SOLID MECHANICS

TUTORIAL – BALANCING OF RECIPROCATING MACHINERY

This work covers elements of the syllabus for the Engineering Council Exam D225 Dynamics of Mechanical Systems.

On completion of this tutorial you should be able to do the following.

- Define a reciprocating machine.
- Derive the equations for the acceleration of the piston in a reciprocating machine.
- Explain primary and secondary force balance.
- Explain primary and secondary moment balance.
- Determine the state of balance of machines with multiple pistons.
- Explain the use of contra-rotating masses to produce primary and secondary balance.
- Solve problems using analytical and graphical methods of solution.

It is assumed that the student has already completed the tutorial on balancing.

1. WHAT IS A RECIPROCATING MACHINE?

Reciprocating machines here means a piston reciprocating in a cylinder and connected to a crank shaft by a connecting rod. You can skip the derivation of the acceleration by going to the next page.

First let's establish the relationship between crank angle, and the displacement, velocity and acceleration of the piston.

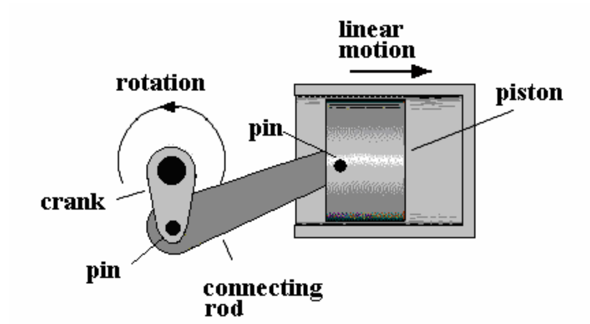

Figure 1

2 DERIVATION OF ACCELERATION EQUATION

A crank, con rod and piston mechanism is shown below. Determine the maximum acceleration of the piston when the crank speed is 30 rev/min clockwise.

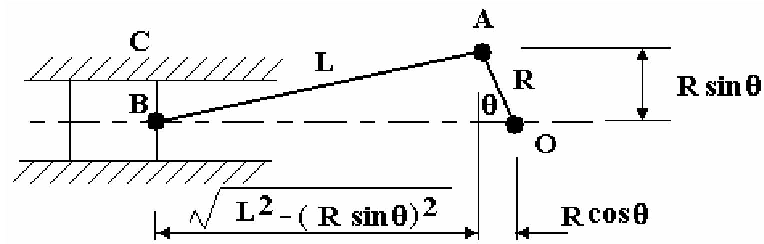

Figure 2

When θ = 0 the piston will be furthest left at a distance of L+R from point O. Take this as the reference point and measure displacement x from there. Remember that θ = ωt and ω = 2π x N. The displacement is then

$$x = (L+R) - R\cos(\theta) - \sqrt{L^2 - (R\sin(\theta))^2} \quad \text{put } n = L/R \quad x = R\left[(n+1) - \cos(\theta) - \sqrt{n^2 - (\sin(\theta))^2}\right]$$

Differentiate to get the velocity.

$$v = \omega R\left[\sin(\theta) + \frac{\sin(\theta)\cos(\theta)}{\left[n^2 - \sin^2(\theta)\right]^{\frac{1}{2}}}\right] \quad = \omega R\left[\sin(\theta) + \frac{\sin(2\theta)}{2\left[n^2 - \sin^2(\theta)\right]^{\frac{1}{2}}}\right]$$

Differentiate again and simplify to get the acceleration.

$$a = \omega^2 R\left[\cos(\theta) + \frac{\sin^2(2\theta)}{4\left(n^2 - \sin^2(\theta)\right)^{\frac{3}{2}}} + \frac{\cos(2\theta)}{\left(n^2 - \sin^2(\theta)\right)^{\frac{1}{2}}}\right]$$

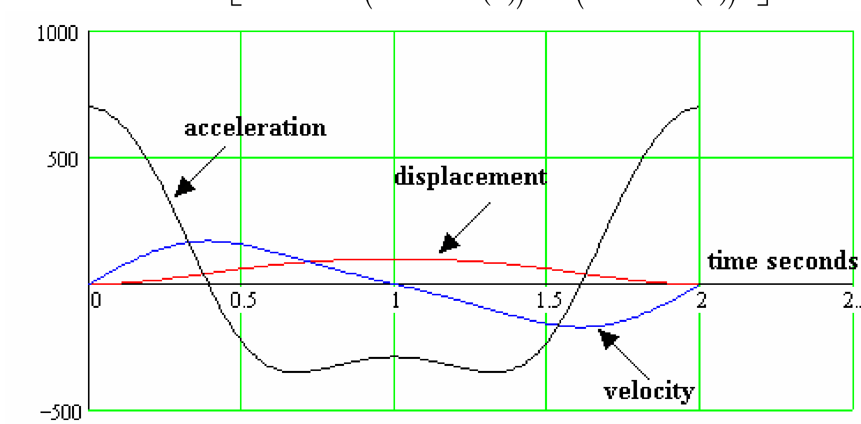

Figure 3

The diagram shows a plot of displacement, velocity and acceleration against angle when L=120 mm, R=50 mm and ω = π rad/s. It should be noted that none of them are sinusoidal and not harmonic (in particular, the acceleration). The larger the value of n, the nearer it becomes to being harmonic.

The equation for acceleration may be expanded as a Fourier series into the form:

$$a = \omega^2 R[\cos(\theta) + A_2 \mathrm{Cos}(2\theta) + A_4 \mathrm{Cos}(4\theta) + A_6 \mathrm{Cos}(6\theta) + \ldots]$$

A is a constant involving n. Except at very high speeds, the following gives a very good approximation.

$$a = \omega^2 R\left[\cos(\theta) + \frac{\cos(2\theta)}{n}\right]$$

3. FORCE

Using the close approximation for acceleration, the inertia force required to accelerate the piston is given by $F = M\omega^2 R\left[\cos(\theta) + \frac{\cos(2\theta)}{n}\right]$

This may be thought of as two separate forces.

$F_p = M\omega^2 R\cos(\theta)$ is called the primary force and

$F_s = M\omega^2 R\left[\frac{\cos(2\theta)}{n}\right]$ is called the secondary force.

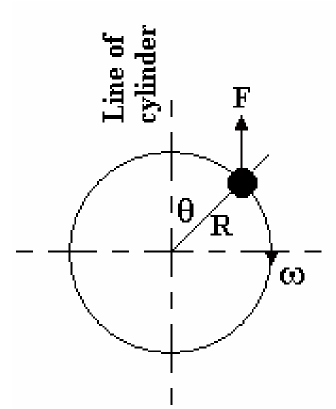

Figure 4

PRIMARY FORCE FOR A SINGLE CYLINDER

The primary force $F_p = M\omega^2 R\cos(\theta)$ must be thought of as a force with peak value $M\omega^2 R$ that varies cosinusoidally with angle θ.

WORKED EXAMPLE No.1

Determine the primary out of balance force for a single cylinder machine with a piston of mass 0.5 kg, with a connecting rod 120 mm long and a crank radius of 50 mm when the speed of rotation is 3000 rev/min.

SOLUTION

$\omega = 2\pi \times 3000/60 = 100\pi$ rad/s

$F_p = M\omega^2 R \cos\theta = 0.5 \times (100\pi)^2 \times 0.05 \cos\theta = 2467.4 \cos\theta$ N

SECONDARY FORCE FOR A SINGLE CYLINDER

The secondary force $F_s = M\omega^2 R \frac{\cos(2\theta)}{n}$ must be thought of as a force with peak value $M\omega^2 R/n$ that varies cosinusoidally with the double angle 2θ.

WORKED EXAMPLE No. 2

Determine the secondary force produced in a single cylinder machine with a piston of mass 0.5 kg, with a connecting rod 120 mm long and a crank radius of 50 mm when the speed of rotation is 3000 rev/min.

SOLUTION

$\omega = 2\pi \times 3000/60 = 100\pi$ rad/s $n = 120/50 = 2.4$

$F_s = M\omega^{2(}R/n) \cos\theta = 0.5 \times (100\pi)^2 \times (0.05/2.4) \cos 2\theta = 1028 \cos 2\theta$ N

PRIMARY FORCE FOR MULTIPLE CYLINDERS

The primary inertia forces produced by reciprocating pistons may be thought of as simply the inline component of the centrifugal force produced by a rotating mass. Each piston may be represented by a rotating mass at the crank radius and angle. This leads us to using the same graphical techniques as for rotating masses except that we must remember it is not the centrifugal force but the component in line with the piston motion that we are finding.

Problems are easier to solve when the radii and masses of all the pistons are the same but the graphical method can be used quite easily with any combination.

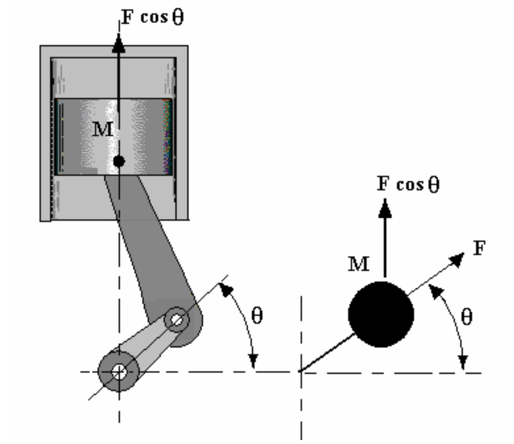

Figure 5

WORKED EXAMPLE No.3a

Two reciprocating pistons have equal mass and crank radii and are placed 180° apart. Determine the primary force.

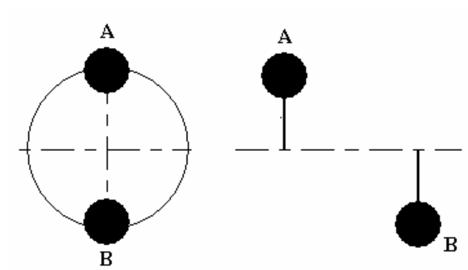

Figure 6

SOLUTION

The force for each piston is $F_p = M\omega^2 R\cos(\theta)$ so what ever the angle of the crank, the vertical components of the forces will be equal and opposite so $F_p = 0$.

WORKED EXAMPLE No.3b

Three reciprocating pistons have equal mass and crank radii and are placed 120° apart from each other. Determine the primary force.

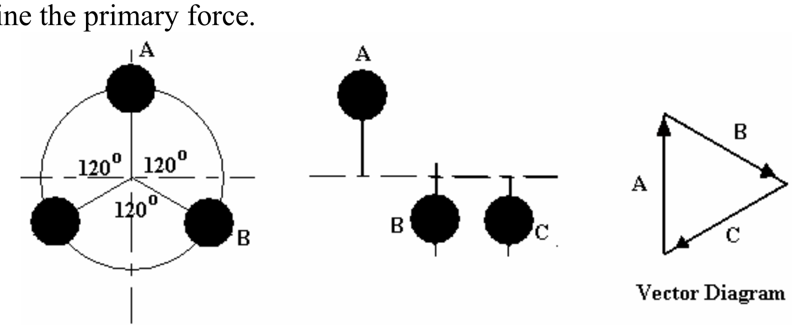

Figure 7

SOLUTION

The force for each piston is $F_p = M\omega^2 R\cos(\theta)$. In order to draw the vectors choose that A is at zero degrees. Each vector has a value $M\omega^2 R$ and adding them we see there is no resultant so there is no resultant vertical component either and so $F_p = 0$ and this will be true what ever the crank angle.

WORKED EXAMPLE No.3c

Four reciprocating pistons have equal mass and crank radii and are placed 90^o apart from each other. Determine the primary force.

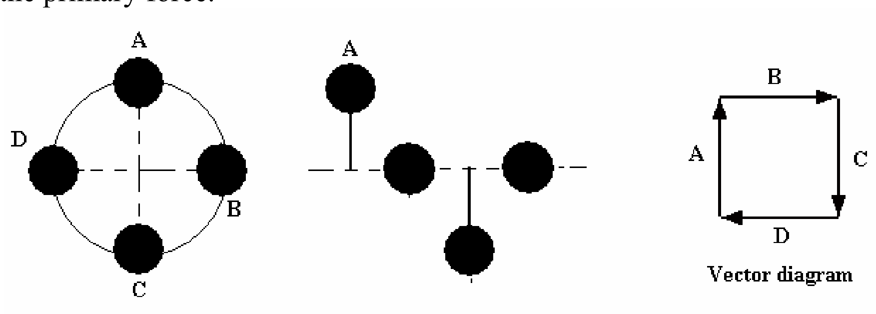

Figure 8

SOLUTION

The force for each piston is $F_p = M\omega^2 R\cos(\theta)$. In order to draw the vectors choose that A is at zero degrees. Each vector has a value $M\omega^2 R$ and adding them we see there is no resultant so there is no resultant vertical component either and so $F_p = 0$ and this will be true what ever the crank angle.

SECONDARY FORCE FOR MULTIPLE CYLINDERS

The secondary force for each piston is given by $F = M\omega^2 \frac{R}{n}[\cos(2\theta)]$. The question is how to represent this as a rotating mass. The area is a bit confusing and the following is my theory.

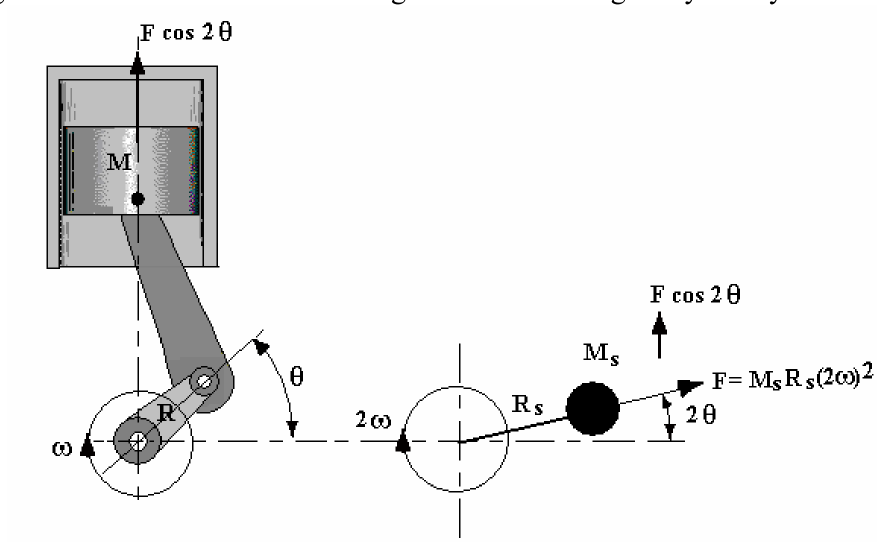

Figure 9

If the crank rotates angle θ, the mass representing the secondary component has rotated 2θ so it needs to rotate at twice the crank speed. The centrifugal force of the mass will be $M_s(2\omega)^2 R_s$.
The suffix s refers to the secondary quantities. If this is to represent the piston then we equate.

$M\,\omega^2(R/n)\cos 2\theta = M_s(2\omega)^2 R_s \cos 2\theta = 4 M_s \omega^2 R \cos 2\theta$

The rotating mass will be $M_s = M\,(R/4R_s n)$ and if the radii are the same $\mathbf{M_s = M/4n}$ This is only needed if we are working out the balancing masses.

WORKED EXAMPLE No.4

Establish the secondary force for the same cases as example 3.

SOLUTION

2 PISTONS. The angle between the two cranks is 180^o so doubling we get 360^o. The vector A may be drawn at any angle 2θ (normally vertically up). Vector B is drawn at 360^o to vector A and added. Each vector has a length $M\omega^2R/n$

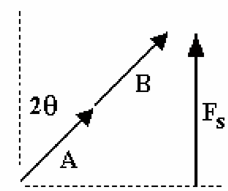

Figure 109

The resulting vertical component is $F_s = 2\ M\omega^2(R/n) \cos 2\theta$ so there is a resultant force that needs to be balanced.

3 PISTONS. The angle between each cranks is 120^o so doubling, vector B will be at 240^o and vector C will be at 480^o all relative to A. (A is drawn vertical for convenience). Adding them we see there is no resultant what ever the angle of vector A so $F_s = 0$ at all angles and the secondary force is balanced.

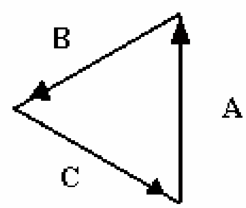

Figure 11

4 PISTONS. The angle between each cranks is 90^o so doubling, vector B will be at 180^o , vector C will be at 360^o and vector D will be 540^o all relative to A. Adding them we see there is no resultant force what ever the angle of vector A so $F_s = 0$ at all angles and the secondary forces are balanced.

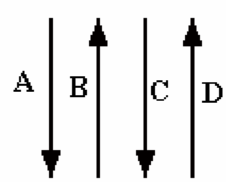

Figure 12

WORKED EXAMPLE No.5

A machine has three reciprocating masses A, B and C with cranks located as shown in the diagram. Determine the primary and secondary forces produced at 600 rev/min.

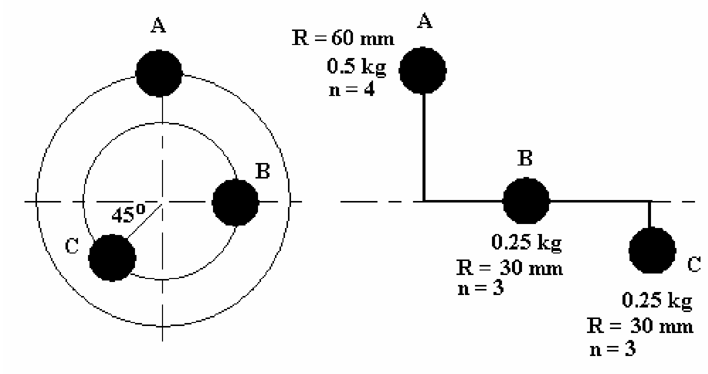

Figure 13

SOLUTION

$\omega = 600 \times 2\pi/60 = 20\pi$ rad/s

PRIMARY FORCE $F_p = M\omega^2 R \cos\theta$ The solution is best done with a table. We only need to draw the MR vectors as the relationship is the same for all values of ω.

	Mass (kg)	R (mm)	MR (kg mm)	θ (deg)
A	0.5	60	30	0
B	0.25	30	7.5	90
C	0.25	30	7.5	225

Draw the MR polygon with A drawn at 0 degrees.

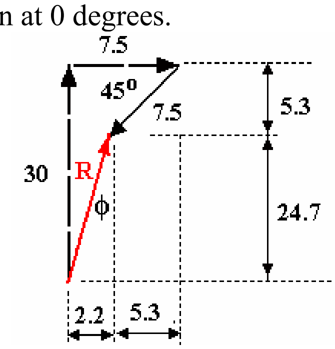

Figure 14

The resultant vector is $R = \sqrt{(2.2^2 + 24.7^2)} = 24.8$ kg mm and $\phi = \tan^{-1}(2.2/24.7) = 5.1^\circ$

The resultant force in line with the pistons is $24.7 \times 10^{-3} \times \omega^2 = 97.5$ N.

For any other angle of crank A, the force is $97.5 \cos\theta$

SECONDARY FORCE $F_p = M\omega^2(R/n)\cos 2\theta$ (or $m_s\omega^2 \cos 2\theta$)

	Mass (kg)	R (mm)	n	MR/n (kg mm)	2θ (deg)
A	0.5	60	4	7.5	0
B	0.25	30	3	2.5	180
C	0.25	30	3	2.5	450

Draw the MR polygon with A drawn at 0 degrees.

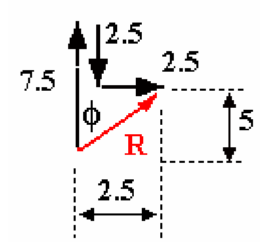

Figure 15

The resulting value of MR/n is R = 5.59 kg mm
The force acting in line with the cylinders is 5 x 10^{-3} x ω^2 = 19.74 N
At any other position of crank A the force is 19.74 cos 2θ
The resulting force is φ degrees clockwise of A. φ = 45°.

4. **MOMENTS**

Each force produces a moment about any point distance x from the centre line of the cylinder along the axis of the crank shaft. Consider the crank below. The distance from the reference plane to the centreline of each crank is x_1, x_2 and so on.

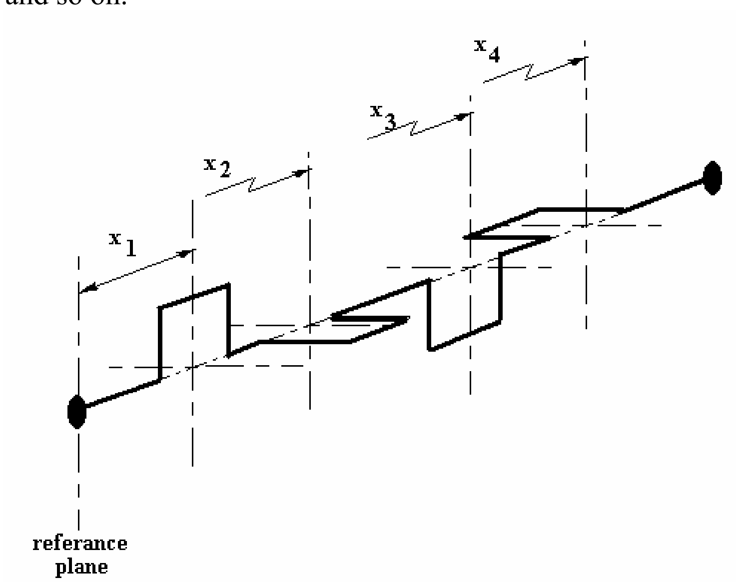

Figure 16

The turning moment about the reference plane is:

$$TM = M\omega^2 R\left[x_1\left\{\cos(\theta)+\frac{\cos(2\theta)}{n}\right\}+x_2\left\{\cos(\theta+\alpha)+\frac{\cos 2(\theta+\alpha)}{n}\right\}+x_3\left\{\cos(\theta+\beta)+\frac{\cos 2(\theta+\beta)}{n}\right\}+\ldots\right]$$

α, β, γ …. are the angles each crank has relative to crank A. This can be separated into primary and secondary moments.

PRIMARY MOMENT

$$TM = M\omega^2 R[x_1\cos(\theta)+x_2\cos(\theta+\alpha)+x_3\cos(\theta+\beta)+\ldots]$$

SECONDARY MOMENT

$$TM = M\omega^2 \frac{R}{n}[x_1\cos(2\theta)+x_2\cos 2(\theta+\alpha)+x_3\cos 2(\theta+\beta)+\ldots]$$

Both may solved with vectors but this time it is MRx and MRx/n that we plot and evaluate.

WORKED EXAMPLE No.6

A machine has three reciprocating masses A, B and C with cranks located as shown in the diagram. Determine the primary and secondary moments produced at 600 rev/min about plane X – X.

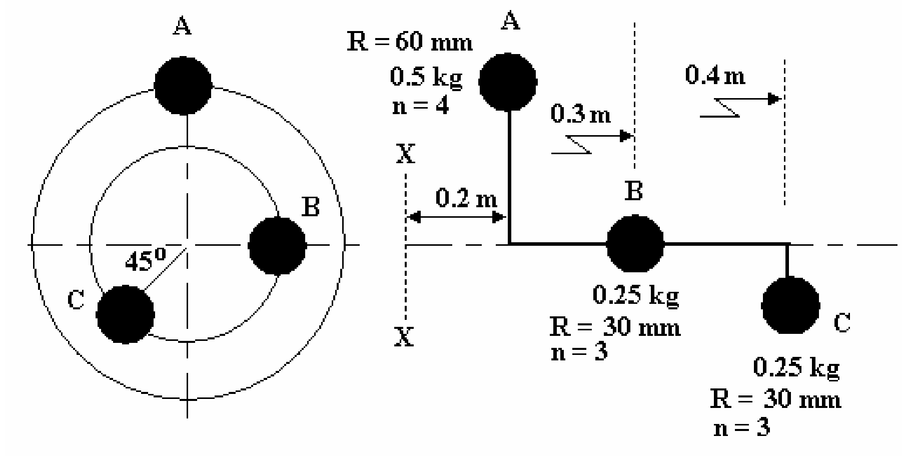

Figure 17

SOLUTION – PRIMARY MOMENTS

$\omega = 600 \times 2\pi/60 = 20\pi$ rad/s

	Mass (kg)	R (m)	x (m)	MRx (kg m^2)	θ (deg)
A	0.5	0.06	0.2	6×10^{-3}	0
B	0.25	0.03	0.3	2.25×10^{-3}	90
C	0.25	0.03	0.4	3×10^{-3}	225

Draw the MRx polygon

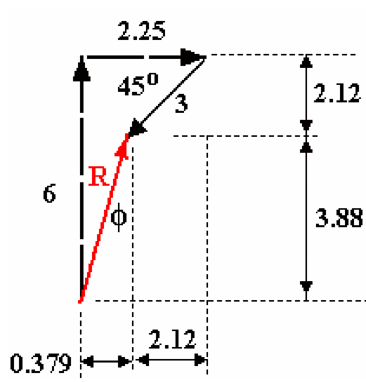

Figure 18

The resultant MRx vector is 3.9×10^{-3} kg m^2 and $\phi = 5.5^{\circ}$

The moment produced in plane XX is $3.88 \times 10^{-3} \times \omega^2 = 15.32$ Nm

At any other position of the crank it is $15.32 \cos \theta$

SOLUTION – SECONDARY MOMENTS

	Mass (kg)	R (m)	n	x (m)	MRx/n (kg m^2)	2θ (deg)
A	0.5	0.06	4	0.2	1.5×10^{-3}	0
B	0.25	0.03	3	0.3	0.1×10^{-3}	180
C	0.25	0.033	3	0.4	0.133×10^{-3}	450

Draw the MRx/n polygon with A drawn at 0 degrees.

The resulting MRx/n vector is 1.406 kg m^2

The angle $\phi = 5.5^o$

The moment about XX in line with the cylinders is
$1.4 \times 10^{-3} \times \omega^2 = 5.53$ Nm

At any other position of the crank it is 5.53 cos 2θ

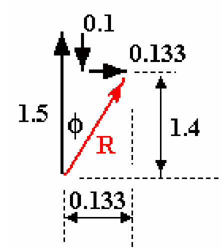

Figure 19

WORKED EXAMPLE No.7

A compressor has three inline pistons of mass 0.4 kg with a crank radius of 40 mm and ratio n of 3. The cranks are equally spaced in angle and positioned as shown. Determine the primary and secondary force and turning moment about the reference plane X when it revolves at 30 rad/s.

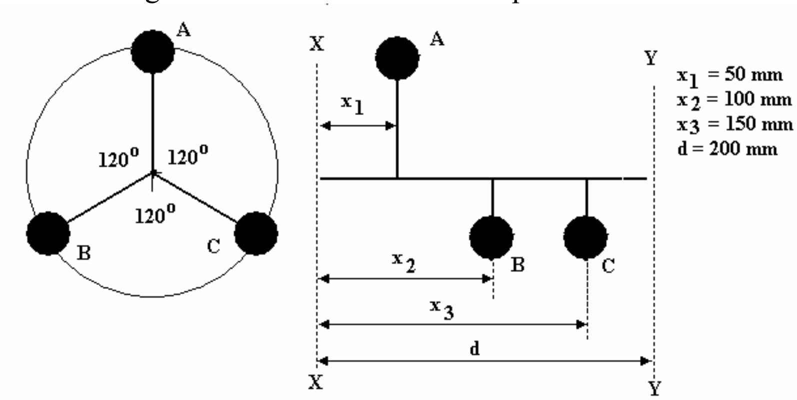

Figure 20

SOLUTION - PRIMARY

	Mass (kg)	Radius (m)	MR (kg m)	x (m)	MRx (kg m^2)	θ (degrees)
Piston A	0.4	0.04	16×10^{-3}	0.050	800×10^{-6}	0
Piston B	0.4	0.04	16×10^{-3}	0.1	1600×10^{-6}	120
Piston C	0.4	0.04	16×10^{-3}	0.15	2400×10^{-6}	240

Drawing the MR polygon with θ = 0 (first is straight up) we see the resultant force is zero as expected.

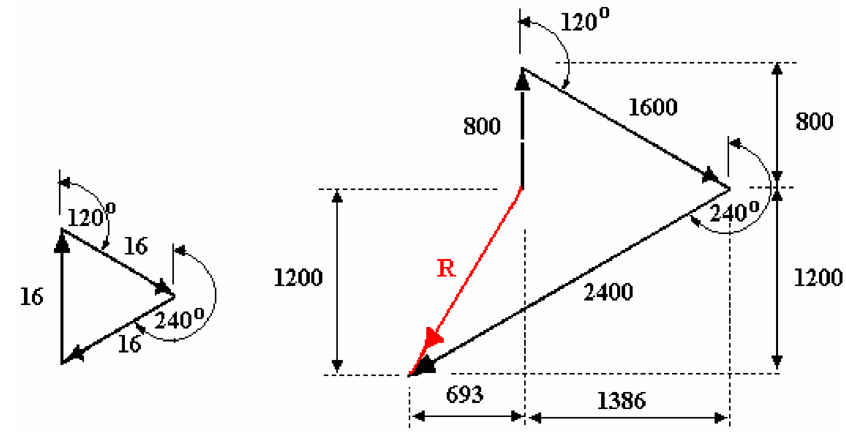

Figure 21

The MRx polygon and a bit of trigonometry shows the resultant is
$R = \sqrt{(1200 \times 10^{-6})^2 + (693 \times 10^{-6})^2} = 1386 \times 10^{-6}$ kg m^2

The resultant makes an angle of $\tan^{-1}(693/1200) = 30^{o}$ with the vertical. The component of the moment in line with cylinder is $\omega^2 \times 1200 \times 10^{-6} = 1.08$ Nm (down)
At any other crank angle it is 1.08 cos θ

SOLUTION - SECONDARY

	MR (kg m)	n	MR/n (kg m)	x (m)	MRx/n (kg m^2)	2θ (degrees)
A	16×10^{-3}	3	5.33×10^{-3}	0.05	266.7×10^{-6}	0
B	16×10^{-3}	3	5.33×10^{-3}	0.1	533.3×10^{-6}	240
C	16×10^{-3}	3	5.33×10^{-3}	0.15	800×10^{-6}	480

Draw the MR/n polygon with double angles and we again get a closed triangle showing that the secondary forces are balanced. Draw the MRx/n polygon with double angles and the resultant vector is $R = 462 \times 10^{-6}$ kg m^2

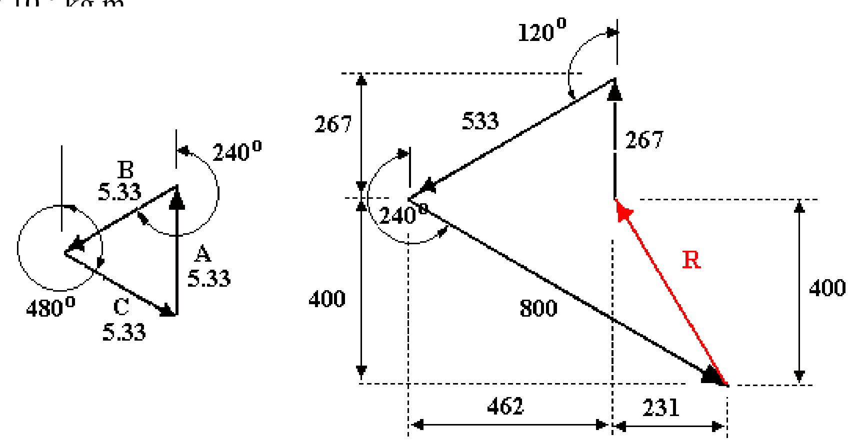

Figure 22

The component in the line of the cylinders is 400×10^{-6} kg m^2

The moment produced is $1200/3 \times 10^{-6} \times 30^2 = 0.36$ Nm about the reference plane (XX).

The resultant is 30^{o} anticlockwise of A.

5. BALANCING

5.1 RECIPROCATING BALANCE

We know from the first balancing tutorial that in order to balance rotors we need to place balancing masses on two planes. Reciprocating machines can be balanced by placing two reciprocating masses on two planes. To balance primary components these would rotate at the crank speed. To balance secondary components, they would have to rotate at double the crank speed in order to produce double angles in a given period of time.

The method so far used is easily adapted to solve the balance. We produce the MRx or MRx/n polygons and deduce the balancing component in the reference plane. Adding this component we then draw the MR polygon to deduce the balancing component in the second reference plane.

WORKED EXAMPLE No.8

Two lines of reciprocating masses at A and B are to be balanced for **PRIMARY** forces and couples by two lines of reciprocating pistons at C and D. Given $M_A = 0.5$ kg and $M_B = 0.75$ kg and that crank B is rotated 70° relative to A, determine the masses M_C and M_D and the angle of their cranks. All crank radii are the same.

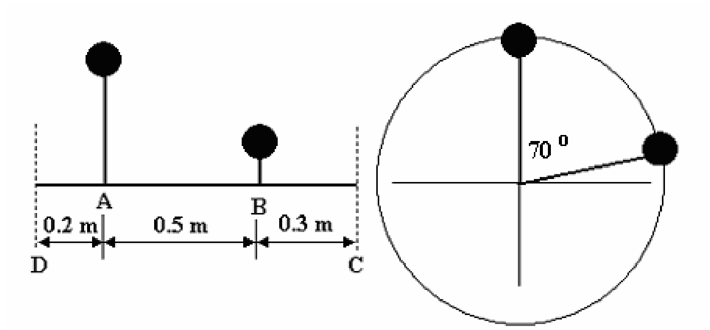

Figure 23

Make D the reference plane.

Mass	M	R	MR	x	MRx
A	0.5	R	0.5R	0.2	0.1R
B	0.75	R	0.75R	0.7	0.525R
C	M_C	R	M_C R	1.0	M_C R
D	M_D	R	M_D R	0	0

Draw the MRx polygon and with a bit of trigonometry M = 0.567R and $\phi = 29.6^\circ$
The mass will be 0.567 kg placed on crank C at 240.4° to crank A.

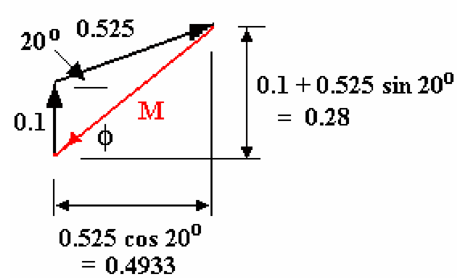

Figure 24

Now draw the MR polygon with M_C R= 0.567 x 1 so M_C = 0.567

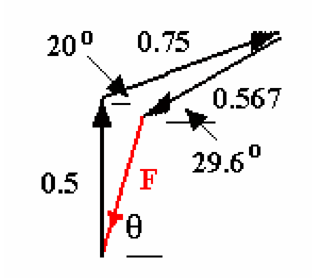

Figure 25

A bit more trigonometry or scaling from the diagram reveals that F = 0.521R so M_D = 0.521 kg and it must be placed at 203.9^o to crank A.

WORKED EXAMPLE No.9

The system described in example 7 is to be balanced by placing a reciprocating mass in planes X and Y with the same crank radius and ratio n. Determine the masses and angles of the cranks for primary and secondary balance.

SOLUTION PRIMARY BALANCING

	Mass kg	Radius m	MR kg m	n	x m	MRx kg m^2	θ
X	M_x	0.04	0.04 M_x	3	0	0	
A	0.4	0.04	16 x 10^{-3}	3	0.05	800 x 10^{-6}	0
B	0.4	0.04	16 x 10^{-3}	3	0.1	1600 x 10^{-6}	120
C	0.4	0.04	16 x 10^{-3}	3	0.15	2400 x 10^{-6}	240
Y	M_y	0.04	0.04 M_y	3	0.2	8 M_y x 10^{-3}	

Draw the MRx polygon. The closing vector is 1386 x 10^{-6} kg m^2 at 30^o as shown.
1386 x 10^{-6} = 8 x $10^{-3}M_y$ M_y = 0.173 kg

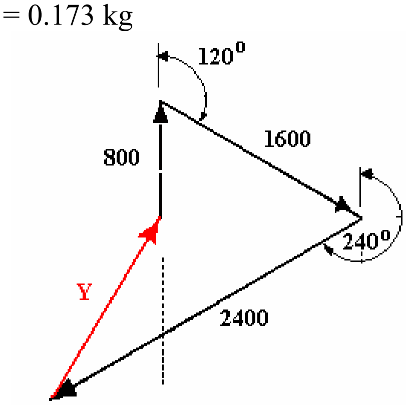

Figure 26

Evaluate 0.04My = 6.93 x 10^{-3} and draw the MR polygon.
The closing vector is 6.93 x 10^{-3} at 30^o as shown.
40 Mx = 6.93   Mx = 0.173 kg

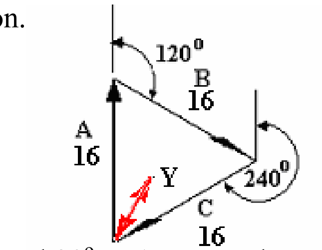

Figure 27

For primary balance we need a piston mass of 173 g placed 30^o to A at Y and another at 210^o to A at X.

SECONDARY COMPONENTS

Mass kg		Radius m	n	MR/n kg m	x m	MRx/n kg m^2	2θ
X	Mx	0.04	3	13.33x10^{-3}Mx	0	0	0
A	0.4	0.04	3	5.33x10^{-3}	0.05	266.7 x 10^{-6}	0
B	0.4	0.04	3	5.33x10^{-3}	0.1	533.3 x 10^{-6}	240
C	0.4	0.04	3	5.33x10^{-3}	0.15	800 x 10^{-6}	480
Y	My	0.04	3	13.33x10^{-3}My	0.2	2.66 x 10^{-3}My	

Draw the MRx/n polygon

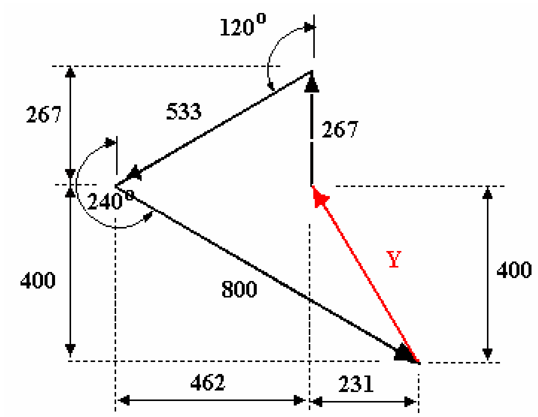

Figure 28

From the MRx/n polygon we get a closing vector representing MRx/n for Y of length 462 x 10^{-6} kg m^2 at 30^o as shown.
Equate 462 x 10^{-6} = 2.66 M_y x 10^{-3} M_y = 0.173 kg
Now evaluate 13.33 x $10^{-3}M_y$ = 2.306 x 10^{-3} and draw the MR/n polygon at double angles.

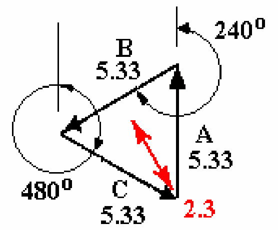

Figure 29

The closing vector is equal and opposite to Y so 13.33 x $10^{-3}M_x$ = 2.3 x 10^{-3} M_x = 0.173 kg.

Earlier it was argued that since the crank will revolve at double speed the mass to be used is M/4 so secondary balance will be produced by a piston of mass 43.3 g. The crank will have an angle of 30^o anticlockwise of A on Y and 43.3 g at 150^o clockwise to A on X.
These cranks to revolve at double speed.

5.2 CONTRA-ROTATING MASSES

A better method is to use equal contra-rotating masses. With these, the centrifugal force produces two components, one in line with the cylinder and one normal to it.

The centrifugal force produced by each is $(M/2)\omega^2R$ and the resolving horizontally and vertically we see the horizontal components cancel and the vertical components add up to $M\omega^2R\cos\theta$ and so cancel the force produced by the piston. The mass and radius can be changes so long as the total product of (MR)/2 is the same.

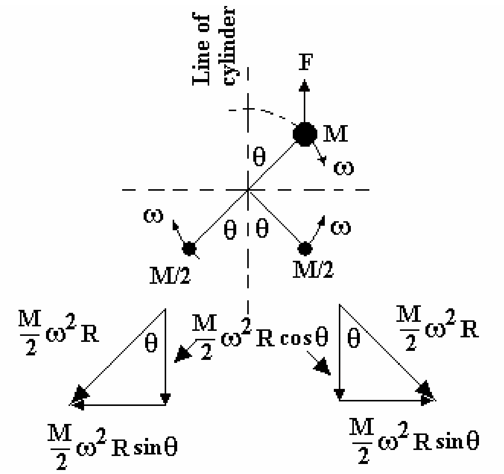

Figure 30

For the balance of primary components, the contra-rotating masses revolve at the crank speed.

For secondary components the contra-rotating cranks must rotate at twice the crank speed (2ω) in order to satisfy the double angle requirement. It was argued earlier that the secondary mass is hence M/4 so the masses on contra rotating wheels must be M/8.

If we balanced the compressor in example 9 in this way , the mass on X and Y would be 173/2 = 86.5 g for primary balance and 21.6 g for secondary balance.

WORKED EXAMPLE No.10

An air compressor has four cylinders in line with cranks as shown. The piston in each cylinder has a mass m of 400 g and each crank is 30 mm radius. The length L of the connecting-rod for each piston is 100 mm. The crankshaft is held in stiff bearings at ends A and B and rotates at Ω rad/s.

In order to balance the primary and secondary components, two pairs of contra-rotating discs, to which balance weights can be attached, are fitted close to plane A and a further two pairs of contra-rotating wheels are fitted close to plane B. One pair at each end rotates at crankshaft speed, Ω and the second pair at each end rotates at 2Ω. Determine the imbalance masses to be added given the radius is 30 mm. You may neglect the small distances between the discs and the bearings.

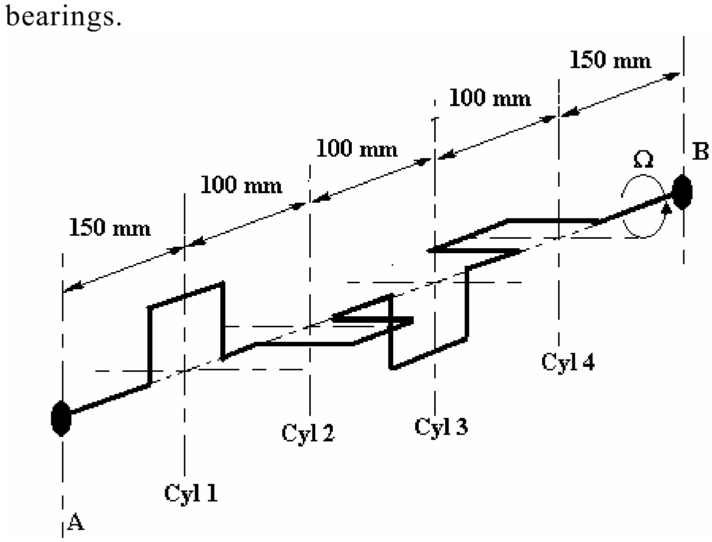

Figure 31

You may assume that the vertical acceleration of the pistons is given by $a = \Omega^2 R\left[\cos(\theta) + \frac{\cos(2\theta)}{n}\right]$

Where θ is the crankshaft angle and n = R/L

SOLUTION

The mass of the piston is M kg so the inertia force F produced is $F = M\Omega^2 R\left[\cos(\theta) + \frac{\cos(2\theta)}{n}\right]$

$M\omega^2 R$ is also the centrifugal force produced by a mass M rotating at radius R and angular velocity Ω. $\Omega = 2\pi \times 500 = 100\,\pi$ rad/s R = 30 mm L = 100 mm n = 100/30=3.333 $\Omega = 100\pi$ rad/s

Both the primary and secondary forces are balanced as the value of MR is the same for each and the resultant is zero in both cases.

The primary turning moment about any reference plane is $M\Omega^2 Rx \cos\theta$ where x is the distance from the reference plane. Taking the reference plane as plane A the table is:

Cylinder	M (kg)	R x 10^3 (m)	MR x 10^3 (kg m)	x (m)	MRx x 10^3 (kg m^2)	Angle (deg)
A	M_A	30	30 M_A	0	0	
1	0.4	30	12	0.15	1.8	0
2	0.4	30	12	0.25	3.0	90
3	0.4	30	12	0.35	4.2	180
4	0.4	30	12	0.45	5.4	270
B	M_B	30	30 M_B	0.60	18 M_B	

Draw the MRx polygon

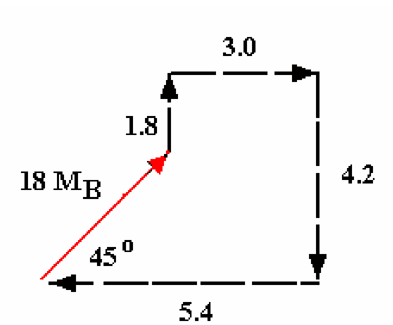

Figure 32

The resultant MRx on plane B is $\sqrt{\{(2.4 \times 10^{-3})^2 + (2.4^2 \times 10^{-3})\}} = 3.394 \times 10^{-3}$ kg m^2

$\phi = 45^o$

$18M_B \times 10^{-3} = 3.394 \times 10^{-3}$ $M_B = 188.6 \times 10^{-3}$ kg

This would be halved to 94.3 g and placed at 45^o either side of crank 1.

We can evaluate 0.03 M_B = 5.657 kg m

Draw the MR polygon

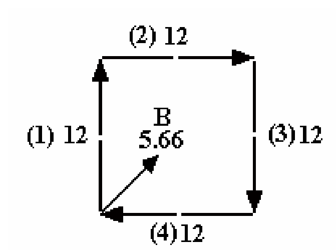

Figure 33

The closing vector is equal and opposite to B and this is the vector for plane A.

$5.66 \times 10^{-3} = 30 \times 10^{-3}\, M_A$ $M_A = 0.188$ kg

This would be halved and placed at 45^o either side of crank 1.

The secondary turning moment about any reference plane is $TM = M\Omega^2 Rx \frac{\cos(2\theta)}{n}$

Taking the reference plane as plane A the turning moment for each cylinder (n = 3.333)

Cylinder	M	R x 10^3	MR/n x 10^3	x	MRx/n x 10^3	Angle
	(kg)	(m)	(kg m)	(m)	(kg m^2)	(deg)
A	M_A	30	9 M_A	0		
1	0.4	30	3.6	0.15	0.54	0
2	0.4	30	3.6	0.25	0.9	180
3	0.4	30	3.6	0.35	1.26	360
4	0.4	30	3.6	0.45	1.62	540
B	M_B	30	9 M_B	0.60	5.4 M_B	

Draw the MRx/n polygon. All the vectors are vertical. The closing vector is hence 0.72 x 10^{-3} kg m^2 vertically up.

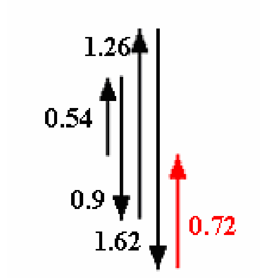

Figure 34

5.4 x $10^{-3}M_B$ = 0.72 x 10^{-3} hence M_B = 0.72/5.4 = 0.1333 kg. The contra rotating mass on B will be 133/8 = 16.6 g and would be placed on the contra-rotating discs at 180° to crank 1.

9 M_B = 1.2 kg m. Now draw the MR/n polygon

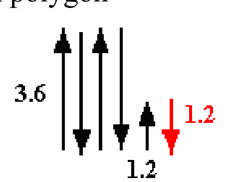

Figure 35

The closing vector is 1.2 x 10^{-3} = 9 M_A x 10^{-3} M_A = 0.133 kg

The contra-rotating mass will be 133/8 = 16.6 g and would be placed on the contra-rotating discs at 100° to crank 1.

SELF ASSESSMENT EXERCISE No.1

1. Two inline reciprocating masses at A and B are to be balanced for primary forces and couples by two reciprocating pistons at C and D in the same line. A is 100 mm from D, B is 150 mm from D and C is 250 mm from D. Given $M_A = 0.25$ kg and $M_B = 0.45$ kg and that crank B is rotated 120^o relative to A, determine the masses M_C and M_D and the angle of their cranks. All crank radii are the same. Outline the procedure to balance the secondary forces and couples.
(0.236 kg at 81.5^o anticlockwise of A for C and 0.167 kg at 69^o clockwise of A)

2. A compressor has three inline pistons A, B and C positioned as shown with crank radii of 80 mm and connecting rods 240 mm long. The compressor is to be balanced for primary and secondary components by placing two sets of contra rotating masses at 50 mm radius at each bearing, one running at the crank speed for the primary balance and one at double the speed for secondary balance. Determine the masses and angles relative to crank A.

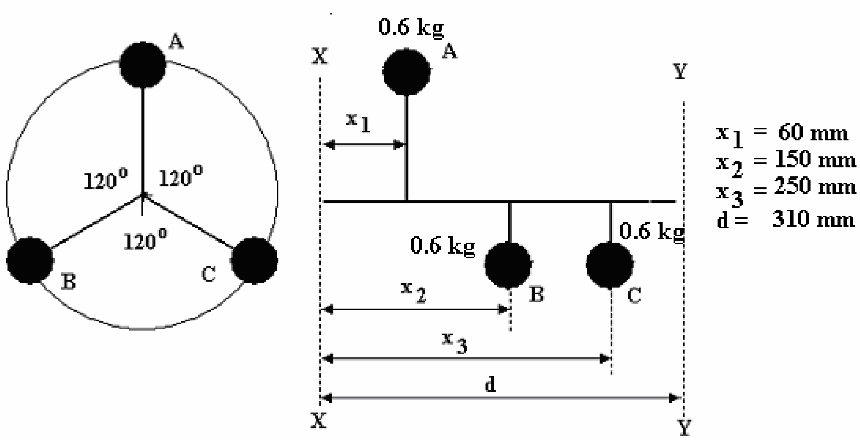

Figure 36

(For primary 255 g on Y at 31.7^o either side of A and 255 g at X at 211.7^o and 148.3^o. For secondary 63.7 g on X and Y at the same angles)

3. An engine has four cylinders in line with cranks equally spaced in order from 1 to 4. The piston in each cylinder has a mass m of 500 g and each crank is 40 mm radius. The length L of the connecting-rod for each piston is 120 mm. The crankshaft is held in stiff bearings at ends A and B and rotates at Ω rad/s. The bearings are 250 mm apart and the cranks are equally spaced at 50 mm intervals with a 50 mm space between the end cranks and the bearings.
In order to balance the primary and secondary components, two pairs of contra-rotating discs, to which balance weights can be attached, are fitted close to plane A and a further two pairs of contra-rotating wheels are fitted close to plane B. One pair at each end rotates at crankshaft speed, Ω and the second pair at each end rotates at 2Ω. Determine the imbalance masses to be added given the radius is 40 mm. You may neglect the small distances between the discs and the bearings.
(141.5 g 45^o either side of crank 1 and 25 g at 180^o to crank 1)

6. AN ANALYTICAL APPROACH

The equations for force and moments developed earlier for multiple masses were

$$F_p = M\omega^2 R[\cos\theta + \cos(\theta+\alpha) + \cos(\theta+\beta) + ...]$$

$$F_s = M\omega^2 \frac{R}{n}[\cos(2\theta) + \cos 2(\theta+\alpha) + \cos 2(\theta+\beta) + ...]$$ secondary force

$$TM_p = M\omega^2 R[x_1\cos(\theta) + x_2\cos(\theta+\alpha) + x_3\cos(\theta+\beta) + ...]$$ primary moment

$$TM_s = M\omega^2 \frac{R}{n}[x_1\cos(2\theta) + x_2\cos 2(\theta+\alpha) + x_3\cos 2(\theta+\beta) + ...]$$ secondary moment

All of these may be expanded using the trigonometry identity cos (A+B) = cos A cos B – sin A sin B

This gives us:

$F_p = M\omega^2 R[\cos\theta\{1 + \cos\alpha + \cos\beta\} - \sin\theta\ \{\sin\alpha + \sin\beta\}]$ primary force

$$F_s = M\omega^2 \frac{R}{n}[\cos(2\theta)\{1 + \cos(2\alpha) + \cos(2\beta)...\} - \sin(2\theta)\{\sin(2\alpha) + \sin(2\beta)..\}]$$ secondary force

$TM_p = M\omega^2 R[x_1 \cos\theta + x_2 \cos\theta \cos\alpha - x_2 \sin\theta \sin\alpha + x_3 \cos\theta \cos\beta - x_3 \sin\theta \sin\beta$ primary moment

$TM_s = M\omega^2(R/n)\ [x_1\cos 2\theta + x_2 \cos 2\theta \cos 2\alpha - x_2 \sin 2\theta \sin 2\alpha +$

$x_3 \cos 2\theta \cos 2\beta - x_3 \sin 2\theta \sin 2\beta....]$ secondary moment

If the system is balanced, these would equate to zero. If the mass M and radius R are the same for all cylinders, we can split each into two expressions that must be equated to zero.

$\{1 + \cos\alpha + \cos\beta....\} = 0$..................(1)	primary force
$\{\sin\alpha + \sin\beta + ...\} = 0$....................(2)	primary force
$1 + \cos(2\alpha) + \cos(2\beta) + ... = 0$ ....(3)	secondary force
$\sin(2\alpha) + \sin(2\beta)..= 0$ ....(4)	secondary force
$x_1 + x_2\cos\alpha + x_3\cos\beta + x_4\cos\gamma + ... = 0$ ...(5)	primary moment
$x_2\sin\alpha + x_3\sin\beta + x_4\sin\gamma + ... = 0$ ....(6)	primary moment
$x_1 + x_2 \cos 2\alpha + x_3 \cos 2\beta .... = 0$ ....(7)	secondary moment
$x_2 \sin 2\alpha + x_3 \sin 2\beta.... = 0$ ....(8)	secondary moment

These may be used to determine how to balance a system.

WORKED EXAMPLE No.11a

Using the criteria just developed determine the state of balance for the 2 crank system in worked example 3a with $x_1 = c$ $x_2 = 2c$

SOLUTION

We must satisfy equation 1,2,5 and 6 with $\alpha = 180^o$

(1)	$1 + \cos\alpha = 1 - 1 = 0$	primary force balanced
(2)	$\sin\alpha = 0$	primary force balanced
(3)	$1 + \cos(2\alpha) = 1 + 1 = 2$	secondary force not balanced
(4)	$\sin(2\alpha) = 0$	secondary force balanced
(5)	$x_1 + x_2\cos\alpha = x_1 - x_2 = -c$	primary moment not balanced
(6)	$x_2\sin\alpha = 0$	primary moment balanced
(7)	$x_1 + x_2 \cos 2\alpha = x_1 + x_2 = 3c$	secondary moment not balanced
(8)	$x_2 \sin 2\alpha = 0$	secondary moment balanced

Only the primary force is completely balanced.

There is a primary moment of $TM_p = -cM\omega^2 R\cos\theta$

a secondary force of $F_s = 2M\omega^2(R/n)\cos(2\theta)$

a secondary moment of $F_s = 3cM\omega^2(R/n)\cos(2\theta)$

WORKED EXAMPLE No.11b

Using the criteria just developed determine the state of balance for the 3 crank system in worked example 3b given $x_1 = c$ $x_2 = 2c$ $x_3 = 3c$

SOLUTION

$\alpha = 120^o$ and $\beta = 240^o$

(1)	$1 + \cos\alpha + \cos\beta = 1 - 0.5 - 0.5 = 0$	primary force balanced
(2)	$\sin\alpha + \sin\beta = 0.866 - 0.866 = 0$	primary force balanced
(3)	$1 + \cos(2\alpha) + \cos(2\beta) = 1 - 0.5 - 0.5 = 0$	secondary force balanced
(4)	$\sin(2\alpha) + \sin(2\beta) = -0.866 + 0.866 = 0$	secondary force balanced
(5)	$x_1 + x_2\cos\alpha + x_3\cos\beta = c - 0.5(2c) - 0.5(3c) = -1.5c$	primary moment not balanced
(6)	$x_2\sin\alpha + x_3\sin\beta = 0.866(2c) - 0.866(3c) = -0.866c$	primary moment not balanced
(7)	$x_1 + x_2\cos2\alpha + x_3\cos2\beta = c - 0.5(2c) - 0.5(3c) = -1.5c$	secondary moment not balanced
(8)	$x_2\sin2\alpha + x_3\sin2\beta = -0.866(2c) + 0.866(3c) = 0.866c$	secondary moment not balanced

There is complete force balance but there are unbalanced moments of

$F_p = 0.866c\,M\omega^2R\sin\theta - 1.5c\,M\omega^2R\cos\theta = c\,M\omega^2R\{0.866\sin\theta - 1.5\cos\theta)$

$F_s = -1.5c\,M\omega^2(R/n)\cos2\theta - 0.866c\,M\omega^2(R/n)\sin2\theta = -c\,M\omega^2(R/n)\{1.5\cos2\theta + 0.866\sin2\theta$

WORKED EXAMPLE No.12

Two lines of reciprocating parts at A and B are to be balanced for primary forces and couples by two lines of reciprocating parts C and D. Given $M_A = 500$ g $M_B = 750$ g and $\alpha = 80^o$, find the masses and angles for C and D. Determine the unbalanced secondary components.

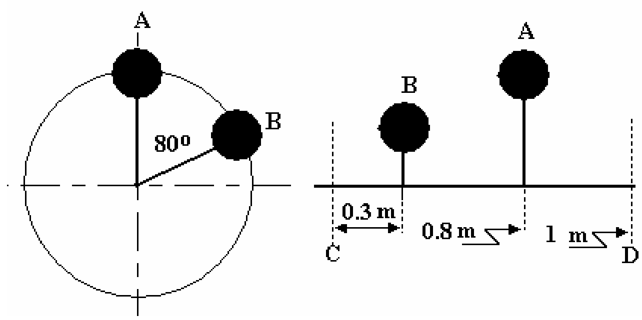

Figure 37

SOLUTION

We must modify equations equation 1,2,5 and 6 to take account of the different masses and distances. They become:

(1)	$M_A + M_B\cos\alpha + M_C\cos\beta + M_D\cos\gamma = 0$	primary force
(2)	$M_B\sin\alpha + M_C\sin\beta + M_D\sin\gamma = 0$	primary force
(5)	$M_A x_A + M_B x_B\cos\alpha + M_C x_C\cos\beta + M_D x_D\cos\gamma = 0$	primary moment
(6)	$M_B x_B\sin\alpha + M_C x_C\sin\beta + M_D x_D\sin\gamma = 0$	primary moment

β is the angle of M_C and γ is the angle of M_D

(1) $0.5 + 0.75\cos80^o + M_C\cos\beta + M_D\cos\gamma = 0$

(2) $0.75\sin80^o + M_C\sin\beta + M_D\sin\gamma = 0$

(5) $(0.5 \times 0.8) + (0.75 \times 0.3)\cos80^o + 0 + M_D\cos\gamma = 0$

(6) $(0.75 \times 0.3)\sin80^o + 0 + M_D\sin\gamma = 0$

Rearrange

(1) $M_C \cos\beta + M_D \cos\gamma = -0.63$
(2) $M_C \sin\beta + M_D \sin\gamma = -0.74$
(5) $M_D \cos\gamma + 0 = -0.439$
(6) $M_D \sin\gamma = -0.222$

From (5) and (6) $\tan\gamma = \sin\gamma/\cos\gamma = 0.22/0.439 = 0.506$ $\gamma = 26.8^o$ or 206.8^o
Since $\sin\gamma$ and $\cos\gamma$ are both negative, γ must lie between 180^o and 270^o
From (6) $M_D = -0.439/\cos\gamma = 0.492$ kg

From (1) $M_C \cos\beta + 0.492 \cos 206.8^o = -0.63$ $M_C \cos\beta = -0.191$
From (2) $M_C \sin\beta + 0.492 \sin 206.8^o = -0.74$ $M_C \sin\beta = -0.518$
$\tan\beta = \sin\beta/\cos\beta = 0.518/0.191 = 2.71$ $\beta = 69.8$ or 249.8
Since $\sin\beta$ and $\cos\beta$ are negative it must be the angle between 180^o and 270^o
It follows that $M_C = -0.518/\sin 249.8^o = 0.552$ kg

SECONDARY COMPONENTS
We must modify equations (3), (4), (7) and (8)

(3) $M_A + M_B\cos(2\alpha) + M_C\cos(2\beta) + M_D\cos(2\gamma) = 0$ secondary force balanced
(4) $M_B\sin(2\alpha) + M_C\sin(2\beta) + M_D\sin(2\gamma) = 0$ secondary force balanced
(7) $M_A x_A + M_B x_B \cos 2\alpha + M_C x_C \cos 2\beta + M_D x_D \cos 2\gamma = 0$ secondary moment not balanced
(8) $M_B x_B \sin 2\alpha + M_C x_C \sin 2\beta + M_D x_D \sin 2\gamma = 0$ secondary moment not balanced

From (3) $M_A + M_B\cos(2\alpha) + 0.492\cos(2\beta) + 0.552\cos(2\gamma)$
$0.5 + 0.75\cos(160^o) + 0.552\cos(413.6^o) + 0.552\cos(499.6^o) = -0.333$ kg

From (4) $M_B\sin(2\alpha) + M_C\sin(2\beta) + M_D\sin(2\gamma)$
$0.75\sin(160^o) + 0.492\sin(499.6^o) + 0.492\sin(413.6^o) = 1.01$ kg

From (7) $M_A x_A + M_B x_B \cos 2\alpha + M_C x_C \cos 2\beta + M_D x_D \cos 2\gamma$
$(0.5 \times 0.8) + (0.75 \times 0.3)\cos(160) + (0.552 \times 0)\cos(499.6^o) + (0.492 \times 1)\cos(413.6^o)$
$= 0.4 - 0.211 + 0 + 0.29 = 0.481$ kg m

From (8) $M_B x_B \sin 2\alpha + M_C x_C \sin 2\beta + M_D x_D \sin 2\gamma = 0$

$(0.75 \times 0.3) \sin(160) + (0.552 \times 0)\sin(499.6^o) + (0.492 \times 1)\sin(413.6^o)$
$= 0.0769 + 0 + 0.396 = 0.473$ kg m

The unbalanced moment is $M\omega^2(R/n)(0.481 \cos 2\theta - 0.473 \sin 2\theta)$

Further studies in this area would include cylinders not in one line such as the Vee configuration and radial engines but this is probably not needed in the EC exam.