SOLID MECHANICS
DYNAMICS

TUTORIAL – DAMPED VIBRATIONS

- Define a free damped oscillation.
- Explain the purpose of damping.
- Define damping coefficient.
- Define damping ratio.
- Derive formulae that describe damped vibrations.
- Determine the natural frequency and periodic time for damped systems.
- Define amplitude reduction factor.
- Calculate damping coefficients from observations of amplitude.

Contents

1. Introduction

In the tutorial on free vibrations it was explained that once set vibrating, the system would carry on oscillating for ever because the energy put into the system by the initial disturbance cannot get out of the system. In reality, the oscillation always dies away with time because some form of friction is present. The diagram shows a displacement – time graph of a typical damped oscillation.

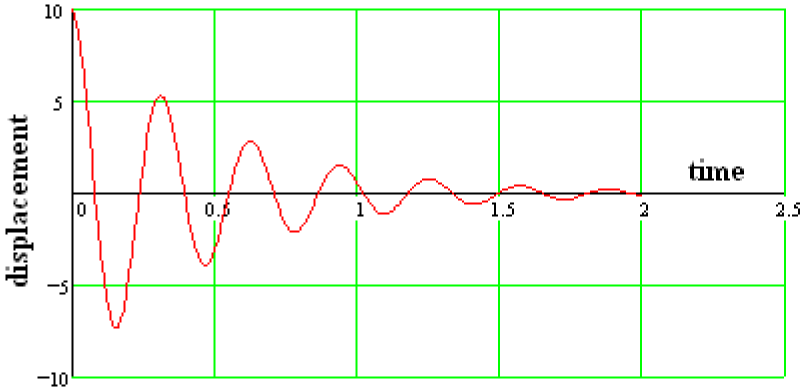

Figure 1

Friction dissipates the energy as heat. Such vibrations are called free and damped. The type of friction that is easiest to deal with mathematically is that created by a dashpot (also called a damper). Let us now examine two kinds of dashpots.

2 Dashpots

Here we will examine dashpots for linear motion. The diagram shows two types of dashpot, air and oil.

Air-The piston moves inside the cylinder and pumps or sucks air through the orifice. Because of the restriction, pressure is needed to make the air flow through the orifice and this pressure will produce a force opposing the motion. Often the piston is replaced with a simple diaphragm.

Oil-The oil is contained in the cylinder and motion of the piston pushes the oil through restrictors in the piston to the other side. Again, pressure is needed to force the fluid through the restrictor and this produces a force opposing motion. It can be shown that for both cases, the force opposing motion (the damping force) is directly proportional to the velocity of the piston. The equation for this force is as follows.

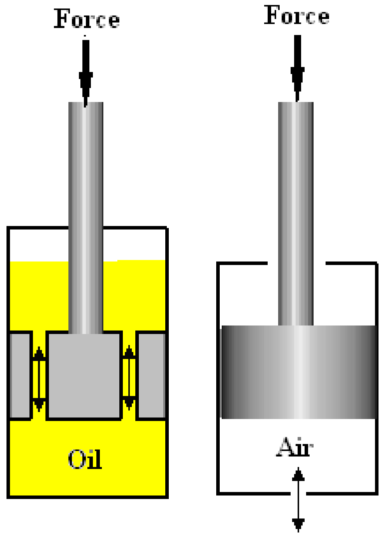

Figure 2

$$\mathbf{F_d = constant \times velocity = c\frac{dx}{dt}}$$

The constant of proportionality 'c' is called the damping coefficient and has units of N s/m. The value of 'c' depends on the size of the restriction.

3. Damped Linear Vibrations

Consider a mass suspended on a spring with the dashpot between the mass and the support.

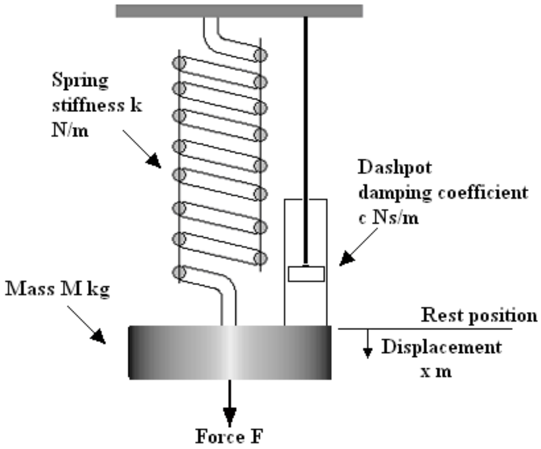

Figure 3

If a force F is applied to the mass as shown, it is opposed by three forces. These are:

$$\text{Inertia force} \quad F_i = \text{mass} \times \text{acceleration} = M\frac{d^2x}{dt^2}$$

$$\text{Damping Force} \quad F_d = \text{constant} \times \text{velocity} = c\frac{dx}{dt}$$

$$\text{Spring Force} \qquad F_s = k\,x$$

The total force is then

$$F = F_i + F_d + F_s = M\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx$$

Because the vibration is free, the applied force must be zero (e.g. when you let go of it).

$$0 = M\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx$$

This is a linear second order differential equation and it is much discussed in most maths books. We make the following changes. First divide each term by k.

$$0 = \frac{M}{k}\frac{d^2x}{dt^2} + \frac{c}{k}\frac{dx}{dt} + x$$

In the work on undamped vibrations it was shown that without a dashpot a natural oscillation occurs with angular frequency $\omega_n = \sqrt{(k/M)}$ rad/s. We may replace (k/M) with ω_n^2.

$$0 = \frac{1}{\omega_n^2}\frac{d^2x}{dt^2} + \frac{c}{k}\frac{dx}{dt} + x$$

Next we examine the term c/k. We need to start with a definition. We will be using a term called the critical damping coefficient and it is defined as follows.

$$c_c = \sqrt{4Mk}$$

This has the same units as the damping coefficient (N s/m). The ration c/c_c is called the damping ratio and this is defined as follows.

$$\delta = \frac{c}{c_c}$$

This may now be developed as follows.

$$\delta = \frac{c}{c_c} = \frac{c}{\sqrt{4Mk}} = \frac{c}{2\sqrt{Mk}} = \frac{c}{2k}\sqrt{\frac{k}{M}} = \frac{c}{2k}\omega_n$$

$$\frac{c}{k} = \frac{2\delta}{\omega_n}$$

Substitute into our equation.

$$0 = \frac{1}{\omega_n^2}\frac{d^2x}{dt^2} + \frac{2\delta}{\omega_n}\frac{dx}{dt} + x$$

Multiply each term by ω_n^2

$$0 = \frac{d^2x}{dt^2} + 2\delta\omega_n\frac{dx}{dt} + \omega_n^2 x$$

This is now a standard equation and the solution may be found in standard text as

$$x = Ae^{at} + Be^{-bt}$$

A and B are constants of integration and

$$a = \{\omega_n(\delta^2 - 1) - 1\} \quad b = \{\omega_n(\delta^2 - 1) + 1\}$$

The resulting graph of displacement x with time t depends upon the damping ratio and hence the value of the damping coefficient c. Three important cases should be considered.

3.1. Under Damped

This occurs when $\delta < 1$ and $c < c_c$. If we assume that t = 0 and x = C at the moment the mass is released we get a decaying cosinusoidal oscillation as shown. The displacement is described by the following equation.

$$x = Ce^{-\delta\omega_n t}\cos(\omega t)$$

The graph shows the result if the mass is pulled down 10 units and released. In this case C = -10.

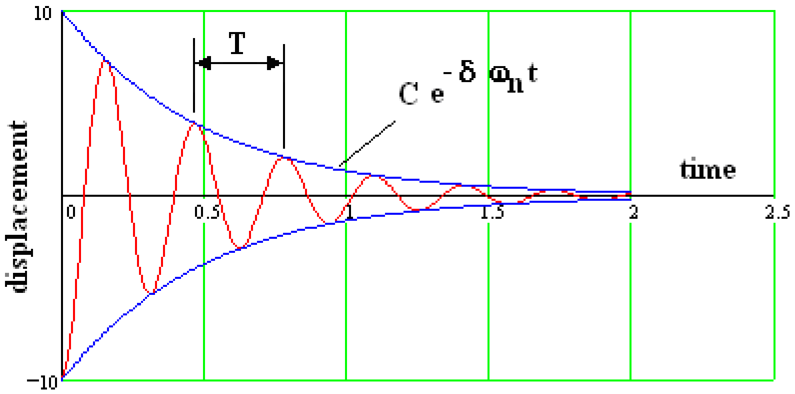

Figure 4

Without proof

$$T = \frac{2\pi}{\omega_n\sqrt{1-\delta^2}} \qquad f = \frac{1}{T} = \frac{\omega_n\sqrt{1-\delta^2}}{2\pi} \qquad \omega = 2\pi f = \omega_n\sqrt{1-\delta^2}$$

Velocity and Acceleration

Starting with displacement velocity and acceleration may be found by successive differentiations using differentiation by parts.

$$x = Ce^{-\delta\omega_n t}\cos(\omega t)$$

Velocity

$$v = \frac{dx}{dt} = Ce^{-\delta\omega_n t}\frac{d\cos(\omega t)}{dt} + \cos(\omega t)\frac{dCe^{-\delta\omega_n t}}{dt}$$

$$v = -Ce^{-\delta\omega_n t}\,\omega\sin(\omega t) + \cos(\omega t)\left(-C\,\delta\omega_n e^{-\delta\omega_n t}\right)$$

$$v = -Ce^{-\delta\omega_n t}\,\omega\sin(\omega t) - C\delta\omega_n e^{-\delta\omega_n t}\cos(\omega t)$$

$$v = -C\left\{\omega e^{-\delta\omega_n t}\ \sin(\omega t) + \delta\omega_n e^{-\delta\omega_n t}\cos(\omega t)\right\}$$

Acceleration

$$a = \frac{dv}{dt} = -C\left\{\left(\omega e^{-\delta\omega_n t}\right)\frac{d\sin(\omega t)}{dt} + \sin(\omega t)\frac{\omega d\left(e^{-\delta\omega_n t}\right)}{dt}\right\}$$

$$-C\left\{\left(\delta\omega_n e^{-\delta\omega_n t}\right)\frac{d\cos(\omega t)}{dt} + \cos(\omega t)\frac{\delta\omega_n d\left(e^{-\delta\omega_n t}\right)}{dt}\right\}$$

$$a = -C\left\{\left(\omega e^{-\delta\omega_n t}\right)\omega\cos(\omega t) + \sin(\omega t)\omega(-\delta\omega_n)\left(e^{-\delta\omega_n t}\right)\right\}$$

$$-C\left\{\left(\delta\omega_n e^{-\delta\omega_n t}\right)\ \left(-\omega\sin(\omega t)\right) + \cos(\omega t)\delta\omega_n(-\delta\omega_n)e^{-\delta\omega_n t}\right\}$$

$$a = -C\left\{\left(\omega^2 e^{-\delta\omega_n t}\right)\cos(\omega t) - \sin(\omega t)\omega\delta\omega_n\left(e^{-\delta\omega_n t}\right)\right\}$$

$$-C\left\{-\left(\omega\delta\omega_n e^{-\delta\omega_n t}\right)\sin(\omega t)\ - \cos(\omega t)\delta^2\omega_n^2 e^{-\delta\omega_n t}\right\}$$

$$a = C\left\{\sin(\omega t)\omega\delta\omega_n\left(e^{-\delta\omega_n t}\right) - \left(\omega^2 e^{-\delta\omega_n t}\right)\cos(\omega t)\right\}$$

$$+C\left\{\left(\omega\delta\omega_n e^{-\delta\omega_n t}\right)\sin(\omega t) + \cos(\omega t)\delta^2\omega_n^2 e^{-\delta\omega_n t}\right\}$$

$$a = C\left[2\omega\omega_n\delta e^{-\delta\omega_n t}\sin(\omega t) + \{\delta^2\omega_n^2 - \omega^2\}e^{-\delta\omega_n t}\cos(\omega t)\ \right]$$

Amplitude Reduction Factor

The amplitude reduction factor governs the rate of decay between successive oscillations.
The natural undamped and damped angular frequencies are

$$\omega_n = \sqrt{\frac{k}{M}} \qquad \omega = \omega_n\sqrt{1-\delta^2}$$

Consider two oscillations, one occurring m cycles after the first. The amplitude of the first oscillation (x_1) at time t and the second one (x_2) at time mT are

$$x_1 = Ce^{-\delta\omega_n t} \text{ and } x_2 = Ce^{-\delta\omega_n(t+mT)}$$

The ratio is

$$\frac{x_1}{x_2} = \frac{Ce^{-\delta\omega_n t}}{Ce^{-\delta\omega_n(t+mT)}} = e^{\delta\omega_n mT} \text{ substitute } T = \frac{2\pi}{\omega} \qquad \delta\omega_n mT = 2\pi\,\delta\,m\frac{\omega_n}{\omega}$$

Substitute

$$\omega = \omega_n\sqrt{1-\delta^2}$$

$$\delta\omega_n mT = \frac{2\pi\,\delta\,m}{\sqrt{1-\delta^2}} \quad \text{hence} \quad \frac{x_1}{x_2} = e^{\frac{2\pi\,\delta\,m}{\sqrt{1-\delta^2}}}$$

Take natural logs

$$\ln\left(\frac{x_1}{x_2}\right) = \frac{2\pi\,\delta\,m}{\sqrt{1-\delta^2}} = \text{amplitude reduction factor}$$

3.2. Critically Damped

If the damping is increased, the oscillations die away quicker and eventually a critical point is reached where the mass just returns to the rest position with no overshoot or oscillation. This occurs when $\delta = 1$ and $c = c_C$. The result is an exponential decay as shown.

3.3. Over Damped

This occurs when $\delta > 1$ and $c > c_C$. The result is an exponential decay with no oscillations but it will take longer to reach the rest position than with critical damping. Figure 5 illustrates examples of these.

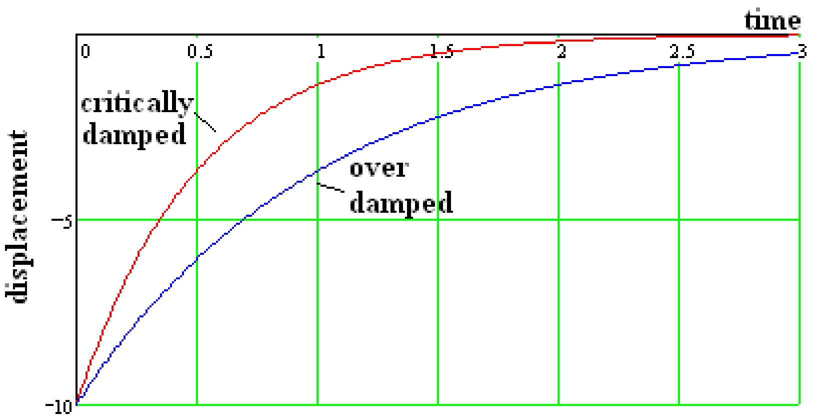

Figure 5

WORKED EXAMPLE No. 1

A vibrating system is analysed and it is found that two successive oscillations have amplitudes of 3 mm and 0.5 mm respectively. Calculate the amplitude reduction factor and the damping ratio.

SOLUTION

For successive amplitudes m = 1 the amplitude reduction factor is:

$$\ln\left(\frac{x_1}{x_2}\right) = \ln\left(\frac{3}{0.5}\right) = \ln 6 = 1.792$$

$$1.792 = \frac{2\pi\,\delta\,m}{\sqrt{1-\delta^2}} \text{ square both sides}$$

$$3.21 = \frac{39.478\,\delta^2}{1-\delta^2} \qquad 1-\delta^2 = 12.298\delta^2$$

$$1 = 13.298\delta^2 \qquad \delta^2 = \frac{1}{13.298} = 0.075 \qquad \delta = \sqrt{0.075} = 0.274$$

WORKED EXAMPLE No. 2

A mass of 5 kg is suspended on a spring and set oscillating. It is observed that the amplitude reduces to 5% of its initial value after 2 oscillations. It takes 0.5 seconds to do them. Calculate the following.

i. The damping ratio.
ii. The natural frequency.
iii. The actual frequency.
iv. The spring stiffness.
v. The critical damping coefficient.
vi. The actual damping coefficient.

SOLUTION

$$\ln\left(\frac{x_1}{x_2}\right) = \ln\left(\frac{100}{5}\right) = \ln 20 = 2.996 = \frac{2\pi\,\delta\,m}{\sqrt{1-\delta^2}} \quad (m = 2)$$

$$\left(\frac{2\pi\,\delta\,\times 2}{\sqrt{1-\delta^2}}\right)^2 = 2.996^2 = 8.976 = 157.9\left(\frac{\delta}{\sqrt{1-\delta^2}}\right)^2$$

$$17.59\,\delta^2 = 1-\delta^2 \qquad 18.59\,\delta^2 = 1 \quad \delta = 0.232$$

$$f = \frac{\text{number of oscillations}}{\text{time taken}} = \frac{2}{0.5} = 4 \text{ Hz}$$

$$f = 4 = f_n\sqrt{1-\delta^2} = f_n\sqrt{1-0.232\times 0.232^2} = 0.9727 f_n$$

$$f_n = \frac{4}{0.9727} = 4.112 \text{ Hz} = \frac{1}{2\pi}\sqrt{\frac{k}{M}} = \frac{1}{2\pi}\sqrt{\frac{k}{5}} \qquad (4.112\times 2\pi)^2 = \frac{k}{5} \qquad k = 3\,338 \text{ N/m}$$

$$c_c = \sqrt{4Mk} = \sqrt{4\times 5\times 3\,338} = 258.4\,\frac{\text{Ns}}{\text{m}}$$

$$c = c_c\delta = 258.4\times 0.232 = 59.94 \text{ Ns/m}$$

WORKED EXAMPLE No. 3

A mass of 5 kg is suspended on a spring of stiffness 4 000 N/m. The system is fitted with a damper with a damping ratio of 0.2. The mass is pulled down 50 mm and released. Calculate the following.

i. The damped frequency.
ii. The displacement, velocity and acceleration after 0.3 seconds.

SOLUTION

You will need to be able to do advanced differentiation in order to follow this solution.

$$\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{4\ 000}{5}} = 28.28\,\frac{\text{rad}}{\text{s}} \qquad f_n = \frac{\omega_n}{2\pi} = 4.5\ \text{Hz}$$

$$f = f_n\sqrt{1-\delta^2} = 4.5\sqrt{1-0.2^2} = 4.41\ \text{Hz (Answer i)}$$

$$\omega = \omega_n\sqrt{1-\delta^2} = 28.28\sqrt{1-0.2^2} = 27.71\ \text{rad/s}$$

Initial displacement C = -50 mm

$$x = Ce^{-\delta\omega_n t}\cos(\omega t) = -50e^{-0.2\times 28.28\times 0.3}\cos(27.71t)$$

$$x = -50 \times 0.183 \times (-0.443) = 4.07\ \text{mm}$$

Velocity

$$v = -C\left\{\omega e^{-\delta\omega_n t}\ \sin(\omega t) + \delta\omega_n e^{-\delta\omega_n t}\cos(\omega t)\right\}$$

$$v = 50\{27.71 \times 0.183\ \sin(27.71 \times 0.3) + 0.2 \times 28.28 \times 0.183\cos(27.71 \times 0.3)\}$$

$$v = 50\{5.071\ \sin(8.313) + 1.035\cos(8.313)\}$$

$$v = 50\{4.546 - 0.4586\} = 204.4\ \text{mm/s}$$

Acceleration

$$a = C\left[2\omega\omega_n\delta e^{-\delta\omega_n t}\sin(\omega t) + \{\delta^2\omega_n^2 - \omega^2\}e^{-\delta\omega_n t}\cos(\omega t)\ \right]$$

$$a = -50[2 \times 27.71 \times 28.28 \times 0.2 \times 0.183\sin(8.313) + \{0.2^2 \times 28.28^2 - 27.71^2\} \times 0.183\cos(8.313)\ ]$$

$$a = -50[57.362\sin(8.313) + \{31.990 - 767.8\} \times 0.183\cos(8.313)\ ]$$

$$a = -50[57.362\sin(8.313) - 134.661 \times \cos(8.313)\ ]$$

$$a = -50[51.424 + 59.664\ ] = -5\ 554\ \text{mm/s}^2$$

SELF ASSESSMENT EXERCISE No. 1

1. A mass of 50 kg is suspended from a spring of stiffness 10 kN/m. It is set oscillating and it is observed that two successive oscillations have amplitudes of 10 mm and 1 mm. Determine the following.

   . The damping ratio (0.188)

   . The damping coefficient (254.5 N s/m)

   . The true frequency (2.21 Hz)

2. A mass of 5 kg is suspended from a spring of stiffness 46 kN/m. A dashpot is fitted between the mass and the support with a damping ratio of 0.3.
   Calculate the following.

   . The undamped frequency (15.26 Hz)

   . The damped frequency (14.56 Hz)

   . The amplitude reduction factor (1.976)

   . The critical damping coefficient (959 N s/m)

3. A mass of 30 kg is supported on a spring of stiffness 60 000 N/m. The system is damped and the damping ratio is 0.4. The mass is raised 5 mm and then released. Calculate the following.

   i. The damped frequency (6.523 Hz)

   ii. The displacement, velocity and acceleration after 0.1 s.
   (-4.812 mm, 36.6 mm/s and -346 mm/s)

4 *Torsional Oscillations*

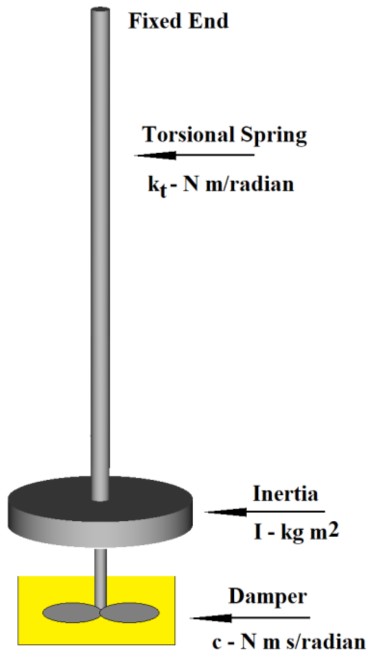

Figure 6

The theory is the same for torsional oscillations. the linear quantities are replaced with angular quantities in the formula. These are

Mass m	2nd moment of inertia I
Force F	Torque T
Distance x	angle θ
Velocity v	angular velocity ω
Acceleration a	angular acceleration α

A dashpot for a torsional system would consist of a vane which rotates inside a pot of oil. The damping torque is directly proportional to the angular velocity such that

Damping Torque = c ω

c is the torsional damping coefficient with units of N m s/radian.

The torsional spring stiffness is torque per unit angle so the units of k_t are Nm/radian.

The critical damping coefficient is

$$c_c = \sqrt{4\,I\,k_t}$$

The natural frequency is

$$\omega_n = \sqrt{\frac{k_t}{I}}$$

Remember that for a shaft the torsional stiffness

$$k_t = \frac{GJ}{L}$$

The polar second moment of area for a circular section is

$$J = \frac{\pi D^4}{32}$$

The actual angular frequency is

$$\omega = \omega_n\sqrt{1-\delta^2}$$

The amplitude reduction factor is

$$\ln\left(\frac{\theta_1}{\theta_2}\right) = \frac{2\pi\,\delta\,m}{\sqrt{1-\delta^2}}$$

θ is the angular displacement.

WORKED EXAMPLE No. 4

A horizontal shaft is fixed at both ends and carries a flywheel at the middle. The shaft is 1 m long either side of the flywheel and is 10 mm diameter. The flywheel has a moment of inertia of 1.9 kg m^2. The system has proportional damping and it is observed that the amplitude reduces by 60% after one oscillation. The shaft material has a modulus of rigidity of 90 GPa.

Calculate the following.

i. The damping ratio.
ii. The natural frequency.
iii. The actual frequency.
iv. The spring stiffness.
v. The critical damping coefficient.
vi. The actual damping coefficient.

SOLUTION

For 1 oscillations m = 1 and θ_1 = 100% θ_2 = 40% θ_1/θ_2= 100/40 = 2.5

$$\ln\left(\frac{\theta_1}{\theta_2}\right) = \ln 2.5 = 0.9163 = \frac{2\pi\,\delta}{\sqrt{1-\delta^2}}$$

Square both sides

$$\frac{(2\pi\,\delta)^2}{1-\delta^2} = 0.9163^2$$

$$\frac{(2\pi\,\delta)^2}{1-\delta^2} = \frac{39.478\,\delta^2}{1-\delta^2} = 0.9163^2 = 0.8396$$

$$47\,\delta^2 = 1-\delta^2$$

$$48\,\delta^2 = 1 \qquad \delta^2 = 0.0208 \quad \delta = 0.1443$$

For two flywheels

$$J = 2\times\frac{\pi D^4}{32} = 2\times\frac{\pi\times 0.01^4}{32} = 0.98\times 10^{-9}\ \text{m}^4$$

$$k_t = \frac{GJ}{L} = \frac{2\times 90\times 10^9\times 0.98\times 10^{-9}}{1} = 1\,944\ \text{N s m/radian}$$

$$f_n = 2\pi\sqrt{\frac{k_t}{I}} = 2\pi\sqrt{\frac{1\,944}{1.9}} = 5.09\ \text{Hz}$$

$$f = f_n\sqrt{1-\delta^2} = 5.09\sqrt{1-0.0208} = 5.04\ \text{Hz}$$

$$c_c = \sqrt{4\,I\,k_t} = \sqrt{4\times 1.9\times 1\,944} = 121.5\ \text{N m s/radian}$$

$$c = c_c\times\delta = 121.5\ \times 0.1443 = 17.54\text{N m s/radian}$$

SELF ASSESSMENT EXERCISE No. 2

1. A flywheel has a moment of inertia of 50 kg m^2. It is suspended on the end of a vertical shaft 2 m long and 40 mm diameter. A torsional damper is fitted with a damping ratio of 0.125. The modulus of rigidity of the shaft is 80 GPa.

   Calculate the following.

   The torsional stiffness of the shaft (10053 N m s/radian)
   The critical damping coefficient for the system (1 418 Nms/rad)
   The frequency of damped oscillations ? (2.24 Hz).
   The amplitude reduction factor (45.6%)

2. A shaft with negligible inertia has a flywheel suspended from the end and a damper to damp the vibrations. The shaft has a torsional stiffness of 5000 N m s/rad. The flywheel has a moment of inertia of 30 kg m^2. The damping ratio is 0.4. The flywheel is rotated 0.02 radian and released. Calculate the following.

   i. The damped frequency (1.833 Hz)
   ii. The angular displacement, velocity and acceleration after 0.5 seconds after being released (0.00141 rad, -8.53×10^{-4} rad/s and 0.226 rad/s^2)