3. DC and AC theory

**DC electrical principles:** Ohm's and Kirchoff's laws; voltage and current dividers; analogue and digital signals; review of motor and generator principles; fundamental relationships (eg resistance, inductance, capacitance; series C-R circuit, time constant, charge and discharge curves of capacitors, L-R circuits)

AC circuits: features of AC sinusoidal wave form for voltages and currents; explanation of how other more complex wave forms are produced from sinusoidal wave forms; R, L, C circuits (eg reactance of R, L and C components, equivalent impedance and admittance for R-L and R-C circuits); high or low pass filters; power factor; true and apparent power; resonance for circuits containing a coil and capacitor connected either in series or parallel; resonant frequency; Q-factor of resonant circuit

Transformers: high and low frequency; transformation ratio; current transformation; unloaded transformer; input impedance; maximum power transfer; transformer losses

### Outcomes and assessment criteria

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<th>Outcomes</th>
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<td>To achieve each outcome a learner must demonstrate the ability to:</td>
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| 1 Analyse static engineering systems | • determine distribution of shear force, bending moment and stress due to bending in simply supported beams.  
• select standard rolled steel sections for beams and columns to satisfy given specifications  
• determine the distribution of shear stress and the angular deflection due to torsion in circular shafts |
| 2 Analyse dynamic engineering systems | • determine the behaviour of dynamic mechanical systems in which uniform acceleration is present  
• determine the effects of energy transfer in mechanical systems  
• determine the behaviour of oscillating mechanical systems |
| 3 Apply DC and AC theory | • solve problems using DC electrical principles  
• recognise a variety of complex wave forms and explain how they are produced from sinusoidal wave forms  
• apply AC theory to the solution of problems on single phase R, L, C circuits and components  
• apply AC theory to the solution of problems on transformers |
| 4 Investigate information and energy control systems | • describe the method by which electrical signals convey information  
• describe the methods by which electrical signals control energy flow  
• select and interface system components to enable chosen system to perform desired operation |
Unless you are already familiar with a.c. theory, you will find outcome 3 and 4 are extremely large and covers a lot of material that takes far more time to study than that nominally allocated. In particular, those not following the electrical level H course will find this requires a lot of study time.

1. **PHASOR DIAGRAMS**

The way a sinusoidal voltage or current varies with time may be represented by the following equations.

\[
V \sin (\omega t) \text{ or } v = V \sin (2\pi f t) \quad \text{and} \quad i = I \sin (\omega t) \text{ or } i = I \sin (2\pi f t)
\]

V and I is the amplitude. f is the frequency in Hz. \( \omega \) is the angular frequency in radian/s \( \omega = 2\pi f \)

\( \omega t \) is an angle in radian and the meaning becomes clear when we look at phasors. We will examine the phasors for voltage but the theory is the same for current.

The diagram shows a phasor of length V rotating anticlockwise at \( \omega \) rad/s. Starting from the horizontal position after a time t it will have rotated an angle \( \theta = \omega t \). The vertical component of the phasor is \( v = V \sin (\theta) \) and this corresponds to the value of the sinusoidal graph at that angle. When \( \theta = \pi/2 \) radian (90°) the peak value is V so V is the amplitude or peak value (not the r.m.s.value).

2. **A.C. AND RESISTANCE**

When a.c. is applied to a pure resistance R, Ohm’s Law applies and since it is non-reactive it applies at all frequencies all moments in time so the phasors for voltage and current must rotate together. They are said to be IN PHASE.
3. **AC AND INDUCTANCE**

The voltage required to drive a current through an inductor is \( v = L \frac{di}{dt} \). \( L \) is the inductance in Henries and \( \frac{di}{dt} \) is the rate of change of current.

Suppose \( i = I \sin \omega t \)

Differentiating \( \frac{di}{dt} = I \cos \omega t \)

It follows that \( v = I L \omega \cos \omega t \) and the maximum value is \( V = I L \omega \)

If \( V \) and \( I \) are plotted together we see that that \( V \) is a \( \frac{1}{4} \) cycle displaced and it is said that the voltage leads the current by 90\(^\circ\). The voltage phasor is 90\(^\circ\) anticlockwise of the current phasor.

4. **AC WITH RESISTANCE AND INDUCTANCE**

Now consider ac applied to a resistor and inductor in series as shown.

The current \( I \) flows through both so this is used as the reference.

The voltage over the resistance is \( V_R = I R \) and on the phasor diagram this must be in the same direction as the current.

The voltage over the inductor is \( V_L = I X_L \) and this must lead the current by 90\(^\circ\) and also \( V_R \) by 90\(^\circ\).

It is not true to say that \( V = V_L + V_R \) because they must be treated as phasors or vectors.

The resultant voltage is \( V_S \) and this is the hypotenuse of a right-angled triangle so \( V = \sqrt{V_R^2 + V_L^2} \)

The angle \( \phi \) is called the phase angle and is always measured from \( V_R \). It follows that \( \phi = \tan^{-1}(V_L/V_R) \)

5. **AC AND CAPACITANCE**

When ac is applied across a capacitor, the voltage is given by the equation

\[ V_C = \frac{q}{C} \]

where \( q \) is the charge stored and \( C \) is the capacitance in Farads. Since \( q = \int i \, dt \) then \( V_C = \frac{\int i \, dt}{C} \)

\( i \) varies sinusoidally so that \( i = I \sin (\omega t) \) \hspace{1cm} \( \int i \, dt = -I \omega \cos(\omega t) \)

Substitute and \( V_C = -\frac{I \omega}{C} \cos(\omega t) \)
The maximum value of $V_C$ is $I\omega/c$ so this will be the length of the phasor representing $V_C$. If we plot $V_C$ and $I$ we find that $V_C$ lags the current by $1/4$ cycle or $90^\circ$. This is opposite to an inductor which leads by $90^\circ$.

6. **AC WITH RESISTANCE AND CAPACITANCE**

Now consider a resistor and capacitor in series as shown. The voltage over the resistance is $IR$ and on the phasor diagram this must be in the same direction as the current. It follows that $V_C$ lags $V_R$ by $90^\circ$. It is not true to say that $V = V_C + V_R$ because they must be treated as vectors.

The resultant voltage is the hypotenuse of a right-angled triangle so

$$V = \sqrt{V_R^2 + V_C^2}$$

The angle $\phi$ is called the phase angle and is always measured from $V_R$. It follows that $\phi = -\tan^{-1}(V_C/V_R)$

The only difference between this and the R L circuit is that $V_C$ lags $V_R$ and $V_L$ leads $V_R$. This means that $V_L$ and $V_C$ are $180^\circ$ out of phase in a series circuit.

7. **R L C IN SERIES**

The 3 voltages $V_R$, $V_L$ and $V_C$ are drawn as 3 phasors and the vector sum is found. It is convenient to draw $V_R$ horizontally but the vector diagram stays the same for all rotations.

Examining the small triangle, we see the vertical height is $V_L - V_R$ and the horizontal length is $V_R$. It follows that the resultant voltage is given by

$$V = \sqrt{(V_L - V_C)^2 + V_R^2} \quad \text{and} \quad \phi = \tan^{-1}\left(\frac{V_L - V_C}{V_R}\right)$$
8. **REACTANCE AND IMPEDANCE REVISITED**

We know from previous studies that the relationship between current and voltage for any component is related as a ratio $X = V/I$. For a resistor this ratio is resistance $R$ but for an inductor it is called inductive reactance $X_L$ and for a capacitor capacitive reactance $X_C$.

Inductive reactance increases with frequency and is given by $X_L = 2\pi fL$

Capacitive reactance decreases with frequency and is given by $X_C = 1/(2\pi fC)$

When current flows in a RLC circuit, the relationship between it and the resulting voltage is called the IMPEDANCE $Z$. $Z = V/I$ where $V$ and $I$ are the resulting r.m.s. volts and current.

Since reactance is $V/I$ it follows that it is also a phasor. The phasor diagram for a series R L C circuit may be drawn as shown with $R$ drawn horizontally to make it easier.

### WORKED EXAMPLE No. 1

A resistor of value 470 $\Omega$ is connected in series with a capacitor of 22 $\mu$F and an inductor of 50 mH and a voltage is applied across it. A current of 100 mA (rms) is produced. Determine the impedance, the phase angle between the voltage and current and the applied voltage when the frequency is 50 Hz

**SOLUTION**

\[
X_L = 2\pi fL = 2\pi \times 50 \times 50 \times 10^{-3} = 15.71 \Omega \\
X_C = 1/(2\pi fC) = 1/(2\pi \times 50 \times 22 \times 10^{-6}) = 144.6 \Omega \\
Z = \sqrt{(X_L - X_C)^2 + R^2} = \sqrt{(15.71 - 144.6)^2 + 470^2} = 487.4 \Omega \\
\phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right) = \tan^{-1}\left(\frac{15.71 - 144.6}{470}\right) = -15.3^\circ \\
V_S = I Z = 0.1 \times 487.4 = 48.7 \text{ V rms}
\]

### SELF ASSESSMENT EXERCISE No. 1

1. A resistor of value 4 $\Omega$ is connected in series with a capacitor of 47 $\mu$F and an inductor of 20 $\mu$H and a voltage is applied across it. A current of 50 mA (rms) is produced. Determine the impedance, the phase angle between the voltage and current and the applied voltage when the frequency is 100 Hz. (34 $\Omega$, -83.3$^\circ$ and 1.7 V)

2. A resistor of value 0.2 $\Omega$ is connected in series with a capacitor of 4.7 $\mu$F and an inductor of 5 mH and 0.5 V rms is applied across the ends. Determine the impedance, the phase angle between the voltage and current and the rms current when the frequency is 1000 Hz. (2.455 $\Omega$, -85.3$^\circ$ and 204 mA)
9. **ADMITTANCE and CONDUCTANCE**

These are mainly used in the solution of parallel circuits.

Conductance is the inverse of resistance and is denoted $G$. $G = \frac{1}{R}$ and it follows that $I_R = V \cdot G$

Admittance is the inverse of impedance and is denoted $Y$. $Y = \frac{1}{Z}$ and it follows that $I = V \cdot Y$

Susceptance is the inverse of reactance and is denoted $B$. $B = \frac{1}{X}$

It follows that $I_C = V \cdot B_C$ for a capacitor and $I_L = V \cdot B_L$ for an inductor.

The units of $Y$, $G$ and $B$ are Siemens symbol $S$. $1 \text{ S} = 1 \Omega^{-1}$

10. **PARALLEL CIRCUITS.**

The main point about parallel circuits is that the voltage is common to each and the current is different.

**PARALLEL RESISTORS**

The circuit shows 2 resistors in parallel. The parallel rule tells us

$$Z = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}}$$

Alternatively

$$\frac{1}{Z} = \frac{1}{R_1} + \frac{1}{R_2} \quad \text{or} \quad Y = G_1 + G_2$$

**PARALLEL RESISTOR, INDUCTOR AND CAPACITOR**

Consider the parallel circuit below. The voltage is common to all components but the current in the inductor is reactive and leads the voltage by $90^\circ$ and the current in the capacitor lags by $90^\circ$. Since the current in the resistor is in phase with the voltage, the phasor diagram for the currents is like this.

It is convenient to draw the phasors with $I_R$ horizontal and the resultant current is as shown.

The resultant current

$$I = \sqrt{\left(I_L - I_C\right)^2 + I_R^2}$$

The phase angle

$$\phi = \tan^{-1}\left(\frac{I_L - X_C}{R}\right)$$

If we substitute $I_R = V \cdot G$ and $I_L = V \cdot B_L$ and $I_C = V \cdot B_C$

$$Y = \sqrt{(B_L - B_C)^2 + G^2} \quad \phi = \tan^{-1}\left(\frac{B_L - B_C}{G}\right)$$

It also follows that we may represent $G$, $B$ and $Y$ as phasors.

You should decide which method you prefer to use.

The parallel circuit may be represented with $G$, $B$ and $Y$ as shown.
WORKED EXAMPLE No. 2

For the three circuits shown below, determine the supply current and phase angle between the supply voltage and current when 50 V rms is applied.

SOLUTIONS

(a) \[ Y = \sqrt{(B_L - B_C)^2 + G^2} = \sqrt{(0 - 0.2)^2 + 0.5^2} = 0.539 \, \text{S} \]
\[ I = V_S \times Y = 50 \times 0.539 = 26.926 \, \text{A} \]
\[ \phi = \tan^{-1} \left( \frac{B_L - B_C}{G} \right) = \tan^{-1} \left( \frac{0 - 0.2}{0.5} \right) = -21.8^\circ \]

(b) \[ Y = \sqrt{(B_L - B_C)^2 + G^2} = \sqrt{(0.125 - 0)^2 + 0.25^2} = 0.28 \, \text{S} \]
\[ I = V_S \times Y = 50 \times 0.28 = 13.98 \, \text{A} \]
\[ \phi = \tan^{-1} \left( \frac{B_L - B_C}{G} \right) = \tan^{-1} \left( \frac{0.125 - 0}{0.25} \right) = 26.6^\circ \]

(c) \[ Y = \sqrt{(B_L - B_C)^2 + G^2} = \sqrt{(0.04 - 0.05)^2 + 0.05^2} = 0.108 \, \text{S} \]
\[ I = V_S \times Y = 50 \times 0.108 = 5.385 \, \text{A} \]
\[ \phi = \tan^{-1} \left( \frac{B_L - B_C}{G} \right) = \tan^{-1} \left( \frac{0.04 - 0.05}{0.1} \right) = 21.8^\circ \]

SELF ASSESSMENT EXERCISE No. 2

1. In a parallel R L C circuit, \( R = 75 \, \text{k}\Omega \), \( L = 5 \, \mu\text{H} \) and \( C = 0.2 \, \text{nF} \). The voltage supply is 0.5 V at 2 Mz. Calculate the admittance, impedance of the circuit, the current and the phase angle.
   \( (0.016 \, \text{S}, 61.7 \, \Omega, 8.1 \, \text{mA and } -34.7^\circ) \)

2. In a parallel R L C circuit, \( R = 2 \, \text{K}\Omega \), \( L = 40 \, \text{mH} \) and \( C = 20 \, \text{nF} \). The voltage supply is 2 V at 100 kHz. Calculate the admittance, impedance of the circuit, the current and the phase angle.
   \( (0.03 \, \text{S}, 33 \, \Omega, 61 \, \text{mA and } 63.9^\circ) \)
11. **RESONANT CIRCUITS.**

**SERIES**

A series circuit is resonant when the inductive reactance is equal and opposite of the capacitive reactance. It follows that the phase angle is zero. At this condition the reactance is equal to R and is a minimum value. For a given circuit, there will be a frequency \( f_0 \) where this occurs.

\[
X_C = X_L = \frac{2\pi f_0 L}{2\pi f_0 C} = \frac{1}{2\pi f_0 L/R}
\]

**Q FACTOR**

It is quite possible to obtain voltages across a capacitor or inductor larger than the supply voltage. We get a magnification. To define this we use the Q factor defined as follows.

\[
Q = \frac{V_C}{V} \quad \text{for a capacitor and} \quad Q_L = \frac{V_L}{V} \quad \text{for an inductor.}
\]

At resonance \( V = IR \) since the capacitive and inductive components are equal and opposite so

\[
V_C = I X_C = \frac{I}{2\pi f_0 C} \quad \text{Q}_C = \frac{1}{2\pi f_0 RC}
\]

\[
V_L = I X_L = \frac{I}{2\pi f_0 L} \quad \text{Q}_L = \frac{2\pi f_0 L}{R}
\]

At any other frequency the Q factor is lower and needs to be worked out the hard way. Note that in both cases, the smaller the value of R the larger the Q factor.

Let’s take some typical values \( V_S = 10 \), \( C = 2\text{mF} \) and \( L = 2\text{mH} \) \( R = 0.1 \Omega \)

The resonant frequency is

\[
f_0 = \frac{1}{2\pi \sqrt{LC}} = \frac{1}{2\pi \sqrt{2 \times 10^{-3} \times 2 \times 10^{-3}}} = 79.6 \text{ Hz} \quad Q_C = \frac{1}{2\pi f_0 RC} = 10 \quad Q_L = \frac{2\pi f_0 L}{R} = 10
\]

If we calculate \( V_L \) and \( V_C \) over a range of frequencies we get the following result.

We see that the voltages peak at resonance is 100 giving \( Q = 10 \) as predicted. If \( R \) is zero, then in theory we get an infinite voltage at resonance. If we increase \( R \), we reduce the peak.
PARALLEL
Resonance in a parallel circuit occurs when $B_L = B_C$ and the resulting admittance is $G = 1/R$

We will find the result is the same and

$$f_o = \frac{1}{2\pi\sqrt{LC}}$$

WORKED EXAMPLE No. 3
A series circuit comprises of a resistance of 5 Ω, a capacitor of 2 nF and an inductor of 5 µH. Calculate the resonant frequency and the current at resonance when 1 V rms is applied. Calculate the Q factor at resonance.

SOLUTION
$$f_o = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{5 \times 10^{-6} \times 2 \times 10^{-9}}} = 1.592 \text{ MHz}$$

$$I = \frac{V}{Z} = \frac{V}{R} = \frac{1}{5} = 0.2 \text{ A}$$

$$Q_C = \frac{1}{2\pi f_o RC} = \frac{1}{(2\pi \times 2.59 \times 10^6 \times 5 \times 2 \times 10^{-9})} = 10$$

$$Q_L = 2\pi f_o L/R = 10$$

SELF ASSESSMENT EXERCISE No. 3

1. A series circuit has $L = 60 \text{ mH}$, $R = 15\Omega$ and $C = 15 \text{ nF}$. The supply is 2V ac. Calculate:
   i. the resonant frequency ($5.3 \text{ kHz}$)
   ii. the voltage over each component. ($V_R = 0.266 \text{ A}$, $V_C = 266 \text{ V}$, $V_L = 266 \text{ A}$)
   iii. the Q factor for the capacitor and inductor at resonance. (133.3)

2. A parallel circuit has $L = 40 \text{ mH}$, $R = 1 \text{ kΩ}$ and $C = 10 \text{ nF}$. The supply is 2V ac. Calculate:
   i. the resonant frequency ($7.96 \text{ kHz}$)
   ii. the current in each component. ($I_R = 2 \text{ mA}$, $I_C = 1 \text{ mA}$, $I_L = 1 \text{ mA}$)
12. POWER FACTOR CORRECTION

Industrial users of electric power often place inductive loads on the supply in the form of large motors and transformers. This may be regarded as an inductor in series with a resistor and produces a current that lags the supply voltage. The true power is that developed across the resistive load and is given by \( P = I^2R \) and is measured in Watts. Consider a resistor in series with an inductor. The inductive voltage phasor leads the resistive voltage phasor by 90°. The resultant voltage is the supply voltage.

The apparent power is the product of \( V \) and \( I \) and is measured in Volt Amps.

\[ Z = \sqrt{R^2 + X_L^2} \]

The POWER FACTOR is the ratio of the true power to the apparent power and is defined as

\[ \text{P.F.} = \frac{I^2R}{V/I} = \frac{I^2}{Z} = \frac{R}{Z} = \cos\phi \]

In order to reduce the power factor a series capacitor is needed to produce capacitive reactance equal and opposite to the inductive reactance. Ideally \( X_C = X_L \).

This is obtained from the relationship \( \frac{X_L}{R} = \tan\phi \)

### WORKED EXAMPLE No. 4

An ac load takes 2.5 kW of power from a supply 110V at 60 Hz. The current is 30 A. Determine the power factor and the size of a capacitor needed in series to correct it.

#### SOLUTION

\[ \text{P.F.} = \frac{\text{True Power}}{\text{Apparent Power}} = \frac{2500}{(110 \times 30)} = 0.758 \]

\[ \phi = \cos^{-1}(\text{P.F.}) = \cos^{-1}0.758 = 40.71° \]

True Power = \( I^2R = 30^2 \times 2500 \) W

\( R = 2500/900 = 2.777 \Omega \)

\( X_L/R = \tan\phi = 0.86 \)

\( X_L = 0.86 \times 2.777 = 2.392\Omega \)

\( X_C = 2.39 = 1/(2\pi fC) \)

\( C = 1/(2\pi \times 60 \times 2.39) = 0.00111 \text{ F} \)
SELF ASSESSMENT EXERCISE No. 4

1. A consumer takes 20 kW of power from an ac supply at 240 V and 50 Hz. Due to an inductive power factor, the current is 100 A. Determine the power factor and the size of a capacitor required to correct it. (0.0024 F)

2. An electrical load comprises of a resistance of 100 Ω and an inductor of 0.6 H in series. The supply is at 240 V and 50 Hz. Determine the Power factor. (0.47)

3. An ac supply to a consumer is at 220V and 50 Hz with a current of 20 A. It is found that there is a lagging phase angle of 20°. Determine the Power Factor, the true power and the size of a capacitor that would make the power factor 1. (0.364, 1.6 kW and 795 μF)