Unit 41: Fluid Mechanics

Unit code: T/601/1445

QCF Level: 4

Credit value: 15

OUTCOME 3

TUTORIAL 5 – DIMENSIONAL ANALYSIS

3 Be able to determine the behavioural characteristics and parameters of real fluid flow

Head losses: head loss in pipes by Darcy's formula; Moody diagram; head loss due to sudden enlargement and contraction of pipe diameter; head loss at entrance to a pipe; head loss in valves; flow between reservoirs due to gravity; hydraulic gradient; siphons; hammerblow in pipes

Reynolds number: inertia and viscous resistance forces; laminar and turbulent flow; critical velocity

Viscous drag: dynamic pressure; form drag; skin friction drag; drag coefficient

Dimensional analysis: checking validity of equations such as those for pressure at depth; thrust on immersed surfaces and impact of a jet; forecasting the form of possible equations such as those for Darcy's formula and critical velocity in pipes

CONTENTS

1. BASIC DIMENSIONS

All quantities used in engineering can be reduced to six basic dimensions. These are the dimensions of

Mass	M
Length	L
Time	T
Temperature	θ
Electric Current	I
Luminous Intensity	J

The last two are not used in fluid mechanics and temperature is only used sometimes.

All engineering quantities can be defined in terms of the four basic dimensions M,L,T and θ. We could use the S.I. units of kilogrammes, metres, seconds and Kelvins, or any other system of units, but if we stick to M,L,T and θ we free ourselves of any constraints to a particular system of measurements.

Let's now explain the above with an example. Consider the quantity *density*. The S.I. units are kg/m^3 and the imperial units are lb/in^3. In our system the units would be Mass/Length^3 or M/L^3. It will be easier in the work ahead if we revert to the inverse indice notation and write it as ML^{-3}.

Other engineering quantities need a little more thought when writing out the basic MLTθ dimensions. The most important of these is the unit of force or the Newton in the S.I. system. Engineers have opted to define force as that which is needed to accelerate a mass such that 1 N is needed to accelerate 1 kg at 1 m/s^2. From this we find that the Newton is a derived unit equal to 1 kg m/s^2. In our system the dimensions of force become MLT^{-2}. This must be considered when writing down the dimensions of anything containing force.

Another unit that produces problems is that of angle. Angle is a ratio of two sides of a triangle and so have neither units nor dimensions at all. This also applies to revolutions which is an angular measurement. Strain is also a ratio and has neither units nor dimensions. Angle and strain are in fact examples of dimensionless quantities which will be considered in detail later.

WORKED EXAMPLE No.1

Write down the basic dimensions of pressure p.

SOLUTION

Pressure is defined as p = Force/Area

The S.I. unit of pressure is the Pascal which is the name for $1N/m^2$.
Since force is MLT^{-2} and area is L^2 then the basic dimensions of pressure are

$$ML^{-1}T^{-2}$$

When solving problems it is useful to use a notation to indicate the MLT dimensions of a quantity and in this case we would write

$$[p] = ML^{-1}T^{-2}$$

WORKED EXAMPLE No. 2

Deduce the basic dimensions of dynamic viscosity.

SOLUTION

Dynamic viscosity was defined in an earlier tutorial from the formula $\tau = \mu du/dy$

τ is the shear stress, du/dy is the velocity gradient and μ is the dynamic viscosity. From this we have $\mu = \tau\, dy/du$

Shear stress is force/area.
The basic dimensions of force are MLT^{-2}
The basic dimensions of area are L^2.
The basic dimensions of shear stress are $ML^{-1}T^{-2}$.
The basic dimensions of distance y are L.
The basic dimensions of velocity v are LT^{-1}.
It follows that the basic dimension of dy/du (a differential coefficient) is T.
The basic dimensions of dynamic viscosity are hence $(ML^{-1}T^{-2})(T) = ML^{-1}T^{-1}$.

$[\mu] = ML^{-1}T^{-1}$.

2. LIST OF QUANTITIES AND DIMENSIONS FOR REFERENCE.

AREA	$(LENGTH)^2$	L^2
VOLUME	$(LENGTH)^3$	L^3
VELOCITY	LENGTH/TIME	LT^{-1}
ACCELERATION	LENGTH/($TIME^2$)	LT^{-2}
ROTATIONAL SPEED	REVOLUTIONS/TIME	T^{-1}
FREQUENCY	CYCLES/TIME	T^{-1}
ANGULAR VELOCITY	ANGLE/TIME	T^{-1}
ANGULAR ACCELERATION	ANGLE/$(TIME)^2$	T^{-2}
FORCE	MASS X ACCELERATION	MLT^{-2}
ENERGY	FORCE X DISTANCE	ML^2T^{-2}
POWER	ENERGY/TIME	ML^2T^{-3}
DENSITY	MASS/VOLUME	ML^{-3}
DYNAMIC VISCOSITY	STRESS/VELOCITY GRADIENT	$ML^{-1}T^{-1}$
KINEMATIC VISCOSITY	DYNAMIC VIS/DENSITY	L^2T^{-1}
PRESSURE	FORCE/AREA	$ML^{-1}T^{-2}$
SPECIFIC HEAT CAPACITY	ENERGY/(MASS X TEMP)	$L^2T^{-2}\theta^{-1}$
TORQUE	FORCE X LENGTH	ML^2T^{-2}
BULK MODULUS	PRESSURE/STRAIN	$ML^{-1}T^{-2}$

3. HOMOGENEOUS EQUATIONS

All equations must be homogeneous. Consider the equation $F = 3 + T/R$
F is force, T is torque and R is radius.
Rearranging we have $3 = F - T/R$

Examine the units.

F is Newton. T is Newton metre and R is metre.

hence $3 = F\,(N) - T/R\,(N\,m)/m)$
$3 = F(N) - T/R\,(N)$

It follows that the number 3 must represent 3 Newton. It also follows that the unit of F and T/R must both be Newton. If this was not so, the equation would be nonsense. In other words all the components of an equation which add together must have the same units. You cannot add dissimilar quantities. For example you cannot say that 5 apples + 6 pears = 11 plums. This is clearly nonsense. When all parts of an equation that add together have the same dimensions, then the equation is homogeneous.

WORKED EXAMPLE No. 3

Show that the equation Power = Force x velocity is homogeneous in both S.I. units and basic dimensions.

SOLUTION

The equation to be checked is $P = F\,v$

The S.I. Unit of power (P) is the Watt. The Watt is a Joule per second. A Joule is a Newton metre of energy. Hence a Watt is 1 N m/s.

The S.I. unit of force (F) is the Newton and of velocity (v) is the metre/second.

The units of F v are hence N m/s.

It follows that both sides of the equation have S.I. units of N m/s so the equation is homogeneous.

Writing out the MLT dimensions of each term we have

$[P] = ML^2T^{-3}$
$[v] = LT^{-1}$
$[F] = MLT^{-2}$

Substituting into the equation we have $ML^2T^{-3} = MLT^{-2}\;LT^{-1} = ML^2T^{-3}$

Hence the equation is homogeneous.

4. INDECIAL EQUATIONS

When a phenomenon occurs, such as a swinging pendulum as shown in figure 14 we observe the variables that effect each other. In this case we observe that the frequency, (f) of the pendulum is affected by the length (l) and the value of gravity (g). We may say that frequency is a function of l and g. In equation form this is as follows.

$f = \phi(l,g)$ where ϕ is the function sign.

When we remove the function sign we must put in a constant because there is an unknown number and we must allocate unknown indices to l and g because we do know not what if any they are. The equation is written as follows. $f = C\, l^a g^b$

C is a constant and has no units. a and b are unknown indices.

This form of relating variables is called an indicial equation. The important point here is that because we know the units or dimensions of all the variables, we can solve the unknown indices.

WORKED EXAMPLE No. 4

Solve the relationship between f, l and g for the simple pendulum.

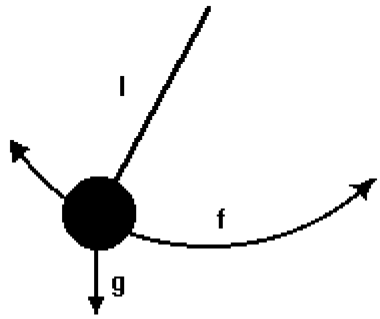


Fig.1

SOLUTION

First write down the indecial form of the equation (covered ov

$f = C\, l^a g^b$

Next write down the basic dimensions of all the variables.

$[f] = T^{-1}$

$[l] = L^{1}$

$[g] = LT^{-2}$

Next substitute the dimensions in place of the variables.

$T^{-1} = (L^1)^a\,(LT^{-2})^b$

Next tidy up the equation. $T^{-1} = L^{1a}\,L^{b}T^{-2b}$

Since the equation must be homogeneous then the power of each dimension must be the same on the left and right side of the equation. If a dimension does not appear at all then it is implied that it exists to the power of zero. We may write them in until we get use to it. The equation is written as follows.

$M^0L^0T^{-1} = L^{1a}\,LT^{-2b}M^0$

Next we equate powers of each dimension. First equate powers of Time.

$T^{-1} = T^{-2b}$ $\quad -1 = -2b \quad b = 1/2$

Next equate powers of Length.

$L^0 = L^{1a}\,L^b$ $\quad 0 = 1a + b$ hence $a = -b = -1/2$

$M^0 = M^0$ yields nothing in this case.

Now substitute the values of a and b back into the original equation and we have the following.

$f = C\, l^{-½}\, g^{½}$ $\quad f = C\,(g/l)^{½}$

The frequency of a pendulum may be derived from basic mechanics and shown to be

$f = (1/2\pi)(g/l)^{½}$

If we did not know how to find $C = (½\ \pi)$ from basic mechanics, then we know that if we conducted an experiment and measured the values f for various values of l and g, we could find C by plotting a graph of f against $(g/l)^{½}$. This is the importance of dimensional analysis to fluid mechanics. We are able to determine the basic relationships and then conduct experiments and determine the remaining unknown constants. We are able to plot graphs because we know what to plot against what.

SELF ASSESSMENT EXERCISE No. 1

1. It is observed that the velocity 'v' of a liquid leaving a nozzle depends upon the pressure drop 'p' and the density 'ρ'. Show that the relationship between them is of the form

$$v = C\left(\frac{p}{\rho}\right)^{\frac{1}{2}}$$

2. It is observed that the speed of a sound in 'a' in a liquid depends upon the density 'ρ' and the bulk modulus 'K'. Show that the relationship between them is

$$a = C\left(\frac{K}{\rho}\right)^{\frac{1}{2}}$$

3. It is observed that the frequency of oscillation of a guitar string 'f' depends upon the mass 'm', the length 'l' and tension 'F'. Show that the relationship between them is $f = C\left(\frac{F}{ml}\right)^{\frac{1}{2}}$

5. DIMENSIONLESS NUMBERS

We will now consider cases where the number of unknown indices to be solved, exceed the number of equations to solve them. This leads into the use of dimensionless numbers.

Consider that typically a problem uses only the three dimensions M, L and T. This will yield 3 simultaneous equations in the solution. If the number of variables in the equation gives 4 indices say a, b, c and d, then one of them cannot be resolved and the others may only be found in terms of it.

In general there are n unknown indices and m variables. There will be m-n unknown indices. This is best shown through a worked example.

WORKED EXAMPLE No. 5

The pressure drop per unit length 'p' due to friction in a pipe depends upon the diameter 'D' , the mean velocity 'v' , the density 'ρ' and the dynamic viscosity 'μ'. Find the relationship between these variables.

SOLUTION

p = function (D v ρ μ) = $K\ D^a\ v^b\ \rho^c\ \mu^d$

p is pressure per metre

$[p] = ML^{-2}T^{-2}$

$[D] = L$

$[v] = LT^{-1}$

$[\rho] = ML^{-3}$

$[\mu] = ML^{-1}T^{-1}$

$$ML^{-2}T^{-2} = L^a (LT^{-1})^b (ML^{-3})^c (ML^{-1}T^{-1})^d$$

$$ML^{-2}T^{-2} = L^{a+b-3c-d} M^{c+d} T^{-b-d}$$

The problem is now deciding which index not to solve. The best way is to use experience gained from doing problems. Viscosity is the quantity that causes viscous friction so the index associated with it (d) is the one to identify. We will resolve a, b and c in terms of d.

TIME $-2 = -b - d$ hence $b = 2 - d$ is as far as we can resolve

MASS $1 = c + d$ hence $c = 1 - d$
LENGTH $-2 = a + b - 3c - d$

$-2 = a + (2 - d) - 3(1-d) - d$ hence $a = -1 - d$

Now put these back into the original formula.

$$p = K\ D^{-1-d}\ v^{2-d}\ \rho^{1-d}\ \mu^d$$

Now group the quantities with same power together as follows :

$$p = K\{\rho v^2 D^{-1}\}\{\mu \rho^{-1} v^{-1} D^{-1}\}^d$$

Remember that p was pressure drop per unit length so the pressure loss over a length L is
$P = K\,L\{\rho v^2 D^{-1}\}\{\mu \rho^{-1} v^{-1} D^{-1}\}^d$

We have two unknown constants K and d. The usefulness of dimensional analysis is that it tells us the form of the equation so we can deduce how to present experimental data. With suitable experiments we could now find K and d.

Note that this equation matches up with Poiseuille's equation which gives the relationship as :

$$p = 32\, \mu\, L\, v\, D^{-2}$$

from which it may be deduced that $K = 32$ and $d = 1$ (laminar flow only)

The term $\{\rho v D \mu^{-1}\}$ has no units. If you check it out all the units will cancel. This is a ***DIMENSIONLESS NUMBER***, and it is named after Reynolds.

Reynolds Number is denoted R_e. The whole equation can be put into a dimensionless form as follows.

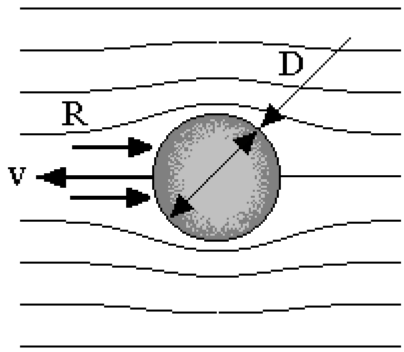
$$\{p\ \rho^{-1}\ L^{-1}\ v^{-2} D^1\} = K\{\mu\ \rho^{-1} v^{-1} D^{-1}\}^d$$

$$\{p\ \rho^{-1}\ L^{-1}\ v^{-2} D^1\} = \text{function}\ (R_e)$$

This is a dimensionless equation. The term $\{p\rho^{-1} L^{-1} v^{-2} D^1\}$ is also a dimensionless number.

Let us now examine another similar problem.

WORKED EXAMPLE No. 6



Consider a sphere moving through an viscous fluid Completely submerged. The resistance to motion R depends upon the diameter D, the velocity v, the density ρ and the dynamic viscosity μ.
Find the equation that relates the variables.

Figure 2

$R = \text{function}(D\ v\ \rho\ \mu) = K\ D^a\ v^b\ \rho^c\ \mu^d$

First write out the MLT dimensions.

$[R] = ML^1T^{-2}$
$[D] = L \qquad ML^1T^{-2} = L^a\,(LT^{-1})^b\,(ML^{-3})^c(ML^{-1}T^{-1})^d$
$[v] = LT^{-1} \qquad ML^1T^{-2} = L^{a+b-3c-d}\,M^{c+d}\,T^{-b-d}$
$[\rho] = ML^{-3}$
$[\mu] = ML^{-1}T^{-1}$

Viscosity is the quantity which causes viscous friction so the index associated with it (d) is the one to identify. We will resolve a,b and c in terms of d as before.

TIME $-2 = -b - d$ hence $b = 2 - d$ is as far as we can resolve b
MASS $1 = c + d$ hence $c = 1 - d$
LENGTH $1 = a + b - 3c - d$
$1 = a + (2 - d) - 3(1-d) - d$ hence $a = 2-d$

Now put these back into the original formula.

$$R = K\ D^{2-d}\,v^{2-d}\,\rho^{1-d}\,\mu^d$$

Now group the quantities with same power together as follows :

$$R = K\{\rho v^2D^2\}\,\{\mu\ \rho^{-1}v^{-1}D^{-1}\}^d$$
$$R\{\rho v^2D^2\}^{-1} = K\,\{\mu\ \rho^{-1}v^{-1}D^{-1}\}^d$$

The term $\{\rho vD\mu^{-1}\}$ is the Reynolds Number R_e and the term $R\{\rho v^2D^2\}^{-1}$ is called the Newton Number N_e. Hence the relationship between the variables may be written as follows.

$R\{\rho v^2D^2\}^{-1} = \text{function}\ \{\rho vD\ \mu^{-1}\}$
$N_e = \text{function}\ (R_e)$

Once the basic relationship between the variables has been determined, experiments can be conducted to find the parameters in the equation. For the case of the sphere in an incompressible fluid we have shown that

$N_e = \text{function}\ (R_e)$ Or put another way $N_e = K\,(R_e)^n$

K is a constant of proportionality and n is an unknown index (equivalent to -d in the earlier lines). In logarithmic form the equation is

$\log(N_e) = \log(K) + n\log(R_e)$

This is a straight line graph from which log K and n are taken. Without dimensional analysis we would not have known how to present the information and plot it. The procedure now would be to conduct an experiment and plot log(Ne) against log(Re). From the graph we would then determine K and n.

6. BUCKINGHAM'S Π (Pi) THEORY

Many people prefer to find the dimensionless numbers by intuitive methods. Buckingham's theory is based on the knowledge that if there are m basic dimensions and n variables, then there are m - n dimensionless numbers. Consider worked example No.12 again. We had the basic equation

$$R = \text{function}\ (D\ v\ \rho\ \mu)$$

There are 5 quantities and there will be 3 basic dimensions ML and T. This means that there will be 2 dimensionless numbers Π_1 and Π_2. These numbers are found by choosing two prime quantities (R and μ).

Π_1 is the group formed between μ and D v ρ
Π_2 is the group formed between R and D v ρ

First taking μ. Experience tells us that this will be the Reynolds number but suppose we don't know this.

The dimensions of μ are $ML^{-1}T^{-1}$

The dimensions of D v ρ must be arranged to be the same.

$$\mu = \Pi_1\ D^a v^b\ \rho^c$$

$$M^1L^{-1}T^{-1} = \Pi_1\ (L)^a\ (LT^{-1})^b\ (ML^{-3})^c$$

Time	**-1 = -b**	**b = 1**
Mass		**c = 1**
Length	-1 = a + b -3c	
	-1 = a + 1 – 3	a=1

$$\mu = \Pi_1\ D^1 v^1\ \rho^1$$

$$\Pi_1 = \frac{\mu}{Dv\rho}$$

The second number must be formed by combining R with ρ,v and D

$R = \Pi_2\ D^a v^b\ \rho^c$
$MLT^{-2} = \Pi_2\ (L)^a\ (LT^{-1})^b (ML^{-3})^c$

Time	**-2 = -b**	**b = 2**
Mass		**c = 1**
Length	1 = a + b -3c	
	1 = a + 2 – 3	a = 2

$R = \Pi_2\ D^2 v^2\ \rho^1$

$$\Pi_2 = \frac{R}{\rho v^2 D^2}$$

***The dimensionless equation is* $\Pi_2 = f(\Pi_1)$**

WORKED EXAMPLE No. 7

The resistance to motion 'R' for a sphere of diameter 'D' moving at constant velocity 'v' through a compressible fluid is dependant upon the density 'ρ' and the bulk modulus 'K'. The resistance is primarily due to the compression of the fluid in front of the sphere. Show that the dimensionless relationship between these quantities is N_e = function (M_a)

SOLUTION

$R = \text{function}(D\ v\ \rho\ K) = C\ D^a\ v^b\ \rho^c\ K^d$

There are 3 dimensions and 5 quantities so there will be 5 –3 = 2 dimensionless numbers. Identify that the one dimensionless group will be formed with R and the other with K.

Π_1 is the group formed between K and D v ρ

Π_2 is the group formed between R and D v ρ

$K = \Pi_2\ D^a\ v^b\ \rho^c$

$[K] = ML^{-1}T^{-2}$

$[D] = L$

$[v] = LT^{-1}$

$[\rho] = ML^{-3}$

$ML^{-1}T^{-2} = L^a\ (LT^{-1})^b\ (ML^{-3})^c$

$ML^{-1}T^{-2} = L^{a+b-3c}\ M^c\ T^{-b}$

Time **-2 = -b** **b = 2**

Mass **c = 1**

Length -1 = a + b -3c

-1 = a + 2 – 3 a = 0

$K = \Pi_2\ D^o\ v^2\ \rho^1$

$$\Pi_2 = \frac{K}{\rho v^2}$$

$R = \Pi_1\ D^a\ v^b\ \rho^c$

$[R] = MLT^{-2}$

$[D] = L$

$[v] = LT^{-1}$

$[\rho] = ML^{-3}$

$MLT^{-2} = L^a\ (LT^{-1})^b\ (ML^{-3})^c$

$ML^1T^{-2} = L^{a+b-3c}\ M^c\ T^{-b}$

Time **-2 = -b** **b = 2**

Mass **c = 1**

Length 1 = a + b -3c

1 = a + 2 – 3 a = 2

$R = \Pi_1\ D^2\ v^2\ \rho^1$

$$\Pi_1 = \frac{R}{\rho v^2 D^2}$$

It was shown earlier that the speed of sound in an elastic medium is given by the following formula.

$$a = (k/\rho)^{½}$$

It follows that $(k/\rho) = a^2$ and so $\Pi_2 = (a/v)^2$

The ratio v/a is called the Mach number (Ma) so $(Ma)^{-2}$

Π_1 is the Newton Number Ne.

The equation may be written as $\Pi_1 = \phi\Pi_2\ N_e$ or $Ne = \phi(M_a)$

SELF ASSESSMENT EXERCISE No. 2

1. The resistance to motion 'R' for a sphere of diameter 'D' moving at constant velocity 'v' on the surface of a liquid is due to the density 'ρ' and the surface waves produced by the acceleration of gravity 'g'. Show that the dimensionless equation linking these quantities is N_e = function(F_r)

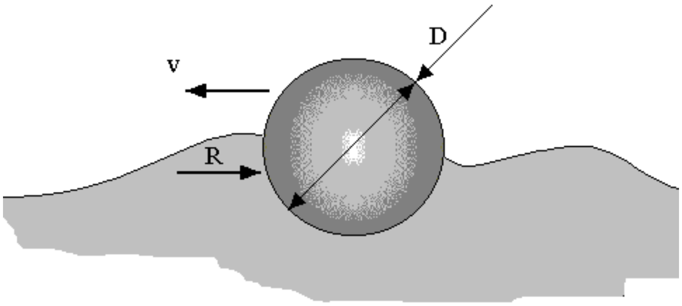


Figure 3

F_r is the Froude number and is given by $F_r = \sqrt{\frac{v^2}{gD}}$

Here is a useful tip. It is the power of g that cannot be found.

2. The Torque 'T' required to rotate a disc in a viscous fluid depends upon the diameter 'D' , the speed of rotation 'N' the density 'ρ' and the dynamic viscosity 'μ'. Show that the dimensionless equation linking these quantities is :

$$\{T\,D^{-5}\,N^{-2}\,\rho^{-1}\} = \text{function}\,\{\rho\,N\,D^{2}\,\mu^{-1}\}$$