SOLID MECHANICS
DYNAMICS

TUTORIAL - SYSTEMS WITH TWO DEGREES OF FREEDOM

Students studying this tutorial should be familiar with matrix mathematics.

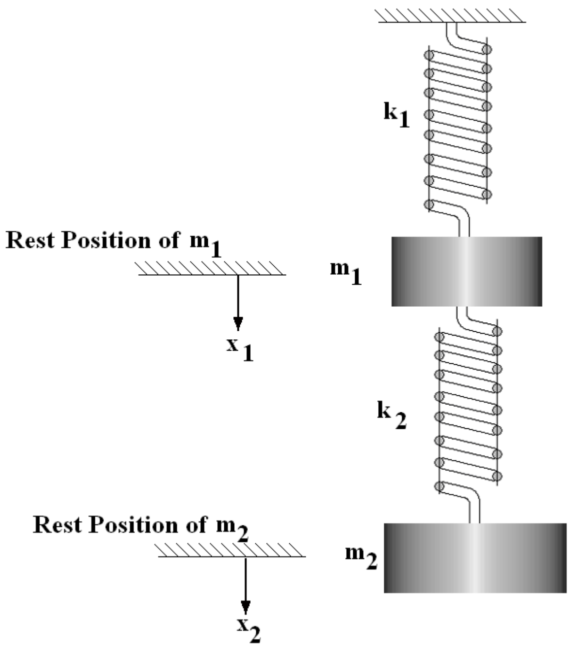
Contents

1. Introduction

This subject required advanced mathematical techniques in particular the use of matrix Algebra. You are advised to study the tutorials on Matrix Algebra in the Maths tutorials before you start.

You will find other tutorials on this subject with animations with a web search. We will start with a mass – spring system with two degrees of freedom.

2. Mass – Spring System



k is the stiffness of the spring (N/m)
m is the mass (kg)
x is the displacement of a mass from its rest position

When this system vibrates freely, there are two natural frequencies, one for each mode of vibration.

Balancing forces on m_1 we have

$$m_1 \frac{d^2x_1}{dt^2} + k_1x_1 - k_2(x_2 - x_1) = 0 \ldots\ldots (1)$$

Balancing forces on m_2 we have

$$m_2 \frac{d^2x_2}{dt^2} + k_2(x_2 - x_1) = 0 \ldots\ldots (2)$$

Knowing the motion of each mass is sinusoidal we can substitute x = X sin(ωt) X is the amplitude.
We can transform this using

$$\sin(\omega t) = ej^{\omega t}$$

x = X sin(ωt) = X $ej^{\omega t}$ and $\frac{d^2x}{dt^2} = \omega^2$ X $ej^{\omega t}$

$x_1 = X_1 e^{j\omega t}$ and $x_2 = X_2 e^{j\omega t}$

$\frac{d^2x_1}{dt^2} = -\omega^2 X_1 e^{j\omega t}$ and $\frac{d^2x_2}{dt^2} = -\omega^2 X_2 e^{j\omega t}$

If these are substituted back into the equation (1) we get

$$-m_1X_1\omega^2e^{j\omega t} + X_1k_1\, e^{j\omega t} - k_2\left(X_2e^{j\omega t} - X_1e^{j\omega t}\right) = 0$$

$$-m_1X_1\omega^2 + X_1k_1 \ - k_2(X_2 - X_1) = 0$$

$$X_1(-m_1\omega^2 + k_1 + k_2) - X_2k_2 = 0 \ldots\ldots (3)$$

Substituting into (2) we get

$$-m_2X_2\omega^2e^{j\omega t} + k_2(x_2 - x_1) = 0$$

$$-m_2X_2\omega^2e^{j\omega t} + k_2(x_2 - x_1) = 0$$

$$-X_2m_2\omega^2 + k_2(X_2 - X_1) = 0$$

$$-X_1k_2 + X_2(-m_2\omega^2 + k_2) = 0 \ldots\ldots (4)$$

Equations (3) and (4) are a pair of simultaneous equations and so X_1 and X_2 are solved. In matrix form we have

$$\begin{bmatrix} -m_1\omega^2 + k_1 + k_2 & -k_2 \\ -k_2 & -m_2\omega^2 + k_2 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

X_1 and X_2 are be solved by equating the determinant to zero.

$$\begin{bmatrix} -m_1\omega^2 + k_1 + k_2 & -k_2 \\ -k_2 & -m_2\omega^2 + k_2 \end{bmatrix} = 0$$

This is called the characteristic equation and yields the values of ω.

$$(-m_1\omega^2 + k_1 + k_2)(-m_2\omega^2 + k_2) - (-k_2)(-k_2) = 0$$

$$(m_1m_2\omega^4 - k_1m_2\omega^2 + \omega^2 m_1k_2 + k_1k_2 - k_2m_2\omega^2 + k_2^2) - k_2^2 = 0$$

$$m_1m_2\omega^4 - \omega^2(m_1k_2 + k_1m_2 + k_2m_2) + k_1k_2 - k_2^2 = 0$$

$$\omega^4 - \omega^2\left(\frac{m_1k_2 + k_1m_2 + k_2m_2}{m_1m_2}\right) + \frac{k_1k_2}{m_1m_2} = 0$$

$$\omega^4 - \omega^2\left(\frac{k_2}{m_2} + \frac{k_1}{m_1} + \frac{k_2}{m_1}\right) + \frac{k_1k_2}{m_1m_2} = 0$$

This gives a solution for ω by solving ω^2 with the quadreatic equation.

$$\omega^2 = \frac{\left(\frac{k_2}{m_2} + \frac{k_1}{m_1} + \frac{k_2}{m_1}\right) \mp \sqrt{\left(\frac{k_2}{m_2} + \frac{k_1}{m_1} + \frac{k_2}{m_1}\right)^2 - 4\frac{k_1k_2}{m_1m_2}}}{2}$$

$$\omega^2 = \frac{1}{2}\left(\frac{k_2}{m_2} + \frac{k_1}{m_1} + \frac{k_2}{m_1}\right) \mp \sqrt{\frac{1}{4}\left(\frac{k_2}{m_2} + \frac{k_1}{m_1} + \frac{k_2}{m_1}\right)^2 - \frac{k_1k_2}{m_1m_2}}$$

This gives two values for ω and these are the natural frequencies. The lower frequency is called the fundamental frequency. Note that $\sqrt{k/m}$ is the natural frequency of a single DOF system so

$$\omega^2 = \frac{1}{2}\left(\omega_{n2}^2 + \omega_{n1}^2 + \frac{k_2}{m_1}\right) \mp \sqrt{\frac{1}{4}\left(\omega_{n2}^2 + \omega_{n1}^2 + \frac{k_2}{m_1}\right)^2 - \omega_{n1}^2\ \omega_{n2}^2}$$

3. Matrix Solutions

The equations for the system using the shorthand form of calculus are

$$m_1\frac{d^2x_1}{dt^2}+k_1x_1-k_2(x_2-x_1)=0\ \ldots\ldots(1)$$

and

$$m_2\frac{d^2x_2}{dt^2}+k_2(x_2-x_1)=0\ \ldots\ldots(2)$$

If we change these to make x the common factor we have

$$m_1\,\ddot{x}_1+k_1x_1-k_2x_2+k_2x_1=0$$

and

$$m_2\ddot{x}_2+k_2x_2-k_2x_1=0$$

In matrix form this is

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}+\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=0$$

The mass matrix is

$$\mathsf{M}=\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

The stiffness matrix is

$$\mathsf{K}=\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

The matrix form may be written as

$$\mathsf{M}\ddot{x}+\mathsf{K}x=0(5)$$

The displacement equations are $x_1 = \mathsf{X}_1 \sin(\omega t+\phi)$ and $x_2 = \mathsf{X}_2 \sin(\omega t+\phi)$

Let the amplitude vector be X

The displacement vectors are $\mathsf{x} = \mathsf{X} \sin(\omega t+\phi)$ ……(6)

The second derivative of this is

$$\ddot{x}-\omega^2\sin(\omega t+\phi)X\ \ldots\ldots(7)$$

Substitute (6) and (7) into (1).

$$\mathsf{M}\{-\omega^2\sin(\omega t+\phi)X\}+K\{\sin(\omega t+\phi)X\}=0$$

$$\{-\mathsf{M}\omega^2+K\}X\{\sin(\omega t+\phi)\}=0$$

$$\{-\mathsf{M}\omega^2+\}X=0\ \ldots\ldots(8)$$

If there is a non trivial solution then

$$\det[-\mathsf{M}\omega^2+K]X=0\ \ldots\ldots(9)$$

Equation (9) gives us the exact same solution for ω as found earlier.

$$\det\begin{bmatrix} -m_1\omega^2+k_1+k_2 & -k_2 \\ -k_2 & -m_2\omega^2+k_2 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}=0$$

Mode shapes are displacement patterns representing the relative positions of both masses. It is normal to take $X_1 = 1$ and solve X_2 so the mode shape is X_1/X_2. There is one mode shape for each degree of freedom so in this case there are two. These are found by substituting ω and finding X_1/X_2 for each frequency.

WORKED EXAMPLE No. 1

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \quad K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} = \begin{bmatrix} 1\,000 & -500 \\ -500 & 500 \end{bmatrix}$$

$$\det \begin{bmatrix} -m_1\omega^2 + k_1 + k_2 & -k_2 \\ -k_2 & -m_2\omega^2 + k_2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 0$$

$$\det \begin{bmatrix} -2\omega^2 + 1\,000 & -500 \\ -500 & -4\omega^2 + 500 \end{bmatrix} = 0$$

$$(-2\omega^2 + 1\,000)(-4\omega^2 + 500) - (-500)(-500) = 0$$

$$8\omega^4 - 1\,000\omega^2 - 4\,000\omega^2 + 500\,000 - 250\,000 = 0$$

$$8\omega^4 - 5\,000\omega^2 + 250\,000 = 0$$

Solving the quadratic for ω^2 we get $\omega^2 = 570.2$ and 54.8

$\omega = 23.9$ rad/s and 7.4 rad/s now substitute these into

$$\begin{bmatrix} -m_1\omega^2 + k_1 + k_2 & -k_2 \\ -k_2 & -m_2\omega^2 + k_2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 0$$

First substitute $\omega = 54.8$

$$\begin{bmatrix} -2(54.8) + 1\,000 & -500 \\ -500 & -4(54.8) + 500 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 890.4 & -500 \\ -500 & 280.8 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 0$$

$890.4X_1 - 500X_2 = 0$ and $-500\,X_1 + 280.8\,X_2 = 0$

Let $X_1 = 1$ and $X_2 = 890.4/500 = 1.781$

or

$500/280.8 = 1.781$. The mode shape is the ratio 1.781/1

Next substitute $\omega = 23.9$

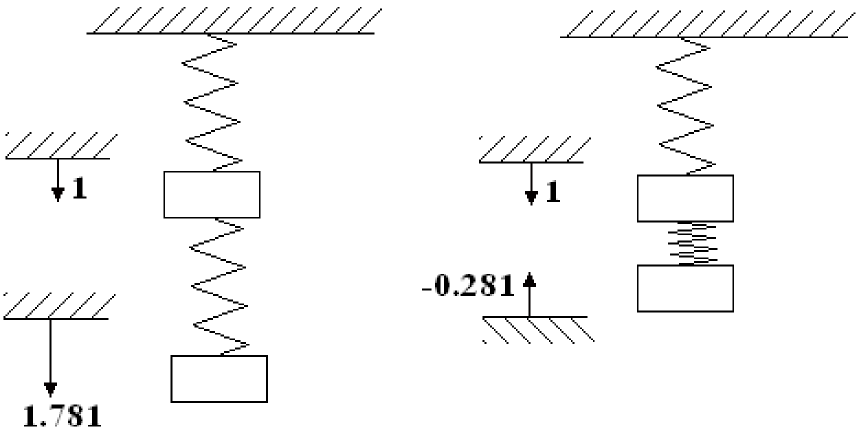
$$\begin{bmatrix} -2(570.2) + 1\,000 & -500 \\ -500 & -4(570.2) + 500 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} -140.4 & -500 \\ -500 & 1\,780.8 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 0$$

$-140.4X_1 - 500X_2 = 0$ and $-500X_1 - 1780.8\,X_2 = 0$

Let $X_1 = 1$ and $X_2 = -140.4/500 = -0.281$ or $-500/1780.8 = -0.281$.
The mode shape is the ratio -0.281/1

In the example the interpretation of the mode shapes is as follows. At the lower frequency they move in phase reaching maximum amplitude with a ratio 1.781/1. At the higher frequency they move in opposite phase with and amplitude -0.281/1. You might puzzle why the frequency is the same for both masses at each mode. Remember that both masses are affected by the other and they are not independent systems. If each were independent the frequency would simply be $\omega = \sqrt{k/m}$ for each (15.8 and 11.2 rad/s in this case)



The shape modes can be determined from the following formulae.

Mode 1

$$\frac{X_2}{X_1} = \frac{-m_1\omega_2^2 + k_1 + k_2}{k_2} = \frac{k_2}{-m_2\omega_1^2 + k_2}$$

Mode 2

$$\frac{X_2}{X_1} = \frac{-m_1\omega_1^2 + k_1 + k_2}{k_2} = \frac{k_2}{-m_2\omega_1^2 + k_2}$$

4. Eigen Values

Equation (9) earlier was

$$\det[-M\omega^2 + K]X = 0$$

Divide through by M

$$[M^{-1} K - \omega^2]X = 0$$

If we let $\lambda = \omega^2$

$$[M^{-1} K - \lambda]\, X = 0$$

$$[M^{-1} K]\, X = \lambda\, X$$

Hence λ is the Eigen value of $[M^{-1} K]$ and this may be found easily with computer packages such as Mathcad.

The method can be used to solve problems with three or more degrees of freedom. There is a natural frequency for each degree.

The method can be used for any two degree of freedom system in which the stiffness and mass matrices are clearly identified.

WORKED EXAMPLE No. 2

Given m_1 = 2 kg , k_1 = 30 000 N/m, m_2 = 30 kg and k_2 = 40 000 N/m determine the two natural frequencies using the mass and stiffness matrices and check with the derived formula. Determine the mode shapes.

SOLUTION

The mass matrix is

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 30 \end{bmatrix}$$

The stiffness matrix is

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} = \begin{bmatrix} 70\ 000 & -40\ 000 \\ -40\ 000 & 40\ 000 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 30 \end{bmatrix}\begin{bmatrix} m_1 \\ m_2 \end{bmatrix} + \begin{bmatrix} 70\ 000 & -40\ 000 \\ -40\ 000 & 40\ 000 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

To avoid trivial solutions we must have

$$\det[-\omega^2 M + K] = 0$$

$$\det\left\{-\omega^2\begin{bmatrix} 2 & 0 \\ 0 & 30 \end{bmatrix} + \begin{bmatrix} 70\ 000 & -40\ 000 \\ -40\ 000 & 40\ 000 \end{bmatrix}\right\} = 0$$

$$\det\begin{bmatrix} -2\omega^2 + 70\ 000 & -40\ 000 \\ -40\ 000 & -30\omega^2 + 40\ 000 \end{bmatrix} = 0$$

$$(-2\omega^2 + 70\ 000)(-30\omega^2 + 40\ 000) - (-40\ 000)(-40\ 000)$$

$$60\omega^4 - 80 \times 10^3\omega^2 - 2.1 \times 10^6\omega^2 + 2.8 \times 10^9 - 1.6 \times 10^9 = 0$$

$$60\omega^4 - 2.18 \times 10^6\omega^2 + 1.2 \times 10^9 = 0$$

Solving the quadratic for ω^2 we get $\omega^2 = 35.77 \times 10^3$ and 560

ω = 189.1 rad/s and 23.6 rad/s

$$\omega^2 = \frac{1}{2}\left(\frac{k_2}{m_2} + \frac{k_1 + k_2}{m_1}\right) \mp \sqrt{\frac{1}{4}\left(\frac{k_2}{m_2} + \frac{k_1 + k_2}{m_1}\right)^2 - \frac{k_1 k_2}{m_1 m_2}}$$

$$\omega^2 = \frac{1}{2}\left(\frac{40\ 000}{30} + \frac{70\ 000}{2}\right) \mp \sqrt{\frac{1}{4}\left(\frac{40\ 000}{30} + \frac{70\ 000}{2}\right)^2 - \frac{1.2 \times 10^9}{60}} = 18\ 167 \mp 17\ 608$$

$$\omega^2 = 35\ 774 \text{ and } 558.7$$

ω = 189.1 and 23.63 rad/s

Next substitute $\omega = 23.63$

$$\begin{bmatrix} -2(558.7) + 70\,000 & -40\,000 \\ -40\,000 & -30(558.7) + 40\,000 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 68\,883 & -40\,000 \\ -40\,000 & 23\,239 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 0$$

$68\,883X_1 - 40\,000X_2 = 0$ and $-40\,000X_1 + 23\,239\,X_2 = 0$

Let $X_1 = 1$ and $X_2 = 68\,883/40\,000 = 1.722$ or $40\,000/23\,239 = 1.722$

The mode shape is the ratio 1.722/1
Next substitute $\omega = 189.1$

$$\begin{bmatrix} -2(35\,774) + 70\,000 & -40\,000 \\ -40\,000 & -30(35\,774) + 40\,000 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 0$$

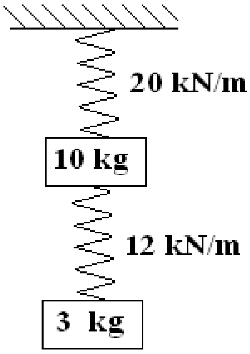
$$\begin{bmatrix} -1\,548 & -40\,000 \\ -40\,000 & -1\,033\,220 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 0$$

$-1\,548\,X_1 - 40\,000\,X_2 = 0$ and $-40\,000\,X_1 - 1\,033\,220\,X_2 = 0$

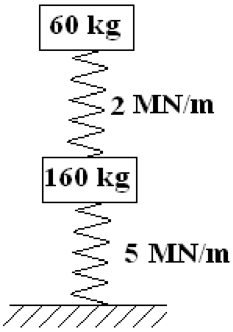
Let $X_1 = 1$ and $X_2 = -1\,548/40\,000 = -0.0387$ or $-40\,000/1\,033\,220 = -0.0387$

SELF ASSESSMENT EXERCISE No. 1

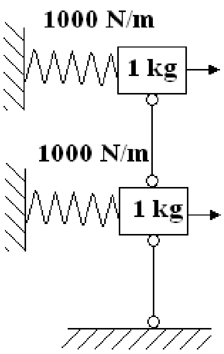
Calculate the natural frequencies and mode shapes for the systems below.



(37 rad/s, 76.3 rad/s, 1.523 and -2.189)



(375 rad/s, 86 rad/s, -0.29 and 1.31)



(51.1 rad/s, 19.5 rad/s, 1.618 and – 0.618)

5. Three Degrees of Freedom

For a three degree of freedom system the mass and stiffness matrices are as follows.

$$M = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \quad K = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}$$

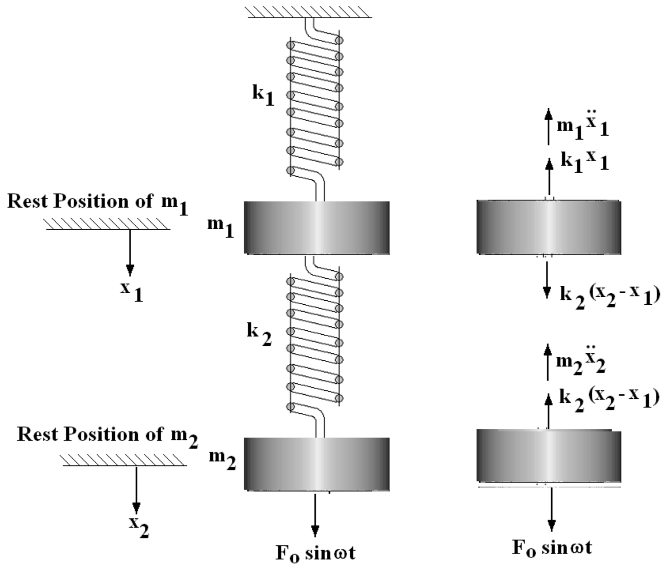
The pattern for higher DOF may be discerned. The solution is difficult without a computer package.

6. *Forced Vibrations*

This section is included for those seeking further studies of the subject but does not seem to be part of the learning outcome. We are not examining cases with no damping so the resonant frequencies will produce very large amplitudes and these will occur at a single frequency. There will be two resonant frequencies for a two DOF system.

Case 1 – Force Applied To m_2

There are more cases of forced vibrations to be examined than for single DOF systems. In this tutorial we will only examine the application of a sinusoidal force. The following is a guide to the general approach and we start with the case when the force is applied to m_2 as shown.



The free body diagram shown yields:-

$m_1 \ddot{x}_1 + k_1 x_1 - k_2(x_2 - x_1) = 0$ and

$m_2 \ddot{x}_2 + k_2 (x_2 - x_1) = F_o \sin(\omega t)$

Change to the complex operator $x = X e^{j\omega t}$

Note that $\ddot{x}$ is $d^2x/dt^2 = -\omega^2 X e^{j\omega t}$

X is the amplitude of the motion.

$-m_1 \omega^2 X_1 e^{j\omega t} + k_1 X_1 e^{j\omega t} - k_2(X_2 - X_1) e^{j\omega t} = 0$

$-m_1 \omega^2 X_1 + k_1 X_1 - k_2(X_2 - X_1) = 0$

$-m_1 \omega^2 X_1 + k_1 X_1 - k_2 X_2 + k_2 X_1 = 0$ …………(1)

$-m_2 \omega^2 X_2 e^{j\omega t} + k_2 (X_2 - X_1) e^{j\omega t} = F_o e^{j\omega t}$

$-m_2 \omega^2 X_2 + k_2 (X_2 - X_1) = F_o$

$-m_2 \omega^2 X_2 + k_2 X_2 - k_2 X_1 = F_o$…………………(2)

From (1) we get

$$X_2 = X_1\left[\frac{k_1 + k_2 - m_1\omega^2}{k_2}\right] \ldots\ldots (3)$$

Note that $k_1/m_1 = \omega_1^2$

$$\frac{X_2}{X_1} = \frac{k_1}{k_2}\left[1 + \frac{k_2}{k_1} - \frac{m_1\omega^2}{k_1}\right] = \frac{k_1}{k_2}\left[1 + \frac{k_2}{k_1} - \frac{\omega^2}{\omega_n^2}\right] \ldots\ldots (4)$$

Substitute (3) into (2)

$$-m_2\omega^2 X_1\left[\frac{k_1 + k_2 - m_1\omega^2}{k_2}\right] + k_2 X_1\left[\frac{k_1 + k_2 - m_1\omega^2}{k_2}\right] - k_2 X_1 = F_o$$

$$X_1\left[\frac{k_1 + k_2 - m_1\omega^2}{k_2}\right]\{-m_2\omega^2 + k_2\} - k_2 X_1 = F_o$$

$$X_1\left\{\left[\frac{k_1 + k_2 - m_1\omega^2}{k_2}\right]\{-m_2\omega^2 + k_2\} - k_2\right\} = F_o$$

$$X_1\left[1 + \frac{k_2}{k_1} - \frac{m_1}{k_1}\omega^2\right]\frac{k_1}{k_2}\{k_2 - m_2\omega^2\} - k_2 X_1 = F_o$$

$$X_1\left[1 + \frac{k_2}{k_1} - \frac{m_1}{k_1}\omega^2\right]\left\{1 - \frac{m_2}{k_2}\omega^2\right\}k_1 - k_2 X_1 = F_o$$

$$X_1\left\{\left[1+\frac{k_2}{k_1}-\frac{\omega^2}{\omega_{n1}^2}\right]\left\{1-\frac{\omega^2}{\omega_{n2}^2}\right\}k_1-k_2\right\}=F_o$$

$$X_1=\frac{F_o}{\left[1+\frac{k_2}{k_1}-\frac{\omega^2}{\omega_{n1}^2}\right]\left\{1-\frac{\omega^2}{\omega_{n2}^2}\right\}k_1-k_2}$$

$$X_2=X_1\left[\frac{k_1+k_2\ -m_1\omega^2}{k_2}\right]$$

ω_{n1} and ω_{n2} are the natural frequencies of the single DOF mass-spring system given by the formula derived earlier.

WORKED EXAMPLE No. 3

A system as shown in the previous diagram has the following data.
m_1 = 10 kg, m_2 = 2 kg, k_1 = 10 000 N/m, k_2 = 6 000 N/m and F_o = 100 N

Plot X_2 and X_1 and the ratio X_2/X_1 against frequency. Calculate the natural frequencies for the two DOF system and the individual one DOF systems. Draw conclusions about the behaviour of the system.

SOLUTION
$\omega_{n1} = \sqrt{(k_1/m_1)}$ = 31.62 rad/s $\quad \omega_{n2} = \sqrt{(k_2/m_2)}$ = 54.77 rad/s

$$\omega^2 = \frac{1}{2}\left(\omega_{n2}^2 + \omega_{n1}^2 + \frac{k_2}{m_1}\right) \mp \sqrt{\frac{1}{4}\left(\omega_{n2}^2 + \omega_{n1}^2 + \frac{k_2}{m_1}\right)^2 - \omega_{n2}^2\omega_{n1}^2}$$

$$\omega^2 = \frac{1}{2}(54.77^2 + 31.62^2 + 600) \mp \sqrt{\frac{1}{4}(54.77^2 + 31.62^2 + 600)^2 - 54.77^2 \times 31.62^2}$$

$$\omega^2 = 2\,300 \mp \sqrt{\frac{1}{4}5\,290\,000 - 3\,000\,000}$$

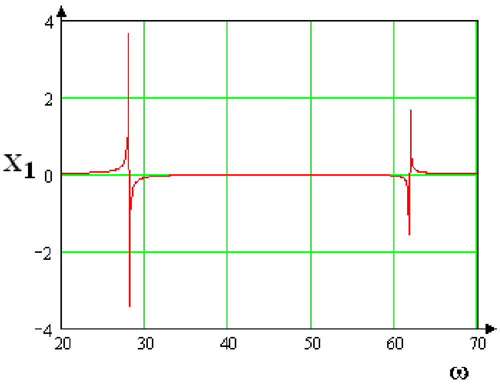
ω^2 = 3813.3 or 786.7 $\qquad\qquad \omega$ = 61.75 or 28 rad/s

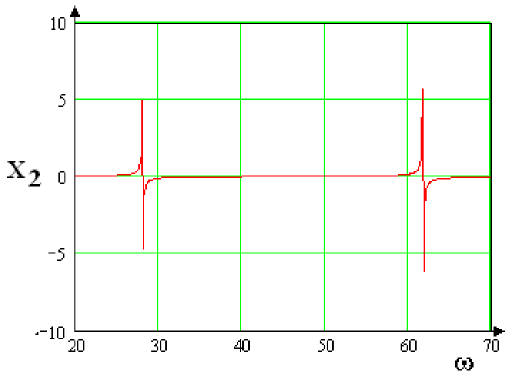
If we tried to evaluate and plot without the aid of a computer we would find it difficult to pin point the resonant frequencies because without damping they are just spikes as shown.

The plots for X_1 and X_2 are shown below.

$$X_1 = \frac{F_o}{\left[1 + \frac{k_2}{k_1} - \frac{\omega^2}{\omega_{n1}^2}\right]\left\{1 - \frac{\omega^2}{\omega_{n2}^2}\right\}k_1 - k_2}$$

$$X_2 = X_1 \frac{k_1}{k_2}\left[1 + \frac{k_2}{k_1} - \frac{\omega^2}{\omega_{n1}^2}\right]$$

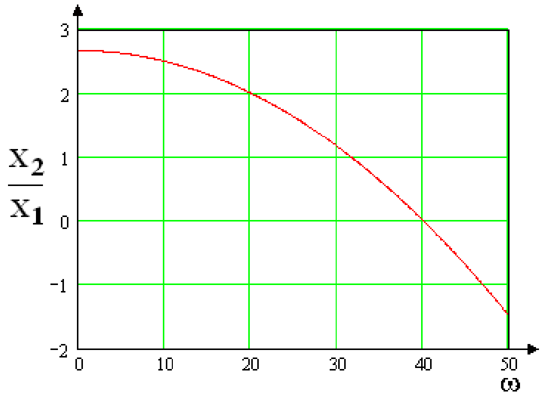




We can see that there are two resonant frequencies that occur at the natural frequencies of the 2 DOF system. Now plot the amplitude ratio.

$$\frac{X_2}{X_1} = \frac{k_1}{k_2}\left[1 + \frac{k_2}{k_1} - \frac{\omega^2}{\omega_{n1}^2}\right]$$

Evaluate and plot



We can see that at ω = 40 rad/s the ratio is 0 and this means that m_2 is not moving. In this case only m_1 is oscillating and both springs are changing length and effectively combine to form one spring of stiffness $k_1 + k_2$.

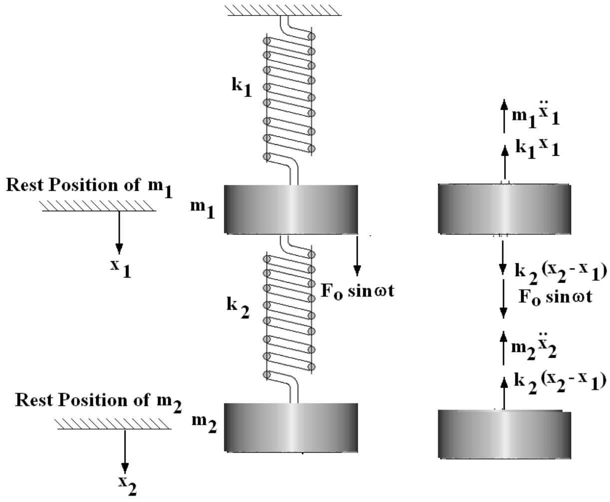
The forcing frequency must then correspond to

$$\omega = \sqrt{\frac{k_1 + k_2}{m_1}}$$

And if we check this out

$$\omega = \sqrt{\frac{1\,600}{10}} = 40 \text{ rad/s}$$

Case 2 – Force Applied to m_1



The free body diagram shown yields:-

$m_1 \ddot{x}_1 + k_1 x_1 - k_2(x_2 - x_1) = F_o\sin(\omega t)$ and
$m_2 \ddot{x}_2 + k_2 (x_2 - x_1) = 0$

Change to the complex operator $x = X e^{j\omega t}$
Note that $\ddot{x}$ is $d^2x/dt^2 = - \omega^2 X e^{j\omega t}$
X is the amplitude of the motion.

$-m_1 \omega^2 X_1 e^{j\omega t} + k_1 X_1 e^{j\omega t} - k_2(X_2 - X_1) e^{j\omega t} = F_o e^{j\omega t}$
$-m_1 \omega^2 X_1 + k_1 X_1 - k_2(X_2 - X_1) = F_o$
$-m_1 \omega^2 X_1 + k_1 X_1 - k_2 X_2 + k_2 X_1 = F_o$ …………(1)

$-m_2 \omega^2 X_2 e^{j\omega t} + k_2 (X_2 - X_1) e^{j\omega t} = 0$
$-m_2 \omega^2 X_2 + k_2 (X_2 - X_1) = 0$
$-m_2 \omega^2 X_2 + k_2 X_2 - k_2 X_1 = 0$………………….(2)
From (2) we get

$$X_1 = X_2\left[\frac{k_2 - m_2\omega^2}{k_2}\right] = X_2\left[1 - \frac{\omega^2}{\omega_{n2}^2}\right] \ldots\ldots (3)$$

Note that $k_2/m2 = \omega_{n2}^2$

$$\frac{X_1}{X_2} = \left[1 - \frac{\omega^2}{\omega_{n2}^2}\right] \ldots\ldots (4)$$

Substitute (3) into (1)

$$-m_1\omega^2 X_2\left[1 - \frac{\omega^2}{\omega_{n2}^2}\right] + k_1 X_2\left[1 - \frac{\omega^2}{\omega_{n2}^2}\right] - k_2 X_2\left[1 - \frac{\omega^2}{\omega_{n2}^2}\right] = F_o$$

$$X_2\left[1 - \frac{\omega^2}{\omega_{n2}^2}\right]\{-m_1\omega^2 + k_1 + k_2\} - k_2 X_2 = F_o$$

$$X_2\left\{\left[1 - \frac{\omega^2}{\omega_{n2}^2}\right]\{k_1 + k_2 - m_1\omega^2\} - k_2\right\} = F_o$$

$$X_2\left\{\left[1 - \frac{\omega^2}{\omega_{n2}^2}\right]\left\{1 + \frac{k_2}{k_1} - \frac{m_1}{k_1}\omega^2\right\}k_1 - k_2\right\} = F_o$$

$$X_2\left\{\left[1 - \frac{\omega^2}{\omega_{n2}^2}\right]\left\{1 + \frac{k_2}{k_1} - \frac{\omega^2}{\omega_{n1}^2}\right\}k_1 - k_2\right\} = F_o$$

$$X_2 = \frac{F_o}{\left\{\left[1 - \frac{\omega^2}{\omega_{n2}^2}\right]\left\{1 + \frac{k_2}{k_1} - \frac{\omega^2}{\omega_{n1}^2}\right\}k_1 - k_2\right\}}$$

$$X_1 = X_2\left[1 - \frac{\omega^2}{\omega_{n2}^2}\right]$$

ω_1 and ω_2 are the natural frequencies of the single DOF mass-spring system.

WORKED EXAMPLE No. 4

A system as shown in the previous diagram has the following data.
m_1 = 10 kg, m_2 = 2 kg, k_1 = 10 000 N/m, k_2 = 6000 N/m and F_o = 100 N.
Plot X_2, X_1 and the ratio X_2/X_1 against frequency.

Calculate the natural frequencies for the two DOF system and the individual one DOF systems. Draw conclusions about the behaviour of the system.

SOLUTION

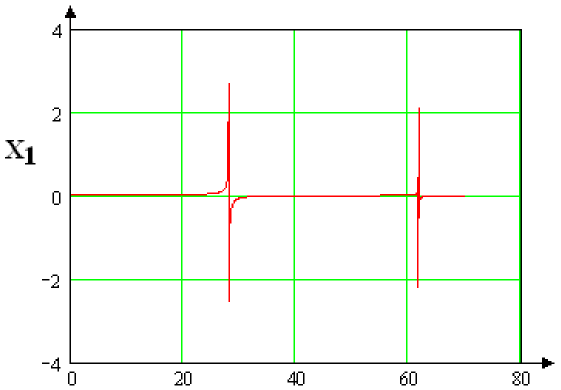
This is the same as the last example but with the force applied to m_1. As before
$\omega_{n1} = \sqrt{(k_1/m_1)}$ = 31.62 rad/s $\quad \omega_{n2} = \sqrt{(k_2/m_2)}$ = 54.77 rad/s
and for the 2 DOF system ω = 61.75 and 28 rad/s

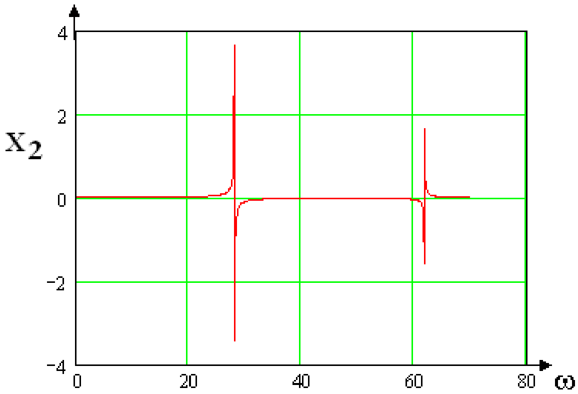
If we tried to evaluate and plot without the aid of a computer we would find it difficult to pin point the resonant frequencies because without damping they are just spikes as shown.
Evaluate

$$X_2 = \frac{F_o}{\left\{\left[1-\frac{\omega^2}{\omega_{n2}^2}\right]\left\{1+\frac{k_2}{k_1}-\frac{\omega^2}{\omega_{n1}^2}\right\}k_1 - k_2\right\}}$$

$$X_1 = X_2\left[1-\frac{\omega^2}{\omega_{n2}^2}\right]$$

Plot and we get the following results.



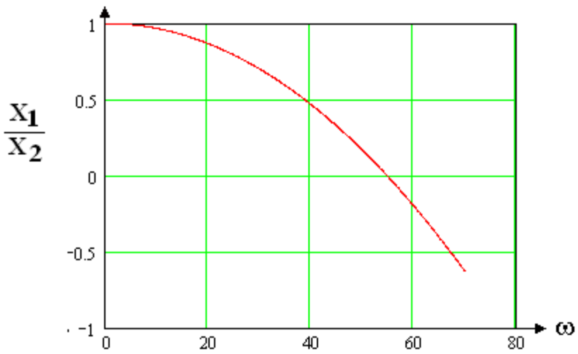


We can see that there are two resonant frequencies that occur at the natural frequencies of the 2 DOF system. Now plot the amplitude ratio.

$$\frac{X_1}{X_2} = \left[1-\frac{\omega^2}{\omega_{n2}^2}\right]$$

Evaluate and plot

We can see that at ω = 54.77 rad/s the ratio is 0 and this means that m_1 is not moving.
In this case only m_1 is oscillating and the frequency must be $\omega_{n2} = \sqrt{k_2/m_2}$= 54.77 rad/s



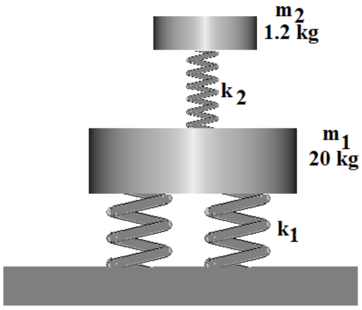
7. *Vibration Absorber*

When a mass and spring is added to a vibrating system with a natural frequency ω_{n1} and tuned so that the motion of the main mass is reduced (to zero), the system is called a vibration absorber. In this case m_2 and k_2 are selected so that ω_{n2} = forcing frequency. In this case

$$\frac{X_1}{X_2} = \left[1 - \frac{\omega^2}{\omega_{n2}^2}\right] = 1 - 1 = 0$$

WORKED EXAMPLE No. 5

A machine of mass 20 kg is mounted on flexible supports and has a significant level of vertical vibration at the resonant frequency of 80 Hz. A second mass is to be attached to the top on a sprung platform to absorb this vibration. If the mass is to be 1.2 kg, calculate the spring rate of the mountings and the absorber spring rate required.



SOLUTION

The initial system has a natural frequency of 80 Hz

$$\omega_n = \frac{80}{2\pi} = \sqrt{\frac{k_1}{m_1}} = 12.732 \text{ rad/s} \qquad \frac{k_1}{m_1} = 162.1$$

$$k_1 = 162.1 \times m_1 = 162.1 \times 20 = 3\ 242 \text{N/m}$$

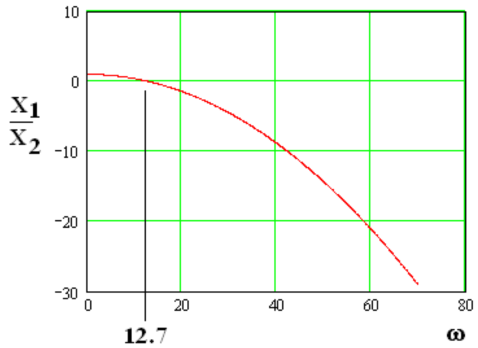
If the lower mass must stay still then the absorber spring is the only one stretching. The absorber mass must absorb the vibrations at 12.732 rad/s so

$$12.732 = \sqrt{\frac{k_2}{m_2}} = \sqrt{\frac{k_2}{1.2}} \quad k_2 = 1.2 \times 12.732^2 = 194.54 \text{ N/m}$$

If we plot

$$\frac{X_1}{X_2} = \left[1 - \frac{\omega^2}{\omega_{n2}^2}\right]$$

We can see that this is zero at 12.7 rad/s

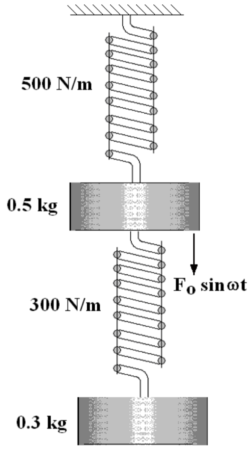


SELF ASSESSMENT EXERCISE No. 2

1. For the system shown, calculate the resonant frequencies when F_o = 4 N. (21.6 and 46.2 rad/s)

Calculate the forcing frequency that results in no motion of the 0.5 kg mass. (31.6 rad/s)

Calculate the absolute amplitudes of both masses at 5 rad/s. (8.34 mm and 8.55 mm)



2. For the system shown, calculate the resonant frequencies when F_o = 12 N. (18 and 42 rad/s)

Calculate the forcing frequency that results in no motion of the 0.5 kg mass. (22.36 rad/s)

Calculate the absolute amplitudes of both masses at 1 rad/s. (10 mm and 40 mm)