FLUID MECHANICS D203
SAE SOLUTIONS TUTORIAL 5 – POTENTIAL FLOW

SELF ASSESSMENT EXERCISE No.1

1.a. Show that the potential function $\phi = A(r + B/r)\cos\theta$ represents the flow of an ideal fluid around a long cylinder. Evaluate the constants A and B if the cylinder is 40 mm radius and the velocity of the main flow is 3 m/s.

b. Obtain expressions for the tangential and radial velocities and hence the stream function ψ.
c. Evaluate the largest velocity in the directions parallel and perpendicular to the flow direction. (6 m/s for tangential velocity)
d. A small neutrally buoyant particle is released into the stream at r = 100 mm and $\theta = 150^o$. Determine the distance at the closest approach to the cylinder. (66.18 mm)

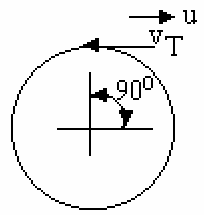
Part (a) is as given in the tutorial. Normally this equation is given as $\phi = (A\,r + B/r)\cos\theta$ but both are the same but the constants represent different values.

Part (b)
The values of the constants depend upon the quadrant selected to solve the boundary conditions. This is because the sign of the tangential velocity and radial velocity are different in each quadrant. Which ever one is used, the final result is the same. Let us select the quadrant from 90^o to 180^o.
At a large distance from the cylinder and at the 90^o position the velocity is from left to right so at this point $v_T = -u$. From equation 4 we have

$$v_T = \frac{d\varphi}{r d\theta} \qquad \varphi = A\left\{r + \frac{B}{r}\right\}\cos\theta$$

$$v_T = -\frac{A}{r}\left\{r + \frac{B}{r}\right\}\sin\theta = -A\left\{1 + \frac{B}{r^2}\right\}\sin\theta$$

$$v_R = \frac{d\varphi}{dr} = A\left(1 - \frac{B}{r^2}\right)\cos\theta$$



Putting r = infinity and $\theta = 90^o$ and remembering that $+v_T$ is anticlockwise +u is left to right, we have $v_T = -3$ m/s. $B/r^2 \to 0$

$$v_T = -A\left\{1 + \frac{B}{r^2}\right\}\sin\theta = -3 = -A\{1 + 0\} \qquad \text{Hence} \quad A = 3$$

At angle 180^o with r = 0.04, $v_R = 0$

$$v_R = A\left(1 - \frac{B}{r^2}\right)\cos\theta = 0 = 3\left(1 - \frac{B}{0.04^2}\right)(-1) \qquad \text{Hence} \quad B = 0.0016$$

$$v_T = -3\left\{1 + \frac{0.0016}{r^2}\right\}\sin\theta \qquad v_R = 3\left(1 - \frac{0.0016}{r^2}\right)\cos\theta$$

$$d\psi = v_R\, r\, d\theta = 3\left(1 - \frac{0.0016}{r^2}\right) r\cos\theta\, d\theta = 3\left(r - \frac{0.0016}{r}\right)\cos\theta\, d\theta \qquad \psi = 3\left(r - \frac{0.0016}{r}\right)\sin\theta$$

Part (c) The maximum velocity is 2u = 6 m/s (proof is in the tutorial)
Part (d) R = 0.1 m $\theta = 150^o$

$$\psi = 3\left(r - \frac{0.0016}{r}\right)\sin\theta = 3\left(0.1 - \frac{0.0016}{0.1}\right)\sin 150 = 3\left(0.1 - \frac{0.0016}{0.1}\right)\sin 150 = 0.244$$

The closest approach is at $\theta = 90^o$

$$\psi = 0.244 = 3\left(r - \frac{0.0016}{r}\right)\sin 90 \qquad 0.0813r = \left(r^2 - 0.0016\right)$$

$r^2 - 0.0813\,r - 0.0016 = 0$ solve the quadratic and r = 0.098 m or 98 mm

2.a. Show that the potential function $\phi = (Ar + B/r)\cos\theta$ gives the flow of an ideal fluid around a cylinder. Determine the constants A and B if the velocity of the main stream is u and the cylinder is radius R.

b. Find the pressure distribution around the cylinder expressed in the form $(p - p')/(\rho u^2/2)$ as a function of angle.

c. Sketch the relationship derived above and compare it with the actual pressure profiles that occur up to a Reynolds number of 5×10^5.

Part (a) is in the tutorial.

Part (b) $\quad v_T = \frac{d\varphi}{r d\theta} \qquad v_T = -\frac{1}{r}\left\{\frac{B}{r} + Ar\right\}\sin\theta \qquad v_T = -\left\{\frac{B}{r^2} + A\right\}\sin\theta$

Putting r = infinity and $\theta = 90^o$ and remembering that $+v_T$ is anticlockwise +u is left to right, we have

$$v_T = -u = -\left\{\frac{B}{r^2} + A\right\}\sin\theta = -\{0 + A\}x1$$

Hence $v_T = -A = -u$ so A = u as expected from earlier work.

At angle 180^o with r = R, the velocity is only radial in directions and is zero because it is arrested.

From equation 3 we have $\qquad v_R = \frac{d\varphi}{dr} = \left(-\frac{B}{r^2} + A\right)\cos\theta$

Putting r = R and $v_R = 0$ and $\theta = 180$ we have

$$0 = \left(-\frac{B}{R^2} + A\right)(-1) = \left(\frac{B}{R^2} - A\right)$$

Put A = u $\qquad 0 = \frac{B}{R^2} - u \qquad B = uR^2$

Substituting for $B = uR^2$ and A = u we have

$$\varphi = \left\{\frac{B}{r} + Ar\right\}\cos\theta = \left\{\frac{uR^2}{r} + ur\right\}\cos\theta$$

At the surface of the cylinder r = R the velocity potential is

$$\varphi = \{uR + uR\}\cos\theta = 2uR\cos\theta$$

The tangential velocity on the surface of the cylinder is then

$$v_T = \frac{d\varphi}{rd\theta} = -\left\{\frac{B}{r^2} + A\right\}\sin\theta \qquad v_T = -\left\{\frac{uR^2}{r^2} + u\right\}\sin\theta \qquad v_T = -2u\sin\theta$$

This is a maximum at $\theta = 90^o$ where the streamlines are closest together so the maximum velocity is 2u on the top and bottom of the cylinder.

The velocity of the main stream flow is u and the pressure is p'. When it flows over the surface of the cylinder the pressure is p because of the change in velocity. The pressure change is p - p'.

The dynamic pressure for a stream is defined as $\rho u^2/2$

The pressure distribution is usually shown in the dimensionless form

$$2(p - p')/(\rho u^2)$$

For an infinitely long cylinder placed in a stream of mean velocity u we apply Bernoulli's equation between a point well away from the stream and a point on the surface. At the surface the velocity is entirely tangential so :

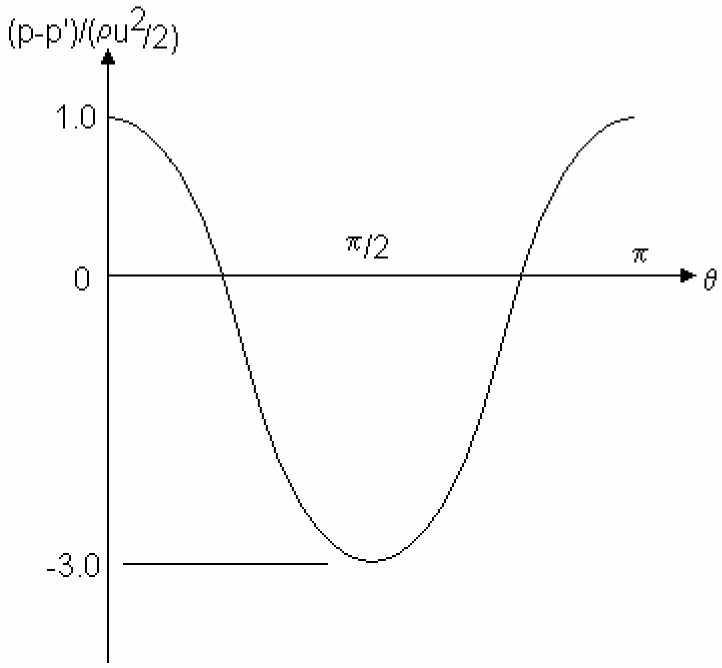
$$p' + \rho u^2/2 = p + \rho v_T^2/2$$

From the work previous this becomes

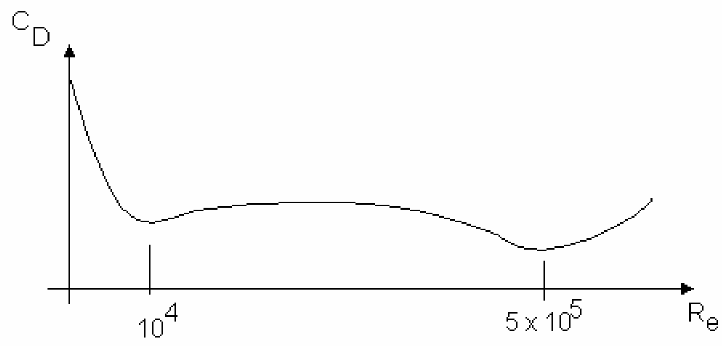
$$p' + \rho u^2/2 = p + \rho(2u\sin\theta)^2/2$$

$$p - p' = \rho u^2/2 - (\rho/2)(4u^2\sin^2\theta) = (\rho u^2/2)(1 - 4\sin^2\theta)$$
$$(p - p')/(\rho u^2/2) = 1 - 4\sin^2\theta$$

If this function is plotted against angle we find that the distribution has a maximum value of 1.0 at the front and back, and a minimum value of -3 at the sides.

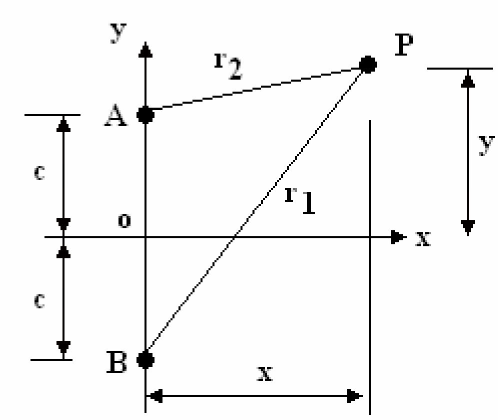


Research shows that the drag coefficient reduces with increased stream velocity and then remains constant when the boundary layer achieves separation. If the mainstream velocity is further increased, turbulent flow sets in around the cylinder and this produces a marked drop in the drag. This is shown below on the graph of C_D against Reynolds's number. The point where the sudden drop occurs is at a critical value of Reynolds's number of 5 x10^5.



3.Show that in the region y>0 the potential function $\phi = a \ln[x^2 + (y-c)^2] + a \ln[x^2 + (y+c)^2]$ gives the 2 dimensional flow pattern associated with a source distance c above a solid flat plane at y=0.

b. Obtain expressions for the velocity adjacent to the plane at y = 0. Find the pressure distribution along this plane.
c. Derive an expression for the stream function ϕ.



The key to this problem is knowing that two identical sources of strength m equal distance above and below the origin produces the pattern required.

$\phi_A = -(m/2\pi)\ln r_2$

$\phi_B = -(m/2\pi)\ln r_1$ Now use pythagoras

$r_2 = \{x^2 + (y-c)^2\}^{1/2}$ $r_1 = \{x^2 + (y+c)^2\}^{1/2}$

$\phi_p = \phi_A + \phi_B$

$$\varphi_p = -\frac{m}{2\pi}\left[\ln\left\{x^2 + (y+c)^2\right\}^{1/2} + \ln\left\{x^2 + (y+c)^2\right\}^{1/2}\right]$$

$$\varphi_p = -\frac{m}{4\pi}\left[\ln\left\{x^2 + (y+c)^2\right\} + \ln\left\{x^2 + (y+c)^2\right\}\right]$$

$$\varphi_p = a\left[\ln\left\{x^2 + (y+c)^2\right\} + \ln\left\{x^2 + (y+c)^2\right\}\right] \quad a = -m/2\pi$$

At y = 0 $\phi = 2a \ln(x^2 + c^2)$

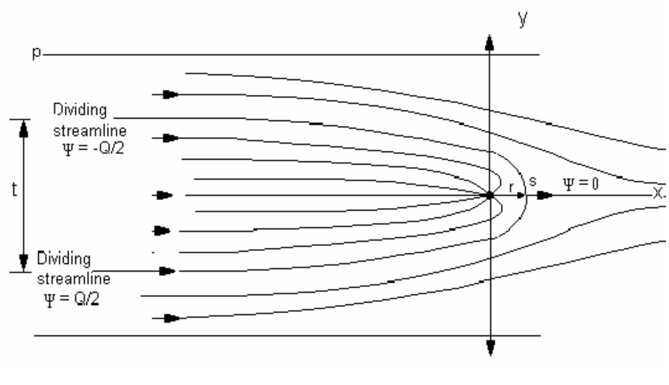
v = $-d\phi/dy = 0$ at all values of x so it is the same as an impervious plane. u = $-d\phi/dx = -\dfrac{4ax}{x^2 + c^2}$

At very large values of x, u = 0 and p = p_o

Apply Bernoulli and $p_o = p + \rho u^2/2 = p_o + \dfrac{\rho}{2}\left(\dfrac{4ax}{x^2 + c^2}\right)^2$

4. A uniform flow has a sink placed in it at the origin of the Cartesian co-ordinates. The stream function of the uniform flow and sink are
$\psi_1 = Uy$ and $\psi_2 = B\theta$

Write out the combined stream function in Cartesian co-ordinates.



Given U=0.001 m/s and B= -0.04 $m^3/3$ per m thickness, derive the velocity potential.
Determine the width of the flux into the sink at a large distance upstream.

$\psi_1 = u\,y$ $\psi_2 = B\theta$ $\psi = u\,y + B\theta$ $B = Q/2\pi$ for a sink

$$-\frac{d\psi}{r d\theta} = -\frac{ur\cos\theta + B}{r} \qquad d\phi = \left(-u\cos\theta + \frac{B}{r}\right)dr$$

$$\phi = (-u\,r\cos\theta + B\ln r) \qquad \phi = (-u\,x + B\ln r) \qquad \phi = \left(-u\,x + B\ln\left\{x^2 + y^2\right\}^{1/2}\right)$$

u = 0.001 B = -0.04

$$\phi = \left(-0.001\,x + 0.04\ln\left\{x^2 + y^2\right\}^{1/2}\right) \qquad \phi = \left(-0.001\,x + 0.02\ln\left\{x^2 + y^2\right\}\right)$$

Q=ut t = Q/u Q = 2π x 0.04 u = 0.001 t = (2π x 0.04)/0.001 = 80π metres

<u>**SELF ASSESSMENT EXERCISE No.2**</u>

1. Define the following terms.

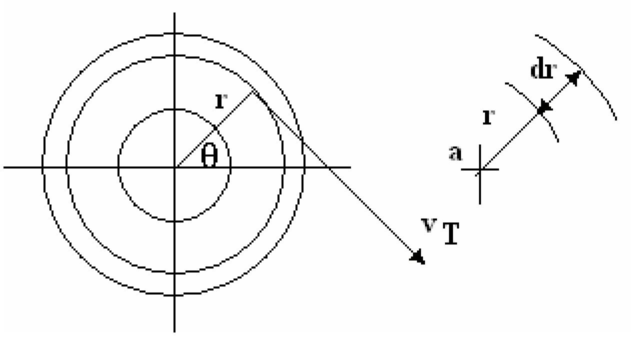
Stream function.
Velocity potential function.
Streamline
Stream tube
Circulation
Vorticity.

All these definitions are in the tutorial.

2.A free vortex of with circulation $K = 2\pi v_T R$ is placed in a uniform flow of velocity u.

Derive the stream function and velocity potential for the combined flow.

The circulation is 7 m^2/s and it is placed in a uniform flow of 3 m/s in the x direction. Calculate the pressure difference between a point at x = 0.5 and y = 0.5.
The density of the fluid is 1000 kg/m^3.
(Ans. 6695 Pascal)



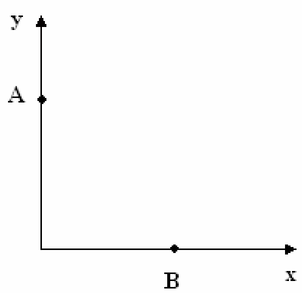
Free Vortex $v_T = k/2\pi r$ $\quad d\psi = v_T\, dr = (k/2\pi r)\, dr$

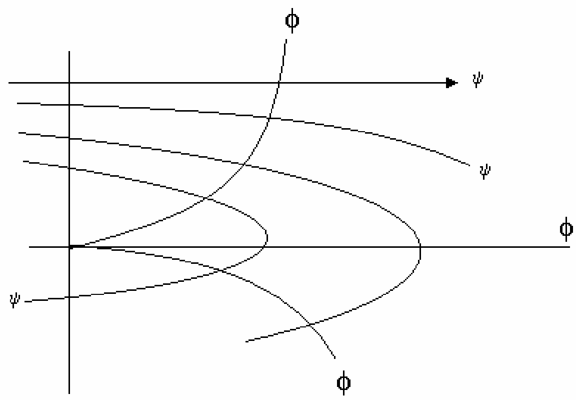
$$\psi = \int_a^r \frac{k}{2\pi\pi} dr = \frac{k}{2\pi}\ln\left(\frac{r}{a}\right) \qquad \phi = \int_0^\theta v_T r\, d\theta = \int_0^\theta \frac{k}{2\pi} r d\theta = \frac{k}{2\pi}\theta$$

Uniform flow
$\psi = -u\,y = -\,u\,r\sin\theta$ $\qquad \phi = u\,r\cos\theta$

Combined Flow

$$\psi = \frac{k}{2\pi}\ln\left(\frac{r}{a}\right) - u\,r\sin\theta \qquad \phi = \frac{k}{2\pi}\theta + u\,r\cos\theta$$





$v_\theta = \sqrt{(v_T^2 + v_R^2)}$ k = 7 u = 3

$v_R = d\phi/dr = u \cos\theta$ $v_T = d\psi/dr = k/2\pi r - u \sin\theta$

Point A $v_T = 7/(2\ \pi\ 0.5) - 3 \sin 90^o = -0.7718$ m/s

$v_R = 0$ $v_\theta = \sqrt{(v_T^2 + v_R^2)} = 0.7718$ m/s

Pont B $\theta = 0^o$ $v_R = u \cos\theta = 3$ $v_T = 7/(2\ \pi\ 0.5) - 3 \sin 0^o = 2.228$ m/s

$v_\theta = \sqrt{(v_T^2 + v_R^2)} = 3.74$ m/s

Bernoulli between stream and A $p = p_A + \rho\ v_\theta{}^2/2$

$p_A - p_\theta = (\rho/2)(v_{\theta B}^2 - v_{\theta A}^2) = (1000/2)(3.74^2 - 0.7718^2) = 6695$ Pa