion 'R' for a sphere of diameter 'D' moving at constant velocity 'v' on the surface of a liquid is due to the density 'ρ' and the surface waves produced by the acceleration of gravity 'g'. Show that the dimensionless equation linking these quantities is N_e = function(F_r)



F_r is the Froude number and is given by

$$F_r = \sqrt{\frac{v^2}{gD}}$$

R = function (D v ρ g) = $C\ D^a\ v^b\ \rho^c\ g^d$

There are 3 dimensions and 5 quantities so there will be 5 –3 = 2 dimensionless numbers. Identify that the one dimensionless group will be formed with R and the other with K.

Π_1 is the group formed between g and D v ρ

Π_2 is the group formed between R and D v ρ

$g = \Pi_2\ D^a\ v^b\ \rho^c$

$[g] = L\ T^{--2}$

$[D] = L$

$[v] = LT^{-1}$

$[\rho] = ML^{-3}$

$LT^{--2} = L^a\ (LT^{-1})^b\ (ML^{-3})^c$

$LT^{--2} = L^{a+b-3c}\ M^c\ T^{-b}$

Time -2 = -b b = 2

Mass c = 0

Length 1 = a + b -3c

1 = a + 2 – 0 a = -1

$g = \Pi_2\ D^1\ v^2\ \rho^0$

$$\Pi_2 = \frac{gD}{v^2} = F_r^{-2}$$

$R = \Pi_1\ D^a\ v^b\ \rho^c$

$[R] = MLT^{-2}$

$[D] = L$

$[v] = LT^{-1}$

$[\rho] = ML^{-3}$

$MLT^{-2} = L^a\ (LT^{-1})^b\ (ML^{-3})^c$

$ML^1T^{-2} = L^{a+b-3c}\ M^c\ T^{-b}$

Time -2 = -b b = 2

Mass c = 1

Length 1 = a + b -3c

1 = a + 2 – 3 a = 2

$R = \Pi_1\ D^2\ v^2\ \rho^1$

$$\Pi_1 = \frac{R}{\rho v^2 D^2} = Ne$$

$\Pi_1 = \phi\Pi_2$ Ne = $\phi(F_r)$

2. The Torque 'T' required to rotate a disc in a viscous fluid depends upon the diameter 'D' , the speed of rotation 'N' the density 'ρ' and the dynamic viscosity 'μ'. Show that the dimensionless equation linking these quantities is $\{T\ D^{-5}\ N^{-2}\ \rho^{-1}\}$ = function $\{\rho\ N\ D^2\ \mu^{-1}\}$

Use the other method here. Identify d as the unknown power.

$T = f(D\ N\ \rho\ \mu) = C\ D^a\ N^b\ \rho^c\ \mu^d$

$ML^2T^{-2} = (L)^a\ (T^{-1})^b\ (ML^{-3})^c\ (ML^{-1}T^{-1})^d$

(T) -2 = -b - d b = 2 – d

(M) 1 = c + d c = 1 - d

(L) 2 = a - 3c – d = a – 3(1 - d) – d a = 5 – 2d

$T = C\ D^{5-2d}\ N^{2-d}\ \rho^{1-d}\ \mu^d = C\ D^5\ N^2\ \rho\ (D^{-2}\ N^{-1}\ \rho^{-1}\ \mu^1)^d$

$T\ D^{-5}\ N^{-2}\ \rho^{-1} = f(D^{-2}\ N^{-1}\ \rho^{-1}\ \mu^1)$

$$\frac{T}{D^5N^2\rho} = f\left(\frac{D^2N\rho}{\mu}\right)$$

SELF ASSESSMENT EXERCISE No.3

1.The resistance to motion 'R' of a sphere travelling through a fluid which is both viscous and compressible, depends upon the diameter 'D' , the velocity 'v' , the density 'ρ' , the dynamic viscosity 'μ' and the bulk modulus 'K'. Show that the complete relationship between these quantities is :

$$N_e = \text{function}\{R_e\}\{M_a\}$$

where $N_e = R\,\rho^{-1} v^{-2} D^{-2}$ $R_e = \rho\, v\, D\, \mu^{-1}$ $M_a = v/a$ and $a = (k/\rho)^{0.5}$

This may be solved with Buckingham's method but the traditional method is given here.

$R = \text{function}(D\ v\ \rho\ \mu\ K) = C\, D^a\, v^b\, \rho^c\, \mu^d\, K^e$

First write out the MLT dimensions.

$[R] = ML^1T^{-2}$
$[D] = L$
$[v] = LT^{-1}$
$[\rho] = ML^{-3}$
$[\mu] = ML^{-1}T^{-1}$
$[K] = ML^{-1}T^{--2}$

$ML^1T^{-2} = L^a (LT^{-1})^b (ML^{-3})^c (ML^{-1}T^{-1})^d (ML^{-1}T^{--2})^e$
$ML^1T^{-2} = L^{a+b-3c-d-e}\, M^{c+d+e}\, T^{-b-d-2e}$

Viscosity and Bulk Modulus are the quantities which causes resistance so the unsolved indexes are d and e.

TIME $-2 = -b - d - 2e$ hence $b = 2 - d - 2e$ is as far as we can resolve b

MASS $1 = c + d + e$ hence $c = 1 - d - e$

LENGTH $1 = a + b - 3c - d - e$ $1 = a + (2 - d - e) - 3(1 - d - e) - d - e$
$1 = a - 1 - d$ $a = 2 - d$

Next put these back into the original formula.

$$R = C\, D^{2-d}\, v^{2-d-2e}\, \rho^{1-d-e}\, \mu^d\, K^e$$

$$R = C\, D^2 v^2 \rho^1 (D^{-1} v^{-1} \rho^{-1} \mu)^d (v^{-2} \rho^{-1} K)^e$$

$$\frac{R}{\rho v^2 D^2} = \left(\frac{\mu}{\rho v D}\right)^d \left(\frac{K}{\rho v^2}\right)^e$$

$$N_e = f(R_e)(M_a)$$

SELF ASSESSMENT EXERCISE No.4

1.(a) The viscous torque produced on a disc rotating in a liquid depends upon the characteristic dimension D, the speed of rotation N, the density ρ and the dynamic viscosity μ. Show that :

$$\{T/(\rho N^2 D^5)\} = f(\rho N D^2/\mu)$$

(b) In order to predict the torque on a disc 0.5 m diameter which rotates in oil at 200 rev/min, a model is made to a scale of 1/5. The model is rotated in water. Calculate the speed of rotation for the model which produces dynamic similarity.
For the oil the density is 750 kg/m^3 and the dynamic viscosity is 0.2 Ns/m^2.
For water the density is 1000 kg/m^3 and the dynamic viscosity is 0.001 Ns/m2.
(c) When the model is tested at 18.75 rev/min the torque was 0.02 Nm. Predict the torque on the full size disc at 200 rev/min.

Part (a) is the same as in SAE 2 whence $\frac{T}{D^5 N^2 \rho} = f\left(\frac{D^2 N \rho}{\mu}\right)$

Part (b) For dynamic similarity

$$\left(\frac{D^2 N\rho}{\mu}\right)_{model} = \left(\frac{D^2 N\rho}{\mu}\right)_{object} \qquad \left(\frac{(0.2D)^2 N_m \text{ x } 1000}{0.001}\right)_{model} = \left(\frac{D^2\, 200 \text{ x } 750}{0.2}\right)_{object}$$

$$N_m = 18.75 \text{ rev/min}$$

$$\left(\frac{T}{D^5 N^2 \rho}\right)_{model} = \left(\frac{T}{D^5 N^2 \rho}\right)_{objectl} \qquad T_o = \left(\frac{T_m D_o^5 N_o^2 \rho_o}{D_m^5 N_m^2 \rho_m}\right) = \left(\frac{0.02 \text{ x } 5^5 200^2 \text{ x } 750}{1^5 \text{ x } 18.75^2 \text{ x } 1000}\right)$$

$$T_o = 5333 \text{ N}$$

2. The resistance to motion of a submarine due to viscous resistance is given by :

$\frac{R}{\rho v^2 D^2} = f\left(\frac{\rho v D}{\mu}\right)^d$ where D is the characteristic dimension.

The submarine moves at 8 m/s through sea water. In order to predict its resistance, a model is made to a scale of 1/100 and tested in fresh water. Determine the velocity at which the model should be tested. (690.7 m/s)
The density of sea water is 1036 kg/m^3
The density of fresh water is 1000 kg/m^3
The viscosity of sea water is 0.0012 N s/m^2.
The viscosity of fresh water is 0.001 N s/m^2.
When run at the calculated speed, the model resistance was 200 N. Predict the resistance of the submarine. (278 N).

$$\left(\frac{\rho v D}{\mu}\right)_m = \left(\frac{\rho v D}{\mu}\right)_o \qquad \left(\frac{1000 \text{ x } v_m \text{ x } D_m}{0.001}\right) = \left(\frac{1036 \text{ x } 8 \text{ x } D}{100 \text{ x } 0.0012}\right)$$

$$v_m = 690.7 \text{ m/s}$$

$$\left(\frac{R}{\rho v^2 D^2}\right)_m = \left(\frac{R}{\rho v^2 D^2}\right)_m \qquad \left(\frac{200}{1036 \text{ x } 8^2 (D_m/100)^2}\right) = \left(\frac{R_o}{1000 \text{ x } 690.7^2 D^2}\right)$$

$$R_o = 278 \text{ N}$$

3. The resistance of an aeroplane is due to, viscosity and compressibility of the fluid. Show that:

$$\left(\frac{R}{\rho v^2 D^2}\right) = f(M_a)(R_e)$$

An aeroplane is to fly at an altitude of 30 km at Mach 2.0. A model is to be made to a suitable scale and tested at a suitable velocity at ground level. Determine the velocity of the model that gives dynamic similarity for the Mach number and then using this velocity determine the scale which makes dynamic similarity in the Reynolds number. (680.6 m/s and 1/61.86)

The properties of air are

sea level	a= 340.3 m/s	μ= 1.7897 x 10^{-5}	ρ = 1.225 kg/m^3
30 km	a= 301.7 m/s	μ= 1.4745 x 10^{-5}	ρ = 0.0184 kg/m^3

When built and tested at the correct speed, the resistance of the model was 50 N. Predict the resistance of the aeroplane.

For dynamic similarity $(M_a)_m = (M_a)_o$ $\quad (v/a)_m = (v/a)_o = 2 \quad v_m = 340.3 = 680.6$ m/s

and $(R_e)_m = (R_e)_o$

$$\left(\frac{\rho v D}{\mu}\right)_m = \left(\frac{\rho v D}{\mu}\right)_o \qquad \left(\frac{D_o}{D}_m\right) = \left(\frac{\rho_m v_m \mu_o}{\rho_o v_o \mu_m}\right) = \frac{1.225 \text{ x } 680.6 \text{ x } 1.4745 \text{ x } 10^{-5}}{0.0184 \text{ x } 603.4 \text{ x } 1.7897 \text{ x } 10^{-5}} = 61.87$$

and $$\left(\frac{R}{\rho v^2 D^2}\right)_o = \left(\frac{R}{\rho v^2 D^2}\right)_m \qquad R_o = \left(\frac{R_m\, \rho_o\, v_o^2 D_o^2}{\rho_m\, v_m^2 D_m^2}\right) = \frac{50 \text{ x } 0.0184 \text{ x } 602.4^2}{1.225 \text{ x } 680.6^2} \text{ x } 61.87^2$$

R_o = 2259 N

4. The force on a body of length 3 m placed in an air stream at 1 bar and moving at 60 m/s is to be found by testing a scale model. The model is 0.3 m long and placed in high pressure air moving at 30 m/s. Assuming the same temperature and viscosity, determine the air pressure which produced dynamic similarity.

The force on the model is found to be 500 N. Predict the force on the actual body.

The relevant equation is $\left(\frac{F}{\rho v^2 D^2}\right) = f(R_e)$

For dynamic similarity $(R_e)_m = (R_e)_o$ $\quad \left(\frac{\rho v D}{\mu}\right)_m = \left(\frac{\rho v D}{\mu}\right)_o \quad \mu_o = \mu_m$

$(\rho v D)_m = (\rho v D)_o$

For a gas $p = \rho RT$ The temperature T is constant so $\rho \propto p = c\,p$

p_m x 30 x 0.3 = p_o x 60 x 3 = 1bar x 60 x 3 $\quad p_m$ = 20 bar

$$\left(\frac{F}{\rho v^2 D^2}\right)_m = \left(\frac{F}{\rho v^2 D^2}\right)_o \qquad \left(\frac{500}{c\,p_m \text{ x } 30^2 \text{ x } 0.3^2}\right) = \left(\frac{F_o}{c\,p_o \text{ x } 60^2 \text{ x } 3^2}\right)$$

F_o = 10 000N

5. Show by dimensional analysis that the velocity profile near the wall of a pipe containing turbulent flow is of the form $u^+ = f(y^+)$

$u^+ = u(\rho/\tau_O)^{1/2}$ and $y^+ = y(\rho\tau_O)^{1/2}/\mu$

When water flows through a smooth walled pipe 60 mm bore diameter at 0.8 m/s, the velocity profile is $u^+ = 2.5\ln(y^+) + 5.5$

Find the velocity 10 mm from the wall.
The friction coefficient is $C_f = 0.079\ R_e^{-0.25}$.

This is best solved by Buckingham's Pi method.

$$\tau_o = \phi(y\ u\ \rho\ \mu)$$

Form a dimensionless group with μ and leave out u that means the group $\left(y^{x_1}u^{y_1}\rho^{z_1}\mu^1\right)$has no dimensions

$$M^0L^0T^0 = (L)^{x_1}\left(LT^{-1}\right)^{y_1}\left(ML^{-3}\right)^{z_1}\left(ML^{-1}T^{-1}\right)^1$$

Time	$0 = 1 - 2\ z_1$	z_1- ½
Mass	$0 = y_1 + z_1 + 1$	$y_1 = -$ ½
Length	$0 = x_1 - 3\ y_1\ - z_1\ - 1$	$x_1 = -1$

The group is $\mu\ y^{-1}\ \rho^{-½}\ \tau_o^{-½}$ or $\dfrac{\mu}{y(\rho\tau_o)^{1/2}}$

Form a dimensionless group with u and leave out μ that means the group $\left(y^{x_2}u^{y_2}\rho^{z_2}u^1\right)$has no dimensions

$$M^0L^0T^0 = (L)^{x_2}\left(LT^{-1}\right)^{y_2}\left(ML^{-3}\right)^{z_2}\left(LT^{-1}\right)^1$$

Time	$0 = 2\ z_2 - 1$	z_2- ½
Mass	$0 = y_2 + z_2$	y_2 = ½
Length	$0 = x_2 - 3\ y_2\ - z_2\ + 1$	$x_2 = 0$

The group is $u\ y^0\ \rho^{½}\ \tau_o^{-½}$ or $u\left(\dfrac{\rho}{\tau_o}\right)^{1/2}$

Hence $$u\left(\frac{\rho}{\tau_o}\right)^{1/2} = f\left(\frac{\mu}{y(\rho\tau_o)^{1/2}}\right) \quad \text{or} \quad u^+ = f(y^+)$$

$C_f = 2\tau_o/(\rho u^2)$= Wall Shear Stress/Dynamic Pressure
$C_f = 0.079\ R_e^{-0.25}$ $\quad 0.079\ (\rho\ u\ D/\mu)^{-0.25} = 5.1879 \times 10^{-3}$
$5.1879 \times 10^{-3} = 2\tau_o/(997 \times 0.8^2)$ $\quad \tau_o = 1.655$ Pa
$u^+ = 2.5\ln y^+ + 5.5$ $\quad u(\rho/\tau_O)^{1/2} = 2.5\ln\{(y/\mu)(\rho\tau_O)^{1/}\} + 5.5$
$u(997/1.6551)^{1/2} = 2.5\ln\{(0.01/0.89\text{x}10^{-3})(1.655 \times 997)^{1/2}\} + 5.5$
$24.543\ u = 2.5\ln 451.42 + 5.5$

u = 0.85 m/s