

**SOLID MECHANICS
DYNAMICS**

TUTORIAL – FORCED VIBRATIONS

- Define a forced vibration in general terms.
- Explain the whirling of shafts and solve problems.
- Solve problems involving mass – spring – damper systems.
- Analyse the case of a harmonic disturbing force.
- Analyse the case of a harmonic disturbance of the support.
- Analyse the frequency response for the same.
- Define and use phasors.

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1. Introduction

In the tutorial on damped oscillations, it was shown that a free vibration dies away with time because the energy trapped in the vibrating system is dissipated by the damping. The equation for the displacement in a damped oscillation was derived and given as

$$x = Ce^{-\delta\omega_n t} \cos(\omega t)$$

δ is the damping ratio and ω_n the natural angular frequency. The following cases were described.

When $\delta > 1$ we have an over damped system.

When $\delta = 1$ we have a critically damped oscillation.

When $\delta < 1$ we have a damped oscillation that dies away with time.

When $\delta = 0$ we have a system with no damping and a steady oscillation occurred

It might be inferred from this pattern that if $\delta < 0$ we get an oscillation that grows with time. The diagram illustrates this pattern.

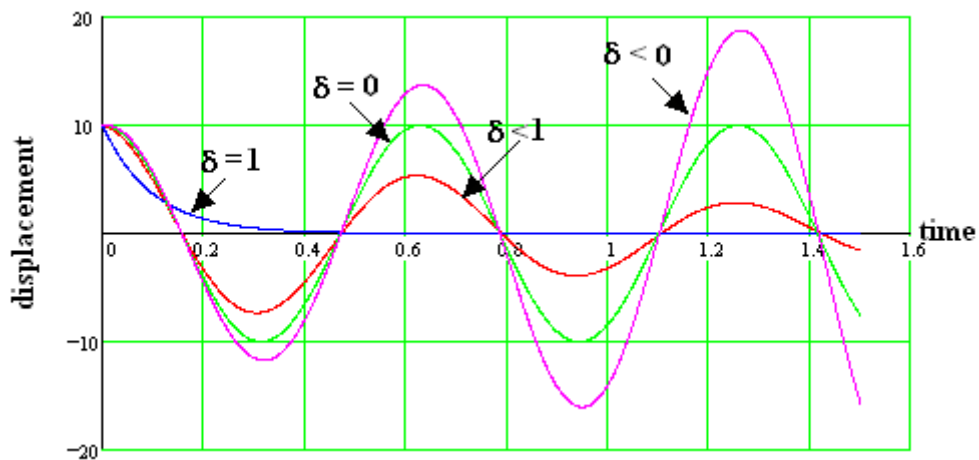


Figure 1

In order for the damping ratio δ to be less than zero, that is, to be negative, we would have the opposite of damping, something that puts energy into the system instead of taking it out. As the energy is added to the system the amplitude grows and grows. The energy is added by an outside source and such oscillations are called forced, (the object of this tutorial). A good example of such an oscillation is a child on a swing. If nothing is done, friction will make the swing come to a halt. If someone gives the swing a small push at the start of each swing, energy is added to the system and the swing goes higher and higher. This phenomenon is also known as excitation.



Figure 2

In engineering, many structures are prone to vibrate when excited at or near the natural frequency. A good example is what happens to a car when the wheels are out of balance or when you drive along a corrugated surface. If the disturbance is close to the natural frequency of the suspension system the vehicle might bounce out of control. Vehicles are fitted with dampers to prevent this. Other examples are the wind blowing around chimney stacks, cooling towers and suspended cables that can excite them into catastrophic oscillations. Bridges and other structures are also prone to this. You can see many videos on the internet searching for vortex shedding.

The problem is very acute in rotating machines even when balanced as accurately as possible there will be critical speeds when the rotor will vibrate transversely possibly causing catastrophic failure.

2. Forced Vibrations

We must examine two common types of forced vibrations, first when a mass has disturbing force acting on it and second when the spring support is disturbed harmonically.

2.1. Harmonic Disturbing Force

Consider an ideal system as shown. A mass M is suspended on a spring and a damper is placed between the spring and the support. The support does not move. Located on the mass is a small rotating machine that is out of balance. It has the equivalent of a small mass m rotating at radius r that produces an out of balance force due to the centripetal/centrifugal affect. The magnitude of this force is $F_0 = mr\omega^2$. The main mass is constrained in guides so that it will only move up and down (one degree of freedom). At the position shown with rotation through angle θ , the component of F_0 acting vertically is $F = F_0 \sin\theta$.

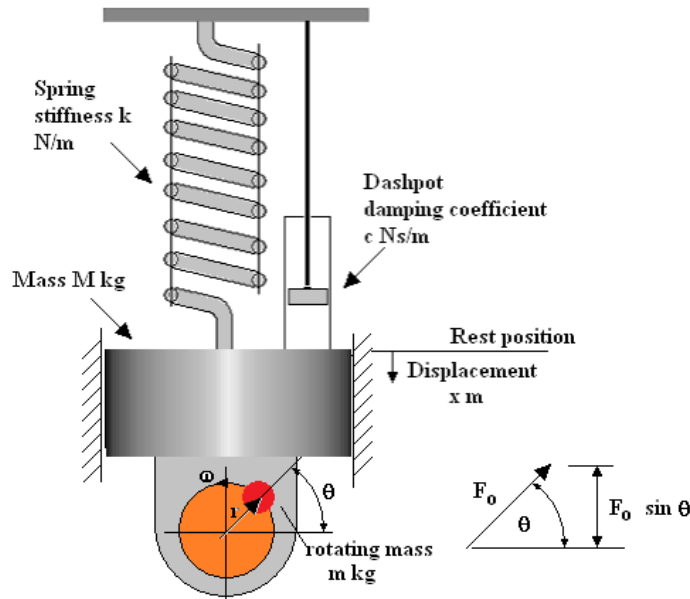


Figure 3

Any force applied to the mass to make it move must overcome the inertia, damping and spring force. The applied force is hence

$$F = F_i + F_d + F_s$$

$$F = M \frac{d^2x}{dt^2} + c \frac{dx}{dt} + kx$$

In this case the mass can only move vertically so the only force applied to it in this direction is the vertical component of the centrifugal force.

$$F_0 \sin(\omega t) = M \frac{d^2x}{dt^2} + c \frac{dx}{dt} + kx$$

F_0 is the amplitude of the disturbing force

2.1.1 Phasor Representation

We may assume (and it is known from observations) that the mass is going to oscillate up and down with a sinusoidal oscillation of amplitude A . Let's assume that time starts when the oscillation passes through the rest position.

The displacement is given by $x = A \sin \omega t$ where A is the amplitude.
 The velocity is then $v = dx/dt = A\omega \cos \omega t$ where $A\omega$ is the amplitude.
 The acceleration is $a = dv/dt = -A\omega^2 \sin \omega t$ where $A\omega^2$ is the amplitude.

The displacement x , velocity v and acceleration a are plotted against time in the diagram below. Each graph may be generated by a vector rotating at ω rad/s and with a length equal to the amplitude. Such vectors are called phasors. At a given moment in time, the tip of each vector is projected across to the appropriate point as shown.

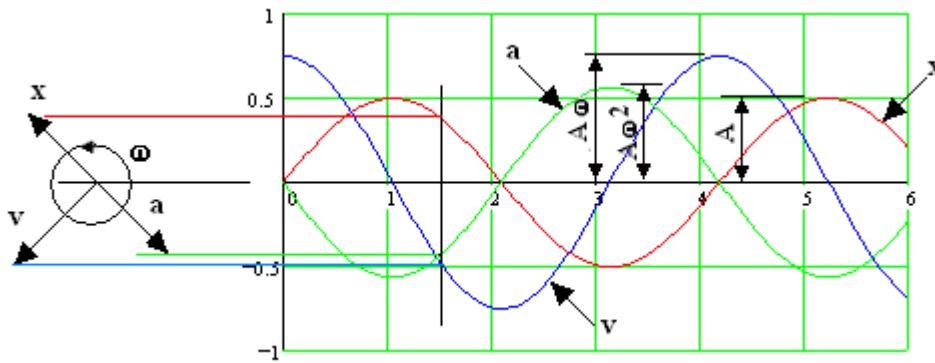


Figure 4

We can see that in order to produce the result, the velocity vector is 90° in front of the displacement and the acceleration is 90° in front of the velocity.

The spring force is directly proportional to displacement x so it must be in phase with x . The damping force is directly proportional to the velocity v so it must be in phase with v . The inertia force is directly proportional to the acceleration a so it must be in phase with a . It follows that the three forces can also be represented by phasors all rotating at angular velocity ω rad/s. We can choose a moment in time when the displacement is horizontal as shown.

The spring force is in phase with the movement so we draw the vector horizontally. The other vectors are 90° and 180° ahead respectively.

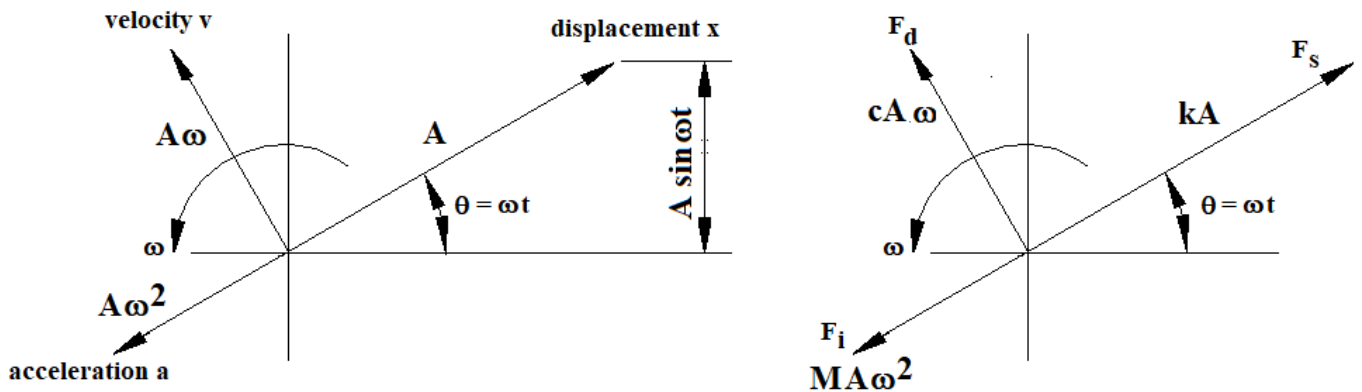


Figure 5

The sum of these three vectors is F_0 so adding them we get a typical vector diagram as shown.

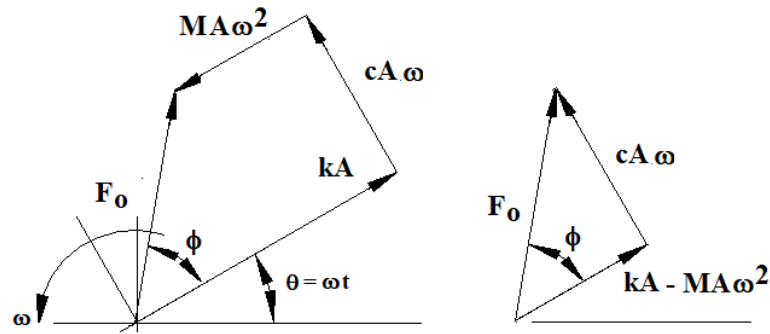


Figure 6

The diagram shows that the applied force F_0 is at an angle ϕ to the spring force so it must be displaced by a phase angle ϕ relative to x . Applying trigonometry (Pythagoras) we have

$$F_0^2 = (kA - MA\omega^2)^2 + (cA\omega)^2$$

$$F_0^2 = A^2(k - M\omega^2)^2 + A^2(c\omega)^2$$

$$F_0^2 = A^2\{(k - M\omega^2)^2 + (c\omega)^2\}$$

Divide through by M^2

$$\frac{F_0^2}{M^2} = A^2 \left\{ \left(\frac{k}{M} - \omega^2 \right)^2 + \left(\frac{c\omega}{M} \right)^2 \right\}$$

It has been shown the tutorial on damped vibrations that

$$\omega_n^2 = \frac{k}{M} \quad \text{and} \quad \frac{c}{M} = 2\delta\omega_n$$

$$A^2 = \left(\frac{F_0}{M} \right)^2 \left\{ \frac{1}{(\omega_n^2 - \omega^2)^2 + (2\delta\omega\omega_n)^2} \right\}$$

The phase angle is

$$\tan\phi = \frac{2\delta\omega\omega_n}{\omega_n^2 - \omega^2}$$

Plotting x and F_0 for a given applied frequency against time produces a graph similar to below.

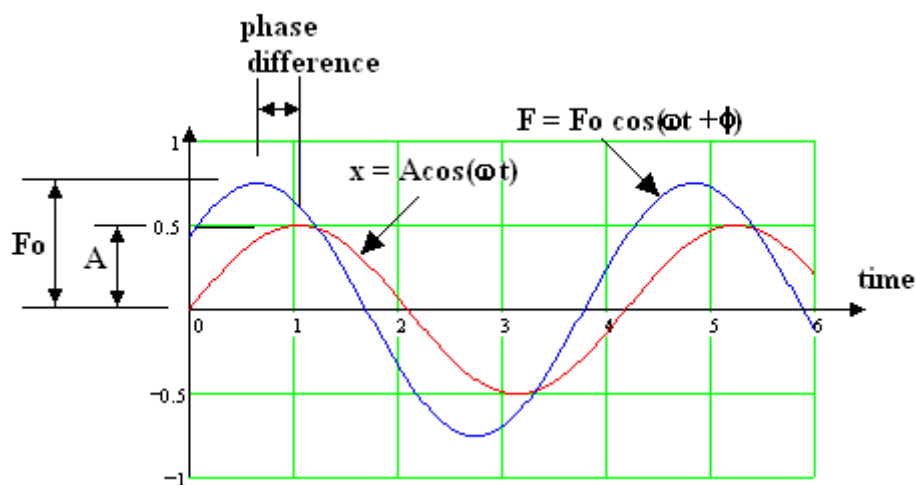


Figure 7

2.1.2 Frequency Response Diagrams

Suppose we start the out of balance machine and gradually increase the speed ω from zero. Taking a typical value of $\omega_n = 10$ and plotting ϕ against ω for various values of δ produces the graph below.

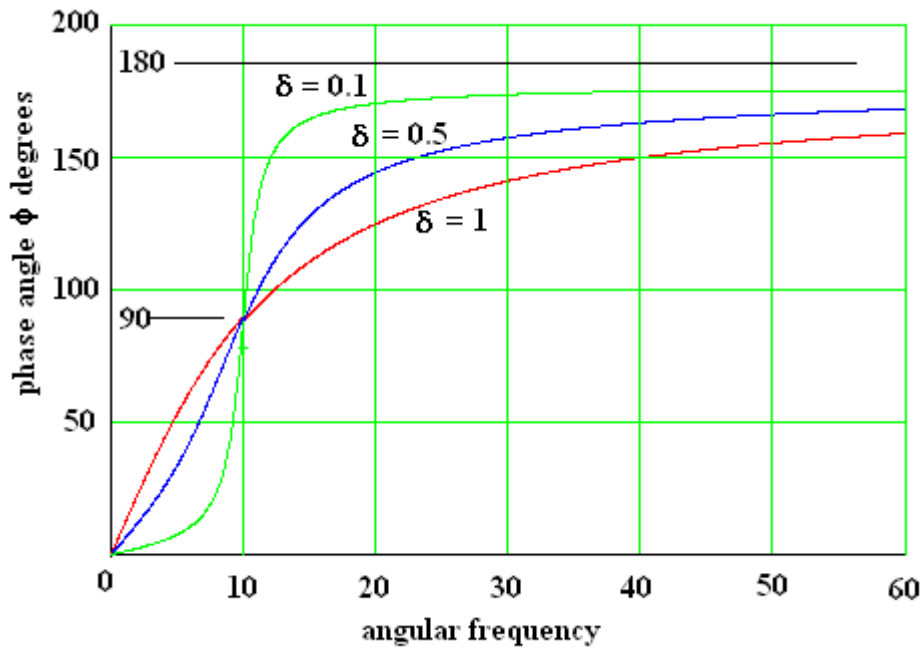


Figure 8

The plot shows that the phase angle ϕ starts at zero and reaches 90° when $\omega = \omega_n$. As the speed increases to large values, the phase angle approaches 180° .

Now consider what happens to the amplitude A . Plotting A against ω for various values of δ gives the graph below.

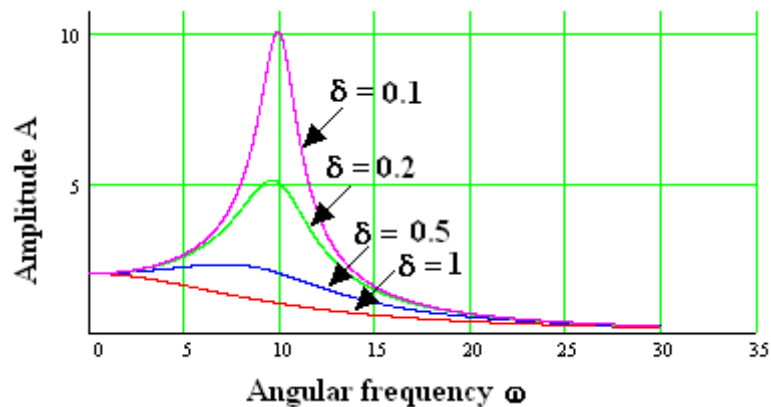


Figure 9

The following analyses the equation for A at the three obvious points

$$A^2 = \left(\frac{F_o}{M}\right)^2 \left\{ \frac{1}{(\omega_n^2 - \omega^2)^2 + (2\delta\omega\omega_n)^2} \right\}$$

When $\omega = 0$ this becomes

$$A = \left(\frac{F_o}{M}\right) \left\{ \frac{1}{\omega_n^2} \right\}$$

This has a value of 2 on the previous diagram and is the same for all values of damping.

When $\omega = \omega_n$ the amplitude is

$$A = \left(\frac{F_o}{M}\right) \left\{ \frac{1}{2\delta\omega^2} \right\}$$

The value is dependent on the damping value.

The smaller the value of the damping ratio, the greater the peak value of A becomes. If $\delta = 0$ then in theory $A = \infty$

When the frequency becomes very large, the amplitude tends to die away to zero for all values of δ .

We may conclude from this, that when the out of balance machine rotates very fast, there is very little disturbance to the system but when it approaches the natural frequency of the system the amplitude might become very large depending on the damping. It should also be noted that the frequency at which the amplitude peaks is called the resonant frequency and this is not quite the same as the natural frequency.

WORKED EXAMPLE No. 1

The diagram shows a mass-spring-dashpot system. The support is fixed. The mass has a harmonic disturbing force applied to it given by the equation $F = 400 \sin(30 t)$ Newton.

Determine the amplitude of the mass and the phase angle.

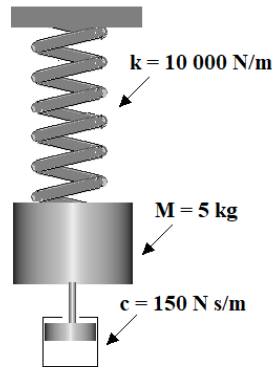


Figure 10

SOLUTION

From the question we know that $k = 10\,000$ N/m, $M = 5$ kg and $c = 150$ N s/m

$$\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{10\,000}{5}} = 44.72 \frac{\text{rad}}{\text{s}} \quad c_c = \sqrt{4Mk} = \sqrt{4 \times 5 \times 10\,000} = 447.21 \text{ N s/m}$$

$$\delta = \frac{c}{c_c} = \frac{150}{447.21} = 0.335$$

From the equation of motion $F_o = 400$ N and $\omega = 30$ rad/s

$$A^2 = \left(\frac{F_o}{M}\right)^2 \left\{ \frac{1}{(\omega_n^2 - \omega^2)^2 + (2\delta\omega\omega_n)^2} \right\}$$

$$A^2 = \left(\frac{400}{5}\right)^2 \left\{ \frac{1}{(44.72^2 - 30^2)^2 + (2 \times 0.335 \times 30 \times 44.72)^2} \right\} = 0.00137 \quad A = 0.056 \text{ m or } 56 \text{ mm}$$

$$\tan\phi = \frac{2\delta\omega\omega_n}{\omega_n^2 - \omega^2} = \frac{2 \times 0.335 \times 30 \times 44.72}{44.72^2 - 30^2} = 0.818 \quad \phi = 39.3^\circ$$

SELF ASSESSMENT EXERCISE No. 1

1. A mass of 12 kg rests on a springy base of stiffness 8 kN/m. There is a damper between the mass and the support with a damping coefficient of 400 N s/m.

The support is subjected to a harmonic disturbing force given by $F = 200 \sin(30 t)$

Calculate the amplitude of the mass and the phase angle.

(16 mm and 103.1 degrees).

2. A mass of 150 kg rests on a springy base of stiffness 60 kN/m. There is a damper between the mass and the support with a damping coefficient of 5 000 N s/m.

The support is subjected to a harmonic disturbing force given by $F = 800 \sin(25 t)$

Calculate the amplitude of the mass and the phase angle.

(6.2 mm and 105.1 degrees).

2.2 Harmonic Movement of the Support

Damper between Mass and Fixed Point

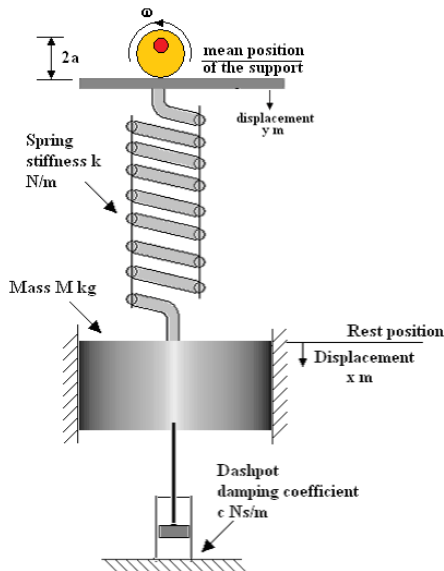


Figure 11A

The diagram shows a mass spring damper system. The mass can only move vertically. The support is made to move up and down by a cam that rotates at ω rad/s with amplitude a . If time starts when the support passes through the mean position, the motion of the support is described by the equation

$$y = a \sin(\omega t)$$

We may assume that the mass is going to move up and down harmonically with an amplitude A but we cannot assume that the motion is in phase with the support so the equation of motion will be given by the equation $x = A \sin(\omega t + \phi)$ where ϕ is the phase angle.

The damper is attached between the mass and a fixed point so the velocity of the damper piston is dx/dt .

At any given moment in time the spring is stretched or shortened by an amount $x - y$ at any time. The spring force is hence $F = k(x - y)$

The three forces acting on the mass are:

| | |
|---------------|------------------------|
| Spring force | $F_s = k(x - y)$ |
| Damping force | $F_d = c \, dx/dt$ |
| Inertia force | $F_i = M \, d^2x/dt^2$ |

$$0 = M \frac{d^2x}{dt^2} + c \frac{dx}{dt} + k(x - y) = M \frac{d^2x}{dt^2} + c \frac{dx}{dt} + kx - ky$$

$$ky = M \frac{d^2x}{dt^2} + c \frac{dx}{dt} + kx$$

$$ka \sin \omega t = M \frac{d^2x}{dt^2} + c \frac{dx}{dt} + kx$$

F_0 is now called the equivalent disturbing force to the previous case (2.1).

$$F_0 \sin \omega t = M \frac{d^2x}{dt^2} + c \frac{dx}{dt} + kx$$

They are the same except the term ka replaces the term F_0 . It follows that the solutions are the same with this substitution.

$$A^2 = \left(\frac{ka}{M}\right)^2 \left\{ \frac{1}{(\omega_n^2 - \omega^2)^2 + (2\delta\omega\omega_n)^2} \right\}$$

$$\tan \phi = \frac{2\delta\omega\omega_n}{\omega_n^2 - \omega^2}$$

WORKED EXAMPLE No. 2

The diagram shows a mass-spring-dashpot system.
The support is moved with a motion of $y = 6 \sin(40t)$ mm.
Determine the amplitude of the mass and the phase angle.

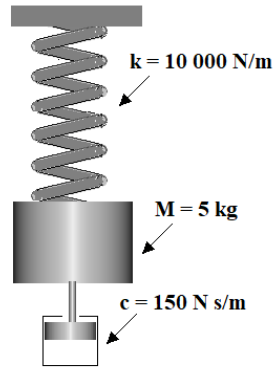


Figure 13

SOLUTION

From the question we know that $k = 10\,000$ N/m, $M = 5$ kg and $c = 150$ N s/m

$$\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{10\,000}{5}} = 44.72 \frac{\text{rad}}{\text{s}} \quad c_c = \sqrt{4Mk} = \sqrt{4 \times 5 \times 10\,000} = 447.21$$

$$\delta = \frac{c}{c_c} = \frac{150}{447.21} = 0.335$$

From the equation of motion $a = 12$ mm and $\omega = 40$ rad/s

$$\frac{A}{a} = \frac{k}{M} \sqrt{\frac{1}{(\omega_n^2 - \omega^2)^2 + (2\delta\omega\omega_n)^2}}$$

$$\frac{A}{6} = \frac{10\,000}{5} \sqrt{\frac{1}{(44.72^2 - 40^2)^2 + (2 \times 0.335 \times 40 \times 44.72)^2}} = 1.581$$

$$A = 1.581 \times 6 = 9.487 \text{ mm}$$

$$\tan\phi = \frac{2\delta\omega\omega_n}{\omega_n^2 - \omega^2} = \frac{2 \times 0.335 \times 40 \times 44.72}{44.72^2 - 40^2} = 3$$

$$\phi = 71.56^\circ$$

2.3 Dashpot between Mass and Support

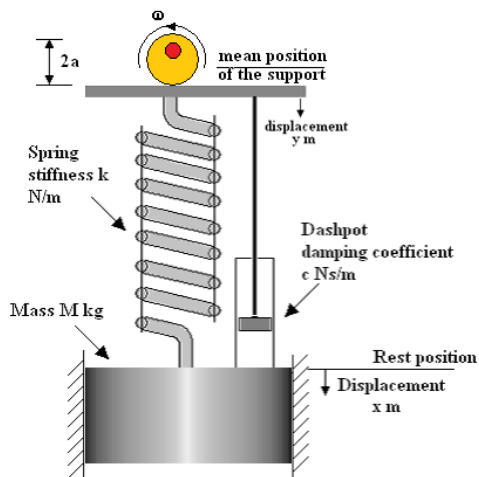


Figure 11B

The diagram shows a mass spring damper system that is a theoretical representation of a car shock absorber. The mass can only move vertically. The support is made to move up and down by a cam that rotates at ω rad/s with amplitude a . If time starts when the support passes through the mean position, the motion of the support is described by the equation

$$y = a \sin(\omega t)$$

We may assume that the mass is going to move up and down harmonically with an amplitude A . The phase angle between the equivalent force F_o and y is ϕ with x is ψ .

At any given moment in time the spring is stretched or shortened by an amount $x - y$. The spring force is hence $F_s = k(x - y)$. The three forces acting on the mass are:

| | |
|---------------|-------------------------------|
| Spring force | $F_s = k(x - y)$ |
| Damping force | $F_d = c \frac{d(x - y)}{dt}$ |
| Inertia force | $F_i = M \frac{d^2x}{dt^2}$ |

In this case the force balance gives

$$0 = M \frac{d^2x}{dt^2} + c \frac{d(x - y)}{dt} + k(x - y)$$

$$M \frac{d^2x}{dt^2} + c \frac{d(x)}{dt} + kx = c \frac{d(y)}{dt} + ky = F_o \text{ (the equivalent disturbing force)}$$

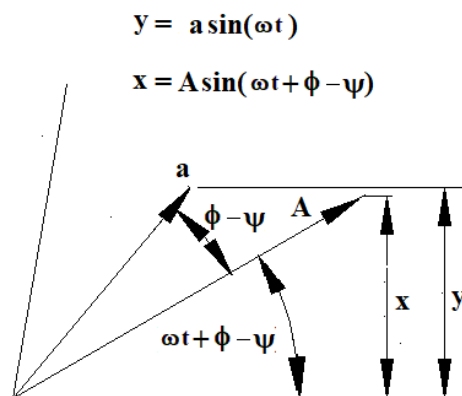
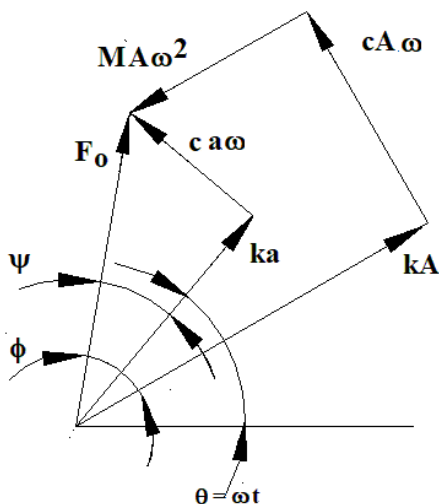
Substitute $y = a \sin(\omega t)$

$$F_o = M \frac{d^2x}{dt^2} + c \frac{d(x)}{dt} + kx = c \frac{d(a \sin(\omega t))}{dt} + k a \sin(\omega t)$$

$$F_o = M \frac{d^2x}{dt^2} + c \frac{d(x)}{dt} + kx = c a \omega \cos(\omega t) + k a \sin(\omega t)$$

In phasor (vector) form

$$F_o = M A \omega^2 + c A \omega + k A \quad \text{and} \quad F_o = c a \omega + k a$$



Note that the phase angle between x and y is $(\phi - \psi)$ hence $x = A \sin[\omega t - (\psi - \phi)] = A \sin(\omega t + \phi - \psi)$

From the phasor (vector) diagram we see that

$$F_o = \sqrt{(ca\omega)^2 + (ka)^2} \quad F_o^2 = (kA - MA\omega^2)^2 + (cA\omega)^2$$

$$\tan \psi = \frac{ca\omega}{ka} = \frac{c\omega}{k} = 2\delta \frac{\omega}{\omega_n}$$

As shown previously

$$A = \sqrt{\left(\frac{F_o}{M}\right)^2 \left\{ \frac{1}{(\omega_n^2 - \omega^2)^2 + (2\delta\omega\omega_n)^2} \right\}} \quad \tan\phi = \frac{2\delta\omega\omega_n}{\omega_n^2 - \omega^2}$$

WORKED EXAMPLE No. 3

The shock absorber is tested on a test rig by harmonically exciting the lower end at an angular frequency of 6π rad/s and amplitude of 50 mm. The upper end supports a load of 500 kg. The damping ratio is 0.25 and the spring stiffness is 50 kN/m. Determine the amplitude of the mass.

SOLUTION

Data

$$\omega = 6\pi \quad a = 50 \text{ mm} \quad \delta = 0.26 \quad M = 500 \text{ kg} \quad k = 50 \text{ kN/m}$$

Natural frequency

$$\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{50\,000}{500}} = 10 \text{ rad/s}$$

Damping Coefficient

$$c_c = \sqrt{4Mk} = \sqrt{4 \times 500 \times 50\,000} = 10\,000 \text{ N s/m}$$

$$c = \delta c_c = 0.25 \times 10\,000 = 2\,500 \text{ N s/m}$$

alternatively

$$c = \frac{2\delta k}{\omega_n} = \frac{2 \times 0.25 \times 50\,000}{10} = 2\,500 \text{ N s/m}$$

Equivalent Disturbing Force

$$F_o = \sqrt{(ca\omega)^2 + (ka)^2} = \sqrt{(2\,500 \times 0.05 \times 6\pi)^2 + (50\,000 \times 0.05)^2} = 3\,435 \text{ N}$$

$$\psi = \tan^{-1}\left(2\delta \frac{\omega}{\omega_n}\right) = \tan^{-1}\left(2 \times 0.25 \frac{6\pi}{10}\right) = \tan^{-1}(0.942) = 0.75 \text{ rad or } 43.3^\circ$$

$$A^2 = \left(\frac{F_o}{M}\right)^2 \left\{ \frac{1}{(\omega_n^2 - \omega^2)^2 + (2\delta\omega\omega_n)^2} \right\}$$

$$A^2 = \left(\frac{3\,435}{500}\right)^2 \left\{ \frac{1}{(10^2 - (6\pi)^2)^2 + (2 \times 0.25 \times 10 \times 6\pi)^2} \right\}$$

$$A^2 = \left(\frac{3435}{500}\right)^2 \left\{ \frac{1}{65\,181 + 8\,882.6} \right\}$$

$$A = 0.0252 \text{ m}$$

2.4 Magnification Factor

The magnification factor is the ratio A/a .

$$A^2 = \left(\frac{F_o}{M}\right)^2 \left\{ \frac{1}{(\omega_n^2 - \omega^2)^2 + (2\delta\omega\omega_n)^2} \right\}$$

For the case with the dashpot between the mass and the support

$$F_o = \sqrt{(c\omega)^2 + (ka)^2}$$

In the case where the dashpot is between the mass and a fixed point $F_o = ka$ and in this case

$$A^2 = \left(\frac{ka}{M}\right)^2 \left\{ \frac{1}{(\omega_n^2 - \omega^2)^2 + (2\delta\omega\omega_n)^2} \right\}$$

$$A = \frac{ka}{M} \sqrt{\left\{ \frac{1}{(\omega_n^2 - \omega^2)^2 + (2\delta\omega\omega_n)^2} \right\}} = a \sqrt{\left\{ \frac{(\omega_n^2)^2}{(\omega_n^2 - \omega^2)^2 + (2\delta\omega\omega_n)^2} \right\}}$$

Since $k/M = \omega_n^2$

$$MF = \frac{A}{a} = \sqrt{\frac{(\omega_n^2)^2}{(\omega_n^2 - \omega^2)^2 + (2\delta\omega\omega_n)^2}} = \sqrt{\frac{1}{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\delta\frac{\omega}{\omega_n}\right)^2}}$$

It follows that

$$MF = \frac{\text{Maximum force in Spring}}{F_o}$$

The response graph is shown below. At low speeds the support and mass move up and down together. As ω approaches ω_n the amplitude of the mass grows and the phase angle approaches 90° . As the speed passes resonance, the amplitude of the mass reduces and eventually becomes almost static. The phase angle tends to 180° at high speeds. The magnification is greatest at resonance and as before, the resonant frequency is not quite the same as the natural frequency.

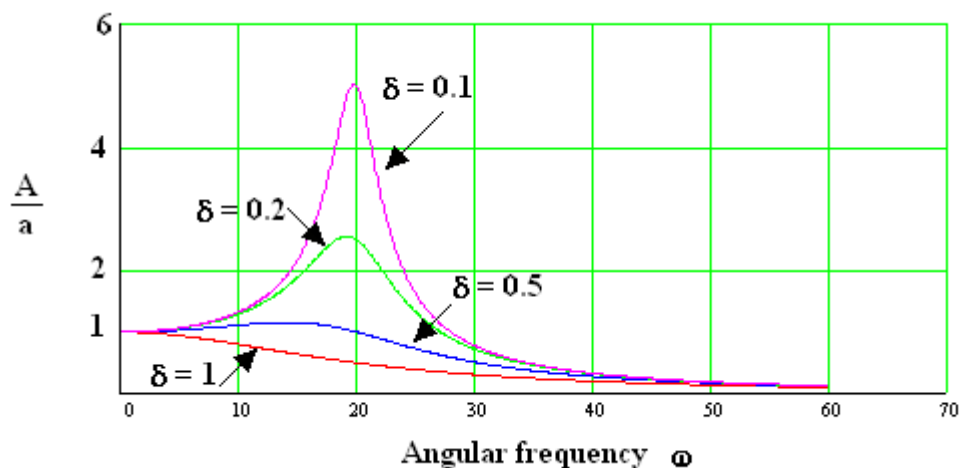


Figure 12

The peak occurs when $\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\delta\frac{\omega}{\omega_n}\right)^2$ is a maximum

This can be found by max and min theory. Simplify this to

$$(1 - r^2)^2 + (2\delta r)^2$$

Differentiate with respect to r

$$\frac{d\{(1 - r^2)^2 + (2\delta r)^2\}}{dr} = 2(1 - r^2)(-2r) + 8\delta^2 r$$

Equate to zero and

$$2(1 - r^2)(-2r) + 8\delta^2 r = 0$$

$$r = \sqrt{1 - 2\delta^2}$$

Peak MF occurs when

$$\omega = \omega_n \sqrt{1 - 2\delta^2}$$

2.5 Transmissibility

When a mass vibrates on an elastic support, a force is transmitted through the spring and damper to the frame or ground. This is the sum of the spring and damping force. This may be illustrated with the vector diagram.

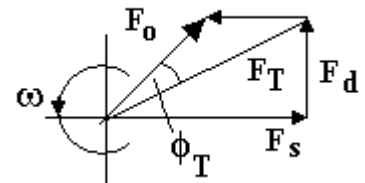


Figure 14

From the vector diagram we deduce that the transmitted force is ($F_s = kA$ and $F_d = cA\omega$)

$$F_T = \sqrt{F_s^2 + F_d^2} = \sqrt{\{(kA)^2 + cA\omega\}}$$

The ratio $\frac{F_T}{F_0}$ is called the transmissibility ratio. The phase angle between the transmitted force and the applied force is

$$\phi_T = \phi - \tan^{-1}\left(\frac{F_d}{F_s}\right)$$

The above work applies to both harmonic disturbing forces and harmonic motion of the support with the substitution $F_0 = ka$

WORKED EXAMPLE No. 4

Calculate the transmitted force and the phase angle of the transmitted force for example No. 2.

SOLUTION

In example 2 we calculated $A = 56 \text{ mm}$ $k = 10 \text{ kN/m}$ and $\omega = 30 \text{ rad/s}$ $\phi = 39.3^\circ$

$$F_s = kA = 10\,000 \times 0.056 = 560 \text{ N}$$

$$F_d = cA\omega = 150 \times 0.056 \times 30 = 252 \text{ N}$$

$$F_T = \sqrt{560^2 + 252^2} = 614.1 \text{ N}$$

$$\phi_T = 39.3 - \tan^{-1}(252/560) = 15.1^\circ$$

SELF ASSESSMENT EXERCISE No. 2

1. A mass of 500 kg rests on a springy support of stiffness 40 kN/m. The damping ratio is 0.25. The support is moved harmonically with amplitude of 0.2 mm at 6 Hz. Calculate the amplitude of the mass and the phase angle.

(0.0118 mm and -7.165 degrees).

Calculate the transmitted force and the phase angle to the motion of the support.

(1.104 N and -71.8°)

2. A mass of 60 kg hangs from a spring of stiffness 100 kN/m. The damping ratio is 0.2. The support is moved harmonically with amplitude of 3 mm at 6 Hz. Calculate the amplitude of the mass and the phase angle.

(7.544 mm and 68.3 degrees).

Calculate the transmitted force and the phase angle to the motion of the support.

(804 N and 48°)

3. A vehicle has a mass of 1 000 kg. There are 4 shock absorbers with a combined spring stiffness of 400 kN/m and damping ratio 0.5. The wheels are excited sinusoidally with an amplitude of 20 mm at a frequency of 5 Hz. Treating the tyres as solid, determine the amplitude of vertical vibration of the body. (17.3 mm)

3. Whirling of Shafts

When a shaft rotates, it may well go into transverse oscillations. If the shaft is out of balance, the resulting centrifugal force will induce the shaft to vibrate. When the shaft rotates at a speed equal to the natural frequency of transverse oscillations, this vibration becomes large and shows up as a whirling of the shaft. It also occurs at multiples of the resonant speed. This can be very damaging to heavy rotary machines such as turbine generator sets and the system must be carefully balanced to reduce this effect and designed to have a natural frequency different to the speed of rotation. When starting or stopping such machinery, the critical speeds must be avoided to prevent damage to the bearings and turbine blades.

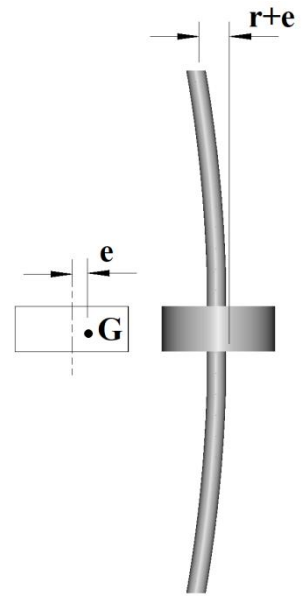


Figure 15

Consider a weightless shaft as shown with an eccentric disc mass M at the middle. Suppose the centre of the mass G is distance e from the centre line. When the shaft rotates, centrifugal force will cause it to bend out. Let the deflection of the shaft be r .

The distance to the centre of gravity is then $r + e$.

The shaft rotates at ω rad/s. The transverse stiffness is k_t N/m

The deflection force is hence $F = k_t r$

The centrifugal force is $M\omega^2(r + e)$

Equating forces we have

$$k_t r = M\omega^2(r + e) \text{ from which } r = \frac{M\omega^2(r + e)}{k_t} = \frac{M\omega^2 r}{k_t} + \frac{M\omega^2 e}{k_t}$$

$$r - \frac{M\omega^2 r}{k_t} = \frac{M\omega^2 e}{k_t}$$

$$r \left(1 - \frac{M\omega^2}{k_t} \right) = \frac{M\omega^2 e}{k_t}$$

Remember that

$$\frac{k_t}{M} = \omega_n^2$$

$$r \left(1 - \frac{\omega^2}{\omega_n^2} \right) = \frac{\omega^2 e}{\omega_n^2}$$

$$\text{so } r = \frac{\omega^2 e}{\omega_n^2 \left(1 - \frac{\omega^2}{\omega_n^2} \right)} = \frac{\omega^2 e}{(\omega_n^2 - \omega^2)} = \frac{e}{\left(\frac{\omega_n}{\omega} \right)^2 - 1}$$

From this we see that when $\omega_n = \omega$ $r = e/0$ which is infinity. This means that no matter how small the value of e , the shaft will whirl at the natural frequency. Balancing does help but can never be perfect so whirling is to be avoided on the best of machines.

The frequencies at which whirling occurs are calculated by the same methods as for transverse vibrations of beams and the derivation is not given here.

Simply Supported – The ends are free to rotate normal to the axis (e.g. self aligning bearings)

$$f = \frac{\pi}{2} n^2 \sqrt{\frac{gEI}{wL^4}} = 1.572 n^2 \sqrt{\frac{gEI}{wL^4}}$$

n is the mode and must be an integer 1, 2, 3

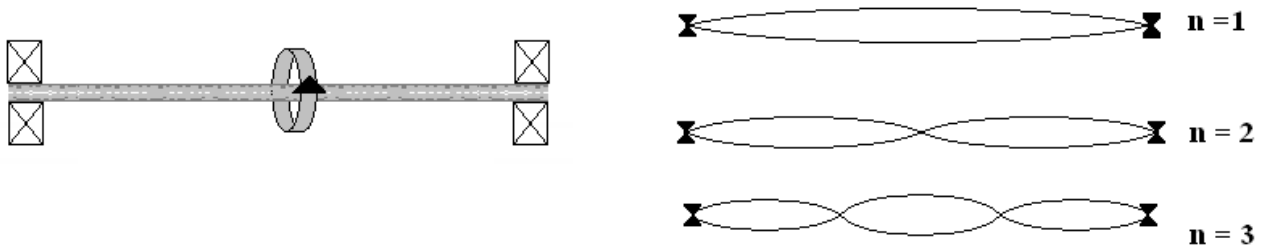


Figure 16

Fixed Ends (e.g. fixed bearings or chucks)

The lowest critical speed is

$$f = 3.562 \sqrt{\frac{gEI}{wL^4}}$$

The higher critical speeds are given by

$$f = \frac{\pi}{2} \left(n + \frac{1}{2} \right)^2 \sqrt{\frac{gEI}{wL^4}}$$

n = 2, 3, ... Note that the lowest speed almost corresponds to n = 1

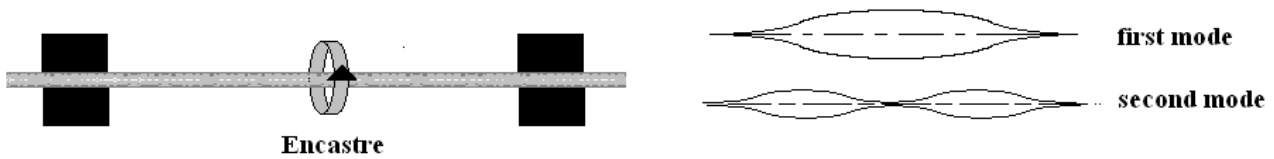


Figure 17

Cantilever (e.g. chuck at one end and free at the other)

The lowest critical speed is

$$f = 0.565 \sqrt{\frac{gEI}{wL^4}}$$

$$f = \frac{\pi}{2} \left(n - \frac{1}{2} \right)^2 \sqrt{\frac{gEI}{wL^4}}$$

n = 2, 3

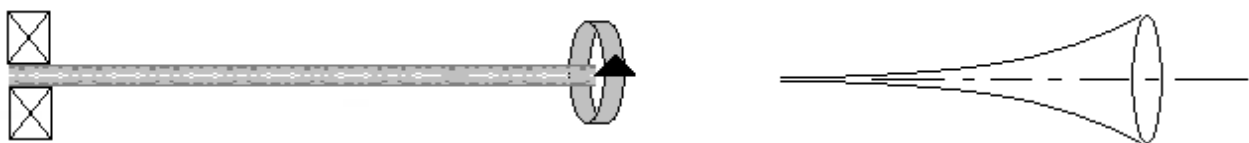


Figure 18

WORKED EXAMPLE No. 5

A shaft is 30 mm diameter and 4 m long and may be regarded as simply supported.
The density is 7 830 kg/m³. E = 205 GPa.
Calculate the first three critical frequencies.

SOLUTION

The essential information is d = 0.03 m L = 4 m ρ = 7 830 kg/m³ E = 205 GPa
First calculate the distributed weight w by calculating the weight of 1 m length.

$$A = \frac{\pi D^2}{4} = \frac{\pi \times 0.03^2}{4} = 706.9 \times 10^{-6} \text{m}^2$$

$$\text{Distributed Weight} = w = \rho A g = 7\,830 \times 706.9 \times 10^{-6} \times 9.81 = 54.3 \text{ N/m}$$

$$I = \frac{\pi D^4}{64} = \frac{\pi \times 0.03^4}{64} = 39.76 \times 10^{-9} \text{m}^4$$

$$f = \frac{\pi}{2} n^2 \sqrt{\frac{gEI}{wL^4}}$$

For the fundamental mode n = 1

$$= 1.572 \sqrt{\frac{gEI}{wL^4}} = 1.572 \sqrt{\frac{9.81 \times 205 \times 10^9 \times 39.76 \times 10^{-9}}{54.3 \times 4^4}} = 3.77 \text{ rev/s}$$

If the shaft took up the second mode the frequency would be $3.77 \times 2^2 = 15.1 \text{ rev/s}$

If the shaft took up the second mode the frequency would be $3.77 \times 3^2 = 33.9 \text{ rev/s}$

WORKED EXAMPLE No. 6

A light shaft carries a pulley at the centre with its centre of gravity on the centre line. The shaft is supported in self aligning bearing at the ends. The shaft deflects 0.5 mm under the static weight of the shaft. Determine the lowest critical speed.

SOLUTION

The frequency corresponds to that of a simply supported beam so

$$f = \frac{1}{2\pi} \sqrt{\frac{g}{x_s}} = \frac{1}{2\pi} \sqrt{\frac{9.81}{0.0005}} = 22.3 \text{ Hz} \quad \text{Critical speed} = 30 \times 60 = 1\,338 \text{ rev/min}$$

WORKED EXAMPLE No. 7

The shaft in the previous example has a distributed weight of 40 N/m and the bearings are 1.2 m apart. The flexural stiffness is 4 500 N m².

What is the critical speed taking into account the distributed mass?

SOLUTION

The frequency due to the distributed weight only is

$$f = \frac{\pi}{2} \sqrt{\frac{gEI}{wL^4}} = 1.572 \sqrt{\frac{9.81 \times 4\,500}{40 \times 1.2^4}} = 36.3 \text{ rev/s}$$

$$N = 60f = 2\,176 \text{ rev/min}$$

The critical speed is N is found from

$$\frac{1}{N^2} = \frac{1}{N_1^2} + \frac{1}{N_2^2} = \frac{1}{1\,338^2} + \frac{1}{2\,176^2} \quad N = 1\,140 \text{ rev/min}$$

WORKED EXAMPLE No. 8

A steel wire 2 mm diameter is held between chucks 1m apart. The wire weighs 0.241 N/m. The flexural stiffness is 0.157 N m². Calculate the first two critical speeds critical speed.

SOLUTION

The lowest frequency due to the distributed weight only is

$$f = 3.562 \sqrt{\frac{gEI}{wL^4}} = 3.562 \sqrt{\frac{9.81 \times 0.157}{0.241 \times 1^4}} = 9 \text{ rev/s} \quad N = 60f = 540 \text{ rev/min}$$

The second critical speed is

$$f = \frac{\pi}{2} \left(n + \frac{1}{2}\right)^2 \sqrt{\frac{gEI}{wL^4}} = \frac{\pi}{2} \left(2 + \frac{1}{2}\right)^2 \sqrt{\frac{9.81 \times 0.157}{0.241 \times 1^4}} = 24.8 \text{ rev/s} \quad N = 60 \times 24.8 = 1\,489 \text{ rev/min}$$

SELF ASSESSMENT EXERCISE No. 3

1. A shaft 30 mm diameter is made from steel with density $7\,830\text{ kg/m}^3$. Calculate the weight per metre (54.3 N/m)

The shaft runs between self aligning bearings 2.2 m apart. The modulus of elasticity E is 200GN/m^2 . Calculate the lowest critical speed. (738 rev/min)

A pulley is mounted at the middle and this makes the shaft deflect 1.5 mm. Calculate the lowest critical speed. (533 rev/min)

2. A thin rod 5 mm diameter is rotated between two chucks 0.8 m apart. The wire weighs 1.508 N/m and the flexural stiffness is 6.136 N m^2 . Calculate the first two critical speeds. (2 110 rev/min and 5 815 rev/min)
3. A shaft is 50 mm diameter and 8 m long is rotated between two self aligning bearings and can be regarded as simply supported. The density is $7\,830\text{ kg/m}^3$. $E = 205\text{ GPa}$. Calculate the first three critical frequencies. (1.571 , 6.279 and 14.13 rev/s)
4. An aluminium rod is held in a chuck with the other end unsupported. It is 12 mm diameter and 400 mm long. The density of aluminium is $2\,710\text{ kg/m}^3$ and the modulus of elasticity E is 71 GPa. Calculate the first two critical speeds. (3 253 rev/min and 20350 rev/min)