SOLID MECHANICS

DYNAMICS

TUTORIAL – TORSIONAL OSCILLATIONS OF LONG UNIFORM SHAFTS

This SHORT tutorial is specifically for the Engineering Council Exam D225 – Dynamics of Mechanical Systems but can be studied by anyone with a suitable background who wants to study the topic.

On completion of this tutorial you should be able to solve the natural frequency of torsional vibrations for long shafts due to their own mass.

You are advised to study the tutorials on free vibrations before commencing on this.

INTRODUCTION

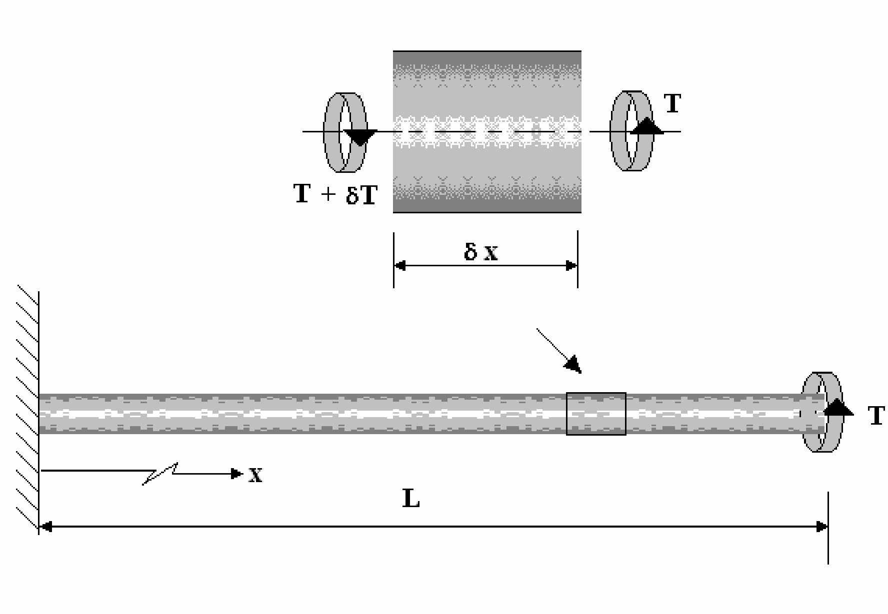
Oscillations can occur in long transmission shafts such as the drill shaft of an oil rig. The theory is similar to that of transverse vibrations and buckling as there can be more than one mode. The derivation uses the wave equation. The work applies only to shafts with a circular cross section.

NOMENCLATURE

θ	Angle of Twist
ρ	Density of material
R	Radius of shaft
L	Length of shaft
A	Cross Section Area
G	Modulus of Rigidity for the material
J	Polar Second Moment of Area $J = \pi R^4/2$
I	Polar Moment of Inertia
α	Angular Acceleration $\alpha = \dfrac{d^2\theta}{dt^2}$
t	Time
ω_n	Natural Angular Frequency
f_n	Natural frequency
n	mode

THEORY

Consider a long shaft fixed a one end and free at the other. Suppose a torque T is applied at the free end.



Consider an element of the shaft length δx. The torque at one end is slightly larger than the torque at the other by δT. Suppose the torque decreases uniformly with x as $\frac{dT}{dx}$

The net torque on the element is $\delta T = \frac{dT}{dx}\delta x$

From the torsion equation we have $T = \frac{GJ\theta}{L}$

For a uniform shaft $\frac{\theta}{L} = \frac{d\theta}{dx}$ $T = GJ\frac{d\theta}{dx}$

Differentiate with respect to x and $\frac{dT}{dx} = GJ\frac{d^2\theta}{dx^2}$

The net torque is now $\delta T = \frac{dT}{dx}\delta x = GJ\frac{d^2\theta}{dx^2}\delta x$

The torque on the element must overcome the inertia of the material only.

$$\delta T = I\alpha = I\frac{d^2\theta}{dt^2} = GJ\frac{d^2\theta}{dx^2}\delta x$$

For a solid circular length of shaft

$$I = \frac{MR^2}{2} = \frac{\rho\, A\, \delta x\, R^2}{2} = \frac{\rho\, \pi\, R^2\, \delta x\, R^2}{2} = \frac{\rho\, \pi\, R\, \delta x\, R^4}{2} = \rho\, \delta x\, J$$

$$\delta T = \rho\, \delta x\, J\frac{d^2\theta}{dt^2} = GJ\frac{d^2\theta}{dx^2}\delta x$$

$$\rho\, \frac{d^2\theta}{dt^2} = G\frac{d^2\theta}{dx^2} \qquad \frac{d^2\theta}{dx^2} = \frac{\rho}{G}\frac{d^2\theta}{dt^2}$$

This is usually expressed as $\frac{d^2\theta}{dx^2} = \frac{1}{c^2}\frac{d^2\theta}{dt^2}$ where c is the velocity of a wave $c = \sqrt{\frac{G}{\rho}}$ m/s

The standard solution for this equation is:

$$\theta = \left[A\sin\left(\frac{\omega x}{c}\right) + B\cos\left(\frac{\omega x}{c}\right)\right]\sin(\omega t)$$ A and B are constants.

Now we put in the boundary conditions for the shaft.
When x = 0, θ = 0 so putting this in the equation

$$0 = [A\sin(0) + B\cos(0)]\sin(\omega t) = [0 + B]\sin(\omega t)$$

It follows that B = 0 and our solution reduces to $\theta = \left[A\sin\left(\frac{\omega x}{c}\right)\right]\sin(\omega t)$

If we differentiate with respect to x $\frac{d\theta}{dx} = \left[\frac{A\omega}{c}\cos\left(\frac{\omega x}{c}\right)\right]\sin(\omega t)$

When x = L, $\frac{d\theta}{dx} = 0$ $\left[\frac{A\omega}{c}\cos\left(\frac{\omega L}{c}\right)\right]\sin(\omega t) = 0$

It follows that $\cos\left(\frac{\omega L}{c}\right) = 0$

This can only occur if $\omega = \left(n - \frac{1}{2}\right)\frac{\pi c}{L}$ where n is an integer 1, 2, 3 ....

This gives the natural frequency of the system.

$$\omega_n = \left(n - \frac{1}{2}\right)\frac{\pi}{L}\sqrt{\frac{G}{\rho}} \qquad f_n = \left(n - \frac{1}{2}\right)\frac{1}{2L}\sqrt{\frac{G}{\rho}}$$

The lowest natural frequency occurs at the fundamental mode n = 1.

WORKED EXAMPLE

A long oil rig drill shaft is modelled as a long uniform shaft fixed at the top and free at the bottom. The shaft is 375 m long and has a material density of 7800 kg/m^3 and Modulus of Rigidity 70 GPa. Determine the fundamental natural frequency.

SOLUTION

$$f_n = \left(n - \frac{1}{2}\right)\frac{1}{2L}\sqrt{\frac{G}{\rho}}$$ Put n = 1

$$f_n = \frac{1}{4L}\sqrt{\frac{G}{\rho}} = \frac{1}{4 \text{ x } 375}\sqrt{\frac{70 \text{ x } 10^9}{7800}} = 2\text{ Hz}$$