SOLID MECHANICS

DYNAMICS

TUTORIAL – TORSIONAL OSCILLATIONS

On completion of this tutorial you should be able to solve the natural frequency of torsional vibrations for shafts carrying multiple moments of inertia. You are advised to study the tutorials on free vibrations before commencing on this.

To do the tutorial fully you must be familiar with the following concepts.

- Torsion theory.
- Moments of Inertia.
- Torsional stiffness of shafts.
- Simple harmonic motion.

The principle explained here is called HOLZER'S METHOD.

Contents

1. Introduction

This tutorial is about finding the natural frequency of torsional vibrations for shafts carrying multiple moments of inertia. A typical problem is illustrated below. The principle to be used is called ***Holzer's Method.***

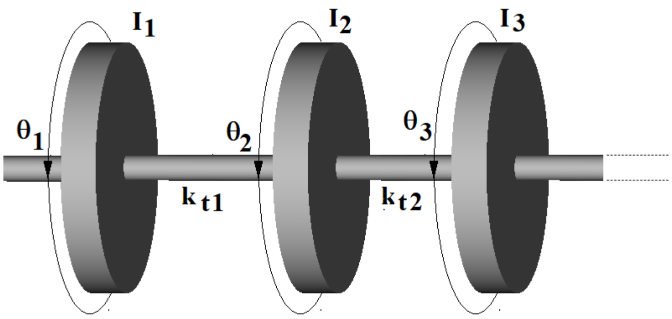


Figure 1

If we have several discs on a shaft as shown, there are several possible modes and natural frequencies. A method of solving this system is due to Holzer. The reasoning goes like this.

Disc 1 twist relative to disc 2. The toque balance gives

$$T = I_1\,\alpha_1 + k_{t1}(\theta_1 - \theta_2) = 0$$

Disc 2 twist relative to discs 1 and 3. The toque balance gives

$$T = I_2\,\alpha_2 + k_{t1}(\theta_2 - \theta_1) + k_{t2}(\theta_2 - \theta_3) = 0$$

Disc 3 twist relative to disc 2. The toque balance gives

$$T = I_3\,\alpha_3 + k_{t2}(\theta_3 - \theta_2) = 0$$

The total torque is

$$T = 0 = I_1\,\alpha_1 + k_{t1}(\theta_1 - \theta_2) + I_2\,\alpha_2 + k_{t1}(\theta_2 - \theta_1) + k_{t2}(\theta_2 - \theta_3) + I_3\,\alpha_3 + k_{t2}(\theta_3 - \theta_2)$$

$$0 = I_1\,\alpha_1 + k_{t1}\theta_1 - k_{t1}\theta_2 + I_2\,\alpha_2 + k_{t1}\theta_2 - k_{t1}\,\theta_1 + k_{t2}\theta_2 - k_{t2}\theta_3 + I_3\,\alpha_3 + k_{t2}\theta_3 - k_{t2}\theta_2$$

$$0 = I_1\,\alpha_1 + I_2\,\alpha_2 + I_3\,\alpha_3$$

For simple harmonic motion we may substitute $\omega^2\theta = -\alpha$ into each equation and rearrange them to give

$$0 = I_1\,\omega_1^2\,\theta_1 + I_2\,\omega_2^2\,\theta_2 + I_3\,\omega_3^2\,\theta_3$$

For any number of discs this may be generalised as $\Sigma(I\,\omega^2\,\theta) = 0$

Holzer's method of solution proposes that we assume any value of ω and make $\theta_1 = 1$ and calculate all the other deflections. The deflection of disc 2 may be found by rearranging

$$I_1\omega_1^2\theta_1 = k_{t1}(\theta_1 - \theta_2) \qquad \theta_2 = \theta_1 - \frac{\omega^2}{k_{t1}}I_1\theta_1 \text{ or } \theta_1 = \theta_2 + \frac{\omega^2}{k_{t1}}I_1\theta_1$$

The deflection of disc 3 may be found by rearranging

$$I_2\omega_2^2\theta_2 = k_{t1}(\theta_2 - \theta_1) + k_{t2}(\theta_2 - \theta_3) \quad \text{substitute from above}$$

$$\omega^2 I_2\theta_2 = k_{t1}\left(\theta_2 - \theta_2 - \frac{\omega^2}{k_{t1}}I_1\theta_1\right) + k_{t2}(\theta_2 - \theta_3)$$

$$\omega^2 I_2 \theta_2 = -\omega^2 I_1 \theta_1 + k_{t2}(\theta_2 - \theta_3)$$

$$\omega^2 I_2 \theta_2 = -\omega^2 I_1 \theta_1 + k_{t2}\theta_2 - k_{t2}\theta_3$$

$$k_{t2}\theta_3 = k_{t2}\theta_2 - \omega^2 I_1 \theta_1 - \omega^2 I_2 \theta_2$$

$$k_{t2}\theta_3 = k_{t2}\theta_2 - \omega^2 (I_1 \theta_1 + I_2 \theta_2)$$

$$\theta_3 = \theta_2 - \frac{\omega^2}{k_{t2}}(I_1 \theta_1 + I_2 \theta_2)$$

If this was continued the pattern for any number of discs would be as follows.

$$\theta_2 = \theta_1 - \frac{\omega^2}{k_{t1}} I_1 \theta_1$$

$$\theta_3 = \theta_2 - \frac{\omega^2}{k_{t2}}(I_1 \theta_1 + I_2 \theta_2)$$

$$\theta_4 = \theta_3 - \frac{\omega^2}{k_{t3}}(I_1 \theta_1 + I_2 \theta_2 + I_3 \theta_3)$$

And so on for as many as exist.

Next we consider the torque produced by the twisting. $T = I\,\alpha$ and $\alpha = \omega^2\theta$ so $T = \omega^2 I\,\theta$

The torque to deflect disc 1 by θ_1 is $\omega^2 I_1 \theta_1$

The torque to deflect disc 2 by θ_2 is $\omega^2 I_2 \theta_2$

The torque to deflect disc 3 by θ_3 is $\omega^2 I_3 \theta_3$

And so on for as many shaft section that exist. Hence

$T_1 = \omega^2 I_1 \theta_1$ $\quad$ $T_2 = T_1 + \omega^2 I_2 \theta_2$ $\quad$ $T_3 = T_2 + \omega^2 I_3 \theta_3$

And so on for as many shaft section that exist.

Since we must satisfy

$$\mathbf{\Sigma I\,\omega^2\,\theta = 0}$$

Then the last T must be zero when the oscillation is free. The problem is to find the values of ω that make this so and these are the natural frequencies of the system.

If a computer programme is used, it is relatively simple to evaluate the displacements and the torques for all values of ω. Before we look at difficult problems let’s consider the case of only two rotors.

2. *Two Inertia Systems*

Consider a shaft with torsional stiffness k_t connecting two inertias I_1 andI_2. If the shaft is free to rotate the torsional oscillation will take the form of both ends twisting but some point in between will not be twisting. This is a node. The shaft must of course be supported in at least two bearings.

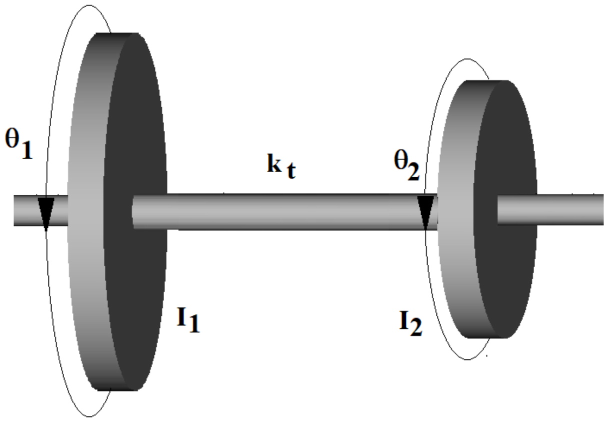


Figure 2

The natural frequency can be derived from the previous work. For two rotors, $T_2 = 0$

$$\theta_2 = \theta_1 - \frac{\omega^2}{k_{t1}} I_1\theta_1$$

$$T_1 = \omega^2 I_1 \theta_1$$

$$T_2 = T_1 + \omega^2 I_2 \theta_2 = 0$$

Substitute for θ_1

$$T_2 = 0 = T_1 + \omega^2 I_2 \theta_2 = \omega^2 I_1 \theta_1 + \omega^2 I_2 \theta_2$$

Substitute for θ_2

$$0 = \omega^2 I_1 \theta_1 + \omega^2 I_2 \left(\theta_1 - \frac{\omega^2 I_1 \theta_1}{k_{t1}}\right)$$

Simplify and rearrange

$$\omega_n^2 = k_{t1}\left(\frac{I_1 + I_2}{I_1 I_2}\right)$$

The node will be somewhere between the two rotors. The next worked example will show how to find the node.

WORKED EXAMPLE No. 1

A shaft free to rotate carries a flywheel with I_1 = 2 kg m^2 at one end and I_2 = 4 kg m^2 at the other. The shaft connecting them has a stiffness of 4 MN m/rad. Calculate the natural frequency and the position of the node.

SOLUTION

$$\omega_n^2 = k_{t1}\left(\frac{I_1 + I_2}{I_1 I_2}\right) = 4 \times 10^6\left(\frac{2+4}{2 \times 4}\right) = 3 \times 10^6$$

$$\omega_n = 1\,732\,\frac{\text{rad}}{\text{s}} \qquad f_n = 275.7\text{ Hz}$$

If we regard the node as a fixed point each rotor will have the same natural frequency about that point. For a single rotor system

$$\omega_n = \frac{k_t}{I}$$

For the first rotor

$$\omega_n^2 = 3 \times 10^6 = \frac{k_{t1}}{2} \qquad k_{t1} = 6 \times 10^6$$

For the other rotor

$$\omega_n^2 = 3 \times 10^6 = \frac{k_{t2}}{4} \qquad k_{t2} = 12 \times 10^6$$

The difference in stiffness is due to the difference in length of the shaft. k_t = GJ/L and GJ is the same for both sections.

$$\frac{k_{t1}}{k_{t2}} = \frac{L_1}{L_2} = \frac{6}{12} \qquad L_2 = \frac{L_1}{2}\text{ and } L_1 + L_2 = L \qquad L_2 = \frac{L - L_2}{2} \quad 2\,L_2 = L - L_2$$

$$3\,L_2 = L \quad L_2 = \frac{L}{3} \quad L_1 = \frac{2L}{3}$$

The node is L/3 from the right. This may be found graphically as shown. Let $\theta_1 = 1$

$$\theta_2 = \theta_1 - \frac{\omega^2}{k_{t1}} I_1\theta_1 = 1 - \frac{3 \times 10^6 \times 2 \times 1}{4 \times 10^6} = 1 - 1.5 = -0.5$$

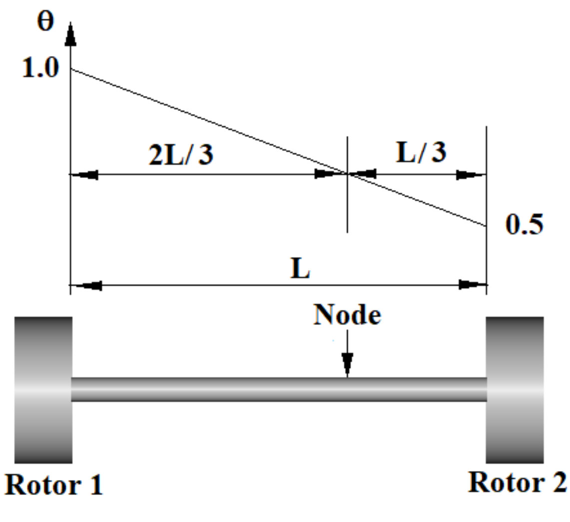


Figure 3

3. Three Inertia Systems

This is best demonstrated with the worked example below.

WORKED EXAMPLE No. 2

A shaft has three inertias on it of 2, 4 and 2 kg m^2 respectively viewed from left to right. The shaft connecting the first two has a stiffness of 3 MN m/radian and the shaft connecting the last two has a stiffness of 2 MN m/radian. The system is supported in bearings at both ends. Ignore the inertia of the shafts and find the natural frequencies of the system.

SOLUTION

$$\theta_1 = 1$$

$$\theta_2 = \theta_1 - \frac{\omega^2}{k_{t1}} I_1 \theta_1 = 1 - \frac{2\omega^2}{3 \times 10^6} = 1 - 0.667 \times 10^{-6}\omega^2$$

$$\theta_3 = \theta_2 - \frac{\omega^2}{k_{t2}}(I_1\theta_1 + I_2\theta_2) = \theta_2 - \frac{\omega^2}{2 \times 10^6}(2 \times 1 + 4\theta_2)$$

$T_1 = \omega^2 I_1 \theta_1 = \omega^2 \times 2 \times 1 = 2\omega^2$

$T_2 = T_1 + \omega^2 I_2 \theta_2 = 2\omega^2 + 4\omega^2 \theta_2$

$T_3 = T_2 + \omega^2 I_3 \theta_3 = 2\omega^2 + 4\omega^2 \theta_2 + 2\omega^2 \theta_3$

These should ideally be evaluated for all values of ω and T_3 plotted against ω. The result is:

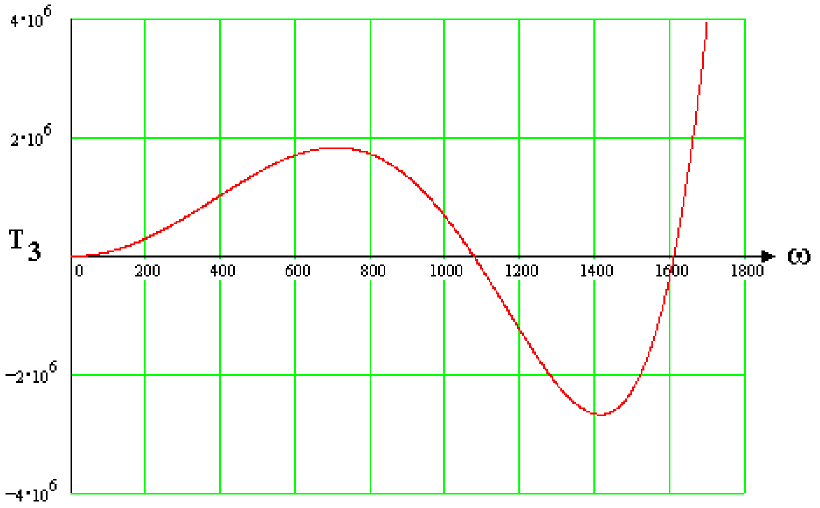


Figure 4

The points where $T_3 = 0$ give the natural frequencies and these are about 1 090 and 1 610 rad/s.

In an examination environment, plotting this graph is not a practical option. We must start by evaluating in large steps of ω and narrowing it down to the points where T_3 change from plus to minus. This can be very tricky as it is quite possible to miss the critical points if the negative area is a small one.

	ω	θ_1	θ_2	θ_3	T_1	T_2	T_3
1	1	1	1	1	2	6	8
2	10	1	0.9999	0.9996	200	600	800
3	100	1	0.9933	0.9635	2×10^4	5.97×10^4	7.9×10^4
4	1000	1	0.3333	-1.333	2×10^6	3.33×10^6	0.6667×10^6
5	1500	1	-0.5	-0.5	4.5×10^6	0	-2.25×10^6

T3 has gone negative so we need to back.

6	1250	1	-.0417	-1.474	3.125×10^6	$2.86\text{x}10^6$	-1.7415×10^6
7	1100	1	0.1933	-1.484	2.42×10^6	$3.36\text{x}10^6$	-0.237×10^6
8	1050	1	0.265	-1.422	2.2×10^6	$3.37\text{x}10^6$	$+0.238\times10^6$
9	1070	1	0.2367	-1.45	$2.29\text{x}10^6$	$3.37\text{x}10^6$	$+0.053\times10^6$
10	1080	1	0.224	-1.46	2.33×10^6	$3.37\text{x}10^6$	-0.042×10^6
11	1075	1	0.23	-1.46	2.31×10^6	$3.37\text{x}10^6$	$+0.0058\times10^6$
12	1076	1	0.228	-1.46	2.31×10^6	$3.37\text{x}10^6$	-0.0037×10^6

Continuing we find the next point at 1 610

	1610	1	-0.728	+0.454	5.18×10^6	-2.36×10^6	-0.009×10^6

The first natural frequency is 1 076 rad/s. We would have to carry on finding the next natural frequency is 1 610 rad/s.

WORKED EXAMPLE No. 3

For the same problem (W. E. 2) determine the approximate nodal points.

SOLUTION

This involves plotting the θ values at the rotor.

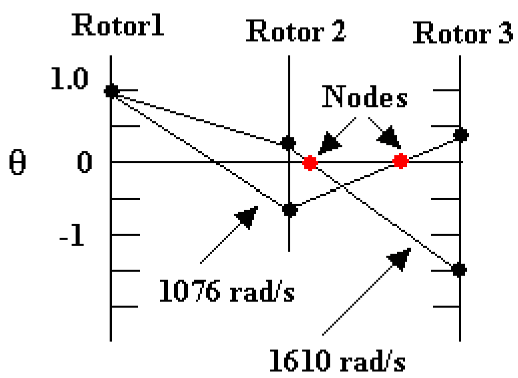


Figure 5

At 1 610 rad/s the node between rotor 2 and 3 and close to rotor 2. At 1 076 rad/s the node is between rotors 2 and 3 and closer to 3 than 2.

SELF ASSESSMENT EXERCISE No. 1

1. A hydraulic motor shaft is supported at the free end in bearings and carries a set of pulley wheels on it. The motor has a moment of inertia of 0.8 kg m^2 and the pulley wheels have a moment of inertia of 2 kg m^2. The shaft has a stiffness of 2 MNm/rad. Calculate the natural frequency of torsional vibrations. (298 Hz)

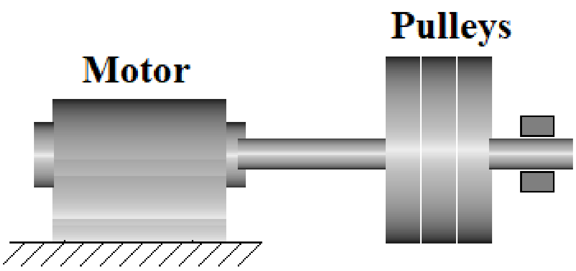


Figure 6

2. A winding motor for raising a lift has the winding wheels mounted on bearings as shown. It is connected with a coupling.

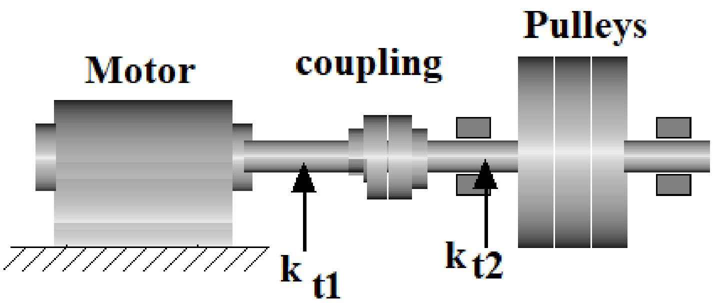


Figure 7

k_{t1} = 80 kN m/rad k_{t2} =60 kN m/rad I_{MOTOR} = 2 kg m^2 $I_{COUPLING}$ = 0.8 kg m^2 I_{WHEEL} = 3 kg m^2

Show that there is a natural frequency of vibration between 100 and 200 rad/s and another between 400 and 500 rad/s.

3. A gas turbine is connected to a compressor and a generator through shafts and couplings with stiffness and moment of inertia as shown on the diagram.

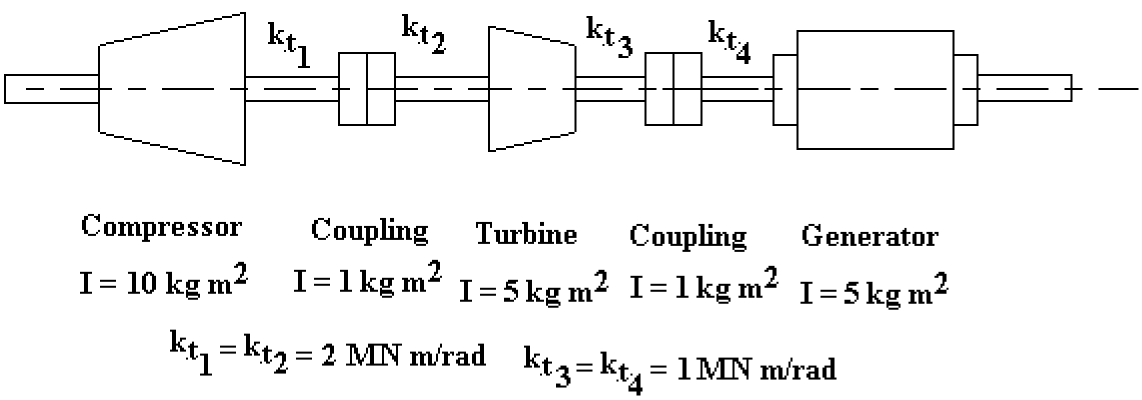


Figure 8

Severe torsional observations occur when running at the normal speed of 50 rev/s. Neglect the inertia of the shafts and determine the fundamental natural frequency and the mode shape. Use Holzer's method to do this.

4. Torsional Oscillations of Very Long Shafts

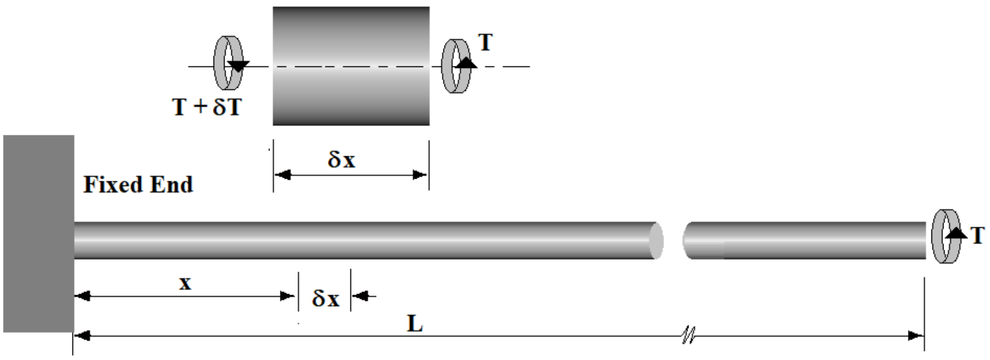
Oscillations can occur in long transmission shafts such as the drill shaft of an oil rig. The theory is similar to that of transverse vibrations and buckling as there can be more than one mode. The derivation uses the wave equation. The work applies only to shafts with a circular cross section. The oscillation is entirely due to the distributed mass of the long shaft.

4.1 Nomenclature

θ Angle of Twist
ρ Density of material
R Radius of shaft
L Length of shaft
A Cross Section Area
G Modulus of Rigidity for the material
J Polar Second Moment of Area $J = \frac{\pi R^4}{2}$
I Polar Moment of Inertia $I = \frac{MR^2}{2}$
α Angular Acceleration $\alpha = \frac{d^2\theta}{dt^2}$
t Time
ω_n Natural Angular Frequency
f_n Natural frequency
n mode

4.2 Theory

Consider a long shaft fixed a one end and free at the other. Suppose a torque T is applied at the free end.



Consider an element of the shaft length δx. The torque at one end is slightly larger than the torque at the other by δT. Suppose the torque decreases uniformly with x as $\frac{dT}{dx}$

The net torque on the element is

$$\delta T = \frac{dT}{dx}\delta x$$

From the torsion equation we have

$$T = \frac{GJ\theta}{L}$$

For a uniform shaft

$$\frac{\theta}{L} = \frac{d\theta}{dx} \qquad T = GJ\frac{d\theta}{dx}$$

Differentiate with respect to x and

$$\frac{dT}{dx} = GJ\frac{d^2\theta}{dx^2}$$

The net torque is now

$$\delta T = \frac{dT}{dx}\delta x = GJ\frac{d^2\theta}{dx^2}\delta x$$

The torque on the element must overcome the inertia of the material only.

$$\delta T = I\alpha = I\frac{d^2\theta}{dt^2} = GJ\frac{d^2\theta}{dx^2}\delta x$$

For a solid circular length of shaft

$$I = \frac{MR^2}{2} = \frac{\rho A\delta x R^2}{2} = \frac{\rho\pi R^2\delta x R^2}{2} = \frac{\rho\pi R^4}{2}\delta x = \rho\ \delta x\ J$$

$$\delta T = \rho\ \delta x\ J\frac{d^2\theta}{dt^2} = GJ\frac{d^2\theta}{dx^2}\delta x$$

$$\rho\ \frac{d^2\theta}{dt^2} = G\frac{d^2\theta}{dx^2}$$

$$\frac{d^2\theta}{dx^2} = \frac{\rho}{G}\frac{d^2\theta}{dt^2}$$

This is usually expressed as

$$\frac{d^2\theta}{dx^2} = \frac{1}{c^2}\frac{d^2\theta}{dt^2}$$

c is the velocity of a wave

$$c = \sqrt{\frac{G}{\rho}}\ m/s$$

The standard solution for this equation is

$$\theta = \left[A\sin\left(\frac{\omega x}{c}\right) + B\cos\left(\frac{\omega x}{c}\right)\right]\sin(\omega t)$$

A and B are constants.

Now we put in the boundary conditions for the shaft. When x = 0, θ = 0 so putting this in the equation

$$0 = [A\sin(0) + B\cos(0)]\sin(\omega t) = [0 + B]\sin(\omega t)$$

It follows that B = 0 and our solution reduces to

$$\theta = \left[A\sin\left(\frac{\omega x}{c}\right)\right]\sin(\omega t)$$

If we differentiate with respect to x

$$\frac{d\theta}{dx} = \left[A\frac{\omega}{c}\ \cos\left(\frac{\omega x}{c}\right)\right]\sin(\omega t)$$

When x = L, $\frac{d\theta}{dx} = 0$

$$\left[A\frac{\omega}{c}\ \cos\left(\frac{\omega L}{c}\right)\right]\sin(\omega t) = 0$$

It follows that

$$\cos\left(\frac{\omega L}{c}\right) = 0$$

This can only occur if

$$\omega = \left(n - \frac{1}{2}\right)\frac{\pi c}{L}$$

n is an integer 1, 2, 3 ....

This gives the natural frequency of the system.

$$\omega_n = \left(n - \frac{1}{2}\right)\frac{\pi}{L}\sqrt{\frac{G}{\rho}} \qquad f_n = \frac{\omega_n}{2\pi} = \left(n - \frac{1}{2}\right)\frac{1}{2L}\sqrt{\frac{G}{\rho}}$$

The lowest natural frequency occurs at the fundamental mode n = 1.

WORKED EXAMPLE No. 4

A long oil rig drill shaft is modelled as a long uniform shaft fixed at the top and free at the bottom. The shaft is 375 m long and has a material density of 7 800 kg/m^3 and Modulus of Rigidity 70 GPa. Determine the fundamental natural frequency.

SOLUTION

$$f_n = \left(n - \frac{1}{2}\right)\frac{1}{2L}\sqrt{\frac{G}{\rho}} \quad \text{Put } n = 1$$

$$f_n = \frac{1}{4L}\sqrt{\frac{G}{\rho}} = \frac{1}{4 \times 375}\sqrt{\frac{70 \times 10^9}{7\,800}} = 2 \text{ Hz}$$

SELF ASSESSMENT EXERCISE No. 2

A long oil rig drill shaft is modelled as a long uniform shaft fixed at the top and free at the bottom. The shaft is 600 m long and has a material density of 7 850 kg/m^3 and Modulus of Rigidity 75 GPa. Determine the fundamental natural frequency. (1.29 Hz)