SOLID MECHANICS

DYNAMICS

TUTORIAL – PULLEY DRIVE SYSTEMS

This work covers elements of the syllabus for the Edexcel module HNC/D Mechanical Principles.

On completion of this tutorial you should be able to

- Solve problems involving speed change with pulleys.
- Solve power transmission problems involving the speed and torque transmission of pulley systems.
- Derive and explain equations governing when pulley belts slip.
- Derive equations governing the maximum power that can be transmitted by pulley systems and solve problems with them.
- Modify equations to include the affect of centrifugal force on the pulley belt.
- Modify equations to show the effect of using Vee section grooves.

It is assumed that the student is already familiar with the following concepts.

- Angular and linear motion.
- Torque and power transmission.
- Coulomb friction.
- The basic centrifugal force equation.
- Higher level mathematics to include the differential calculus.

All the above may be found in the pre-requisite tutorials.

1. INTRODUCTION

Pulley drive systems depend on friction to enable the belt to grip the wheel and pull it around with it. To enable this, the belt must be tensioned, even when the wheels are stationary. This is unlike positive chain drive systems where teeth mesh with the chain and slip is not possible so no initial tension is required. Pulley drives are most often used to produce speed reduction between a motor and the machine being driven (e.g. a motor driving an air compressor). Many other applications exist from small rubber band drives in video recorders to large multi belt systems on heavy industrial equipment. On many modern systems, toothed belts are used (e.g. timing belt on a car engine) to prevent the belt slipping. This tutorial is only concerned with smooth belts.

2. THEORY

2.1 RELATIONSHIP BETWEEN WHEEL SPEEDS

When two wheels are connected by a pulley belt, the surface or tangential velocity must be the same on both and be the same as the velocity of the belt.

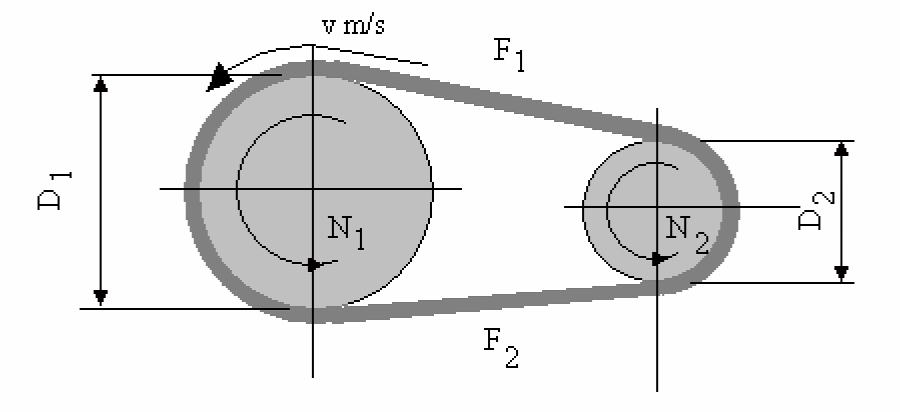

Figure 1

Velocity of belt = v m/s

Velocity of point on wheel in contact with belt = v m/s

The relationship between angular and linear velocity is v = πND

Since this is the same on wheel (1) and wheel (2) then it follows

$$\frac{N_1}{N_2} = \frac{D_2}{D_1}$$

2.2. BELT TENSION

Imagine that both pulley wheels are completely free to turn and that the belt is initially tensioned by stretching the centre distance between the wheels. (In practice a spring loaded wheel pushing against the belt is used to tension the belts). The tension in the belt will be the same along its whole length and equal to F. When transmitting power, the driven wheel will be reluctant to turn and the driving wheel has to pull it and exert a torque on it. Consequently the tension in the side pulling will increase to F_1 but the tension in the other side will decrease to F_2. The increase in tension on the tight side of the belt is equal to the decrease on the slack side so the sum of the tensions remains constant.

It follows that $F_1 + F_2 = \text{constant}$

When stationary both were equal to F so $F_1 + F_2 = F + F = 2F$

So when the wheel is transmitting power $F_1 + F_2 = 2F$

2.3 POWER TRANSMITTED

Mechanical power is the product of force and velocity so $P = Fv$ (Watts).

In this case we have one force pulling in opposition to the other so the net power transmitted is $\mathbf{P = v(F_1 - F_2)}$

Since $v = \pi ND$ $\mathbf{P = \pi ND(F_1 - F_2)}$

Another way to look at this follows.

Torque = Force x Radius and since radius is half the diameter $T = F \times D/2$
Since there are two forces pulling in opposite directions, the net torque on a given wheel is $T = (D/2)(F_1 - F_2)$.

The power of a shaft or wheel under a torque T (Nm) is $P = 2\pi NT$ or $P = \omega T$
In this case we use the net torque so $P = 2\pi N(D/2)(F_1 - F_2)$.

$$\mathbf{P = \pi ND(F_1 - F_2) = v(F_1 - F_2)}$$

You should use which ever formula is the most convenient.

3. FRICTION ON CURVED SURFACES.

Most of us at some time or other have wrapped a rope around a post so that it takes the strain out of the rope. It will be shown that the formula relating the large force being held (F_1) to the small force being exerted (F_2) is given by

$$\frac{F_1}{F_2} = e^{\mu\theta}$$

This formula is also important for ropes passing around drums and pulleys and governs the power that can be transmitted by a pulley drive before the belt slips.

Consider a pulley wheel with a belt passing around it as shown. In order for the belt to produce a torque on the wheel (whether or not it is rotating), there must be tension in both ends. If this was not so, the belt would not be pressed against the wheel and it would slip on the wheel. The belt depends upon friction between it and the wheel in order to grip and produce torque.

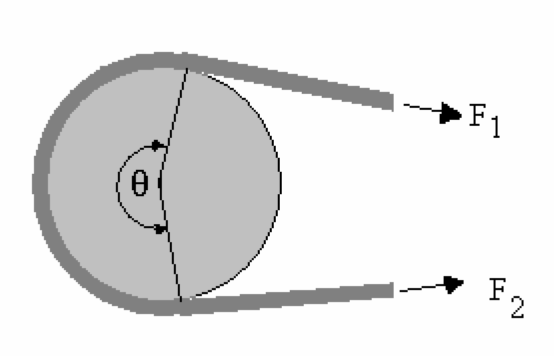

Figure 2

For a belt to produce torque on the wheel, the force in one end must be greater than the force in the other end, otherwise the net torque is zero.

Let F_1 be the larger force and F_2 the smaller force. θ is the angle of lap.

Now consider an elementary length of the belt on the wheel. The force on one end is F and on the other end is slightly larger and is F + dF. The angle made by the small length is $d\theta$.

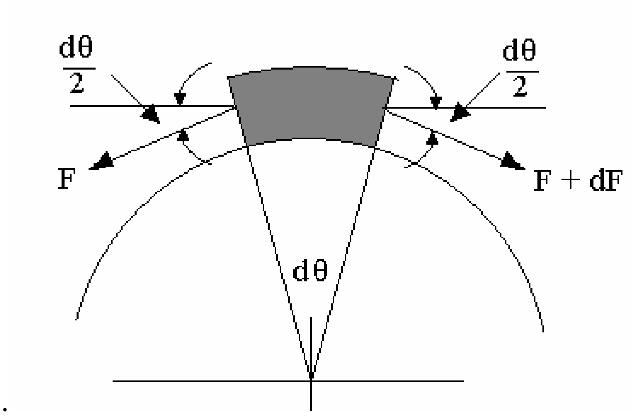

Figure 3

First resolve F radially and tangentially to the wheel.

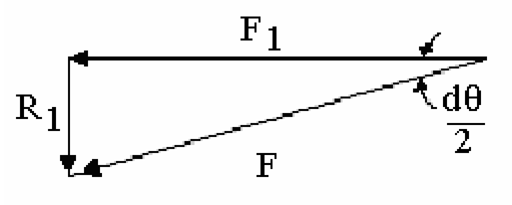

Figure 4

$F_1 = F \cos(½ \, d\theta)$

Since the cos of a small angle is almost 1 then $F_1 = F$

$R_1 = F \sin(½ \, d\theta)$

Since the sin of a small angle is equal to the angle in radians then $R_1 = F\,(½ \, d\theta)$

Next repeat for the other end by resolving F + dF

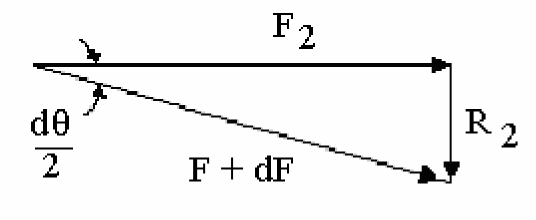

Figure 5

By the same reasoning we get

$F_2 = (F+dF) \cos(½ \, d\theta) = F + dF$

and $R_2 = (F + dF) \sin(½ \, d\theta) = (F+dF)(½ \, d\theta)$

Ignoring the product of two small quantities the total reaction force is

$R = R_1 + R_2 = F \, d\theta$

The resultant tangential force is $F = F_2 - F_1 = dF$

Summarising $F = dF$ $R = Fd\theta$

Now treat the small piece of belt as a small block about to slip on a flat surface.

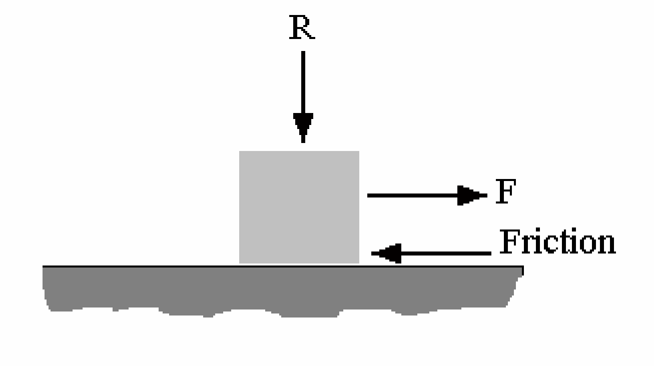

Figure 6

When the block just slips the force F is equal and opposite of the friction force. Coulombs law states that the friction force is directly proportional to the normal force and the constant of proportionality is the coefficient of friction μ.
It follows that $F = \mu R$. In this case the it is dF not F so we get the following.

$$dF = \mu R = \mu F d\theta \text{ and so } dF/F = \mu d\theta$$

Integrating between limits of $\theta = 0$ and $\theta = \theta$ for angle and $F = F_1$ and $F = F_2$ for force, we get

$$\ln F_1 - \ln F_2 = \mu\theta \qquad \ln(F_1/F_2) = \mu\theta \qquad \mathbf{(F_1/F_2) = e^{\mu\theta}}$$

This equation gives us the ratio of (F_1/F_2) when the belt is about to slip on the wheel.

The equation may also be used for other related problems such as ropes wrapped around capstans and bollards. The ratio of the forces increases with the angle of lap. If the rope is wrapped around the capstan several times, then a small force may be used to hold a large force. This is the principle used in winches and capstans.

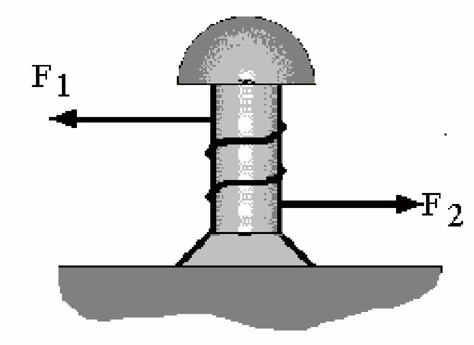

Figure 7

<u>WORKED EXAMPLE No.1</u>

A rope is hung over a stationary drum as shown with weights on each end. The rope is just on the point of slipping. What is the coefficient of friction?

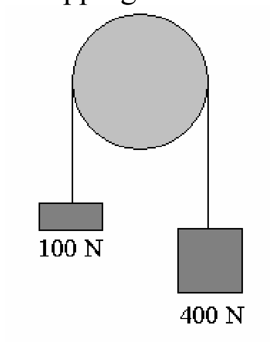

Figure 8

<u>SOLUTION</u>

Angle of lap = $\theta = 180^o = \pi$ radian $\qquad F_1 = 400$ N and $F_2 = 100$ N

$F_1/F_2 = e^{\mu\theta} = 400/100 = e^{\mu\theta} = 4 \qquad \ln 4 = \mu\theta = 1.386$

μ = **0.441**

SELF ASSESSMENT EXERCISE No.1

1. Calculate the number of times a belt must be wrapped around a post in order for a force of 100 N to hold a force of 20 kN. The coefficient of friction is 0.3.

   Note you must calculate θ which will be in radians and convert to turns by dividing the answer by 2π.
   (Ans. 2.81)

2. Calculate the force ratio for a pulley belt on a wheel when it is about to slip given the coefficient of friction is 0.4 and the angle of lap is 200^o. (Don't forget to convert into radians).
   (Ans. 4.04)

3. A pulley is tensioned to 800 N whilst stationary. Calculate the tensions in both sides given that the wheel is 200 mm diameter and 2 kW of power are being transmitted at 300 rev/min.
   (Ans 1118.3 N and 481.7 N).

4. A pulley wheel is 100 mm diameter and transmits 1.5 kW of power at 360 rev/min. The maximum belt tension is 1200 N at this point. Calculate the initial tension applied to the stationary belts.
   (Ans. 802 N).

4. MAXIMUM POWER TRANSMITTED BY PULLEYS

The force (tension) in a pulley belt increases with torque and power. The maximum power that a pulley system can transmit is ultimately limited by the strength of the belt material. If this is a problem then more than one belt should be used to share the load. If the belt does not break then the possibility of the belt slipping exists and this depends upon the angle of lap and the coefficient of friction. If the coefficient of friction is the same on both wheels, then slippage will occur first on the smaller wheel. The power at which the belt slips is not the absolute maximum power that can be transmitted as more power can be transmitted with slippage occurring by using higher wheel speeds.

The friction between the belt and the wheel is further affected by centrifugal force which tends to lift the belt off the wheel. This increases the likelihood of slippage.

The friction between the belt and wheel may be increased by the shape of the belt. A vee section or round section belt in a vee groove will grip better than a flat belt and is less likely to slip.

4.1 MAXIMUM POWER WITH NO BELT SLIP.

The power transmitted by a pulley was shown to be $P = \pi ND (F_1 - F_2) = v(F_1 - F_2)$

When the belt starts to slip the force ratio is $(F_1 / F_2) = e^{\mu\theta}$

Substituting for F_2 we have $P = \pi NDF_1(1 - e^{-\mu\theta})$

It may be easier to use belt velocity instead of wheel speed. $v = \pi ND$

$$\mathbf{P = v\, F_1(1 - e^{-\mu\theta})}$$

This is the maximum power that can be transmitted at a given speed with no slip occurring.

WORKED EXAMPLE No.2

The tension in a pulley belt is 110 N when stationary. Calculate the tension in each side and the power transmitted when the belt is on the point of slipping on the smaller wheel. The wheel is 240 mm diameter and the coefficient of friction is 0.32. The angle of lap is 165^o. The wheel speed is 1500 rev/min.

SOLUTION

Belt velocity $v = \pi ND = \pi \times (1500/60) \times 0.24 = 18.85$ m/s

Lap Angle $\theta = (165/180) \times \pi = 2.88$

Initial tensions $F_1 = F_2 = 110$ N when stationary so $F_1 + F_2 = 220$ N

$F_2 = 220 - F_1$

Tension in belts.

$$\frac{F_1}{F_2} = e^{\mu\theta} = e^{0.32x2.88} = 2.513$$

$$F_1 = 2.513F_2 = 2.513(220 - F_1) = 552.9 - 2.513\,F_1$$

$$F_1 + 2.513\,F_1 = 552.9$$

$$3.513\,F_1 = 552.9$$

$$F_1 = 157.4\text{ N}$$

$$F_2 = \frac{157.4}{2.513} = 62.6\text{ N}$$

Power $P = v(F_1 - F_2) = 18.85(157.4 - 62.6) = 1\ 786$ W

Check $P = v\,F_1(1 - e^{-\mu\theta}) = 18.85 \times 157.4 \times (1 - 0.398)$

$P = 18.85 \times 157.4 \times 0.6 = 1\ 786$ W

SELF ASSESSMENT EXERCISE No.2

1. The tension in a pulley belt is 30 N when stationary. Calculate the tension in each side and the power transmitted when the belt is on the point of slipping on the smaller wheel. the wheel is 300 mm diameter and the coefficient of friction is 0.4. The angle of lap is 160^o. The wheel speed is 1420 rev/min.

   (Ans. 14.79 N, 45.21 N and 678.5 Watts).

2. A belt drive must transmit 10 kW of power at 3000 rev/min. The wheel is 200 mm diameter and the coefficient of friction is 0.35. The lap angle is 150^o. The belt tension must not exceed 50 N. Calculate the minimum number of belts required.

   (Ans. 11 belts).

5. VEE BELTS

Consider a VEE section belt with an included angle of 2β.Vee belts grip on the side and not on the bottom. The wedging affect increases the reaction force between the pulley and the belt from R to R'. Since the friction force is increased, greater power can be transmitted before the belt slips.

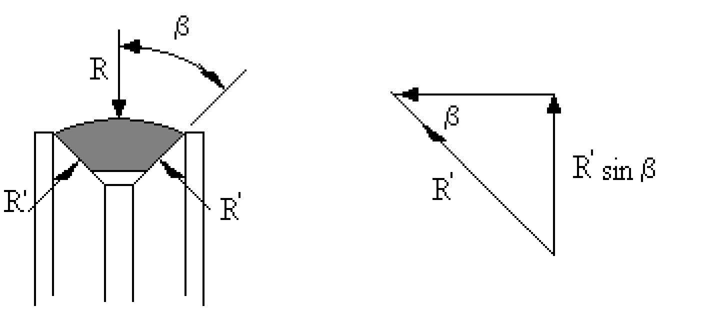

Figure 9

Resolving R' vertically gives an upwards force of R'sinβ on each side of the belt. Balancing vertically $R = 2\,R'\sin\beta$

$$R' = \frac{R}{2\sin\beta}$$

In the original derivation we had dF = μR but this time we must use dF = μ R', hence

$$dF = \mu R' = \mu\frac{R}{2\sin\beta}$$

Since there are two faces in contact with the wheel, the friction force is doubled, hence

$$dF = \mu\frac{R}{\sin\beta}$$

Completing the derivation by integrating between limits as before we get the following result.

$$\frac{F_1}{F_2} = e^{\left(\frac{\mu\theta}{\sin\beta}\right)} = e^{\mu'\theta} \text{ where } \mu' = \frac{\mu}{\sin\beta}$$

6. THE EFFECT OF CENTRIFUGAL FORCE

Consider the element of belt on the wheel once again.

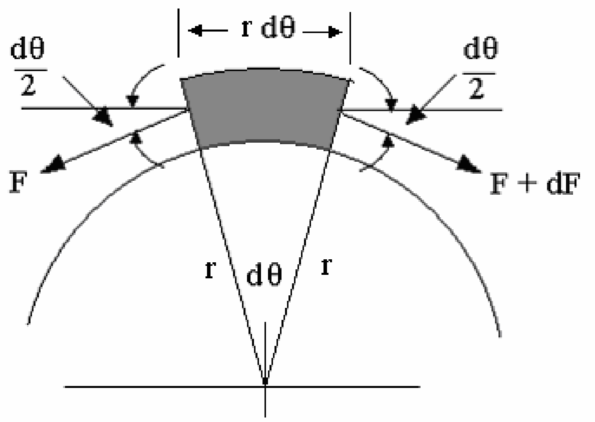

Figure 10

The length of the curved element is $rd\theta$.
The density of the belt material is ρ.
The cross sectional area of the belt is A.
The volume is $Ard\theta$.
The mass is $dm = \rho Ard\theta$
The equation for centrifugal force is $CF = mv^2/r$. The centrifugal force acting on the tiny mass dm is written in calculus form as d(CF)

$d(CF) = dm\, v^2/r$
$d(CF) = \rho Ar\, d\theta\, v^2/r$
$d(CF) = \rho A\, d\theta\, v^2$

The normal force R pressing the element to the wheel was from earlier work

$R = F\, d\theta$

Now this is reduced by the centrifugal force so

$R = F\, d\theta - \rho A d\theta v^2$
$R = d\theta(F - \rho Av^2)$

The change in force from one side of the element to the other was $dF = \mu R$
Substituting for R we have

$dF = \mu d\theta(F - \rho Av^2)$
$dF/(F - \rho Av^2) = \mu d\theta$

Integrating as before we have

$(F_1 - \rho Av^2)/(F_2 - \rho Av^2) = e^{\mu\theta}$

Let $\rho Av^2 = F_c$ which is the centrifugal force term.

$$\mathbf{(F_1 - F_c)/(F_2 - F_c) = e^{\mu\theta}}$$

Note that since the angle of lap is smallest on the small wheel, the belt always slips first on the small wheel (if the coefficient of friction is the same).

POWER TRANSMITTED

The power transmitted is as before

$$P = \pi ND(F_1 - F_2)$$

Subtracting F_c from each term makes no difference.

$$P = \pi ND\{(F_1-F_c)-(F_2-F_c)\}$$

but $(F_2- F_c)= (F_1- F_c)e^{-\mu\theta}$

Substituting for $(F_2 - F_c)$ we have

$$P = \pi ND(F_1- F_c)(1 -e^{-\mu\theta})$$

or $\mathbf{P = v(F_1- F_c)(1 -e^{-\mu\theta})}$

WORKED EXAMPLE No.3

A pulley system uses a flat belt of c.s.a. 500 mm^2 and density 1300 kg/m^3. The angle of lap is 165^o on the smaller wheel. The coefficient of friction is 0.35. The maximum force allowed in the belt is 600 N. Calculate the power transmitted when the belt runs at 10 m/s and

a. centrifugal force is not included.
b. centrifugal force is included.

Calculate the initial tension in the belts.

SOLUTION

$F_c = \rho Av^2 = 1300 \times 500 \times 10^{-6} \times 10^2 = 65$ N

$\theta = (165/180) \times \pi = 2.88$ radian

$P = v(F_1- F_c)(1 -e^{-\mu\theta})$
$P = 10(600 – 65)(1- e^{-0.35 \times 2.88})$
$P = 10 \times 535 \times 0.635 = 3397$ Watts

$F_2 = (F_1 – F_c)e^{-\mu\theta} + Fc = (600 – 65)\ e^{-0.35 \times 2.88} + 65 = 260.3$ N

Initial tension $= (F_1 + F_2)/2 = (600 + 260.3)/2 = 430$ N

SELF ASSESSMENT EXERCISE No. 3

A pulley system uses a flat belt of c.s.a. 400 mm^2 and density 1200 kg/m^3. The angle of lap is 170^o on the smaller wheel. The coefficient of friction is 0.25. The maximum force allowed in the belt is 800 N. Calculate the power transmitted when the belt runs at 16 m/s and

a. centrifugal force is not included. (6.7 kW)
b. centrifugal force is included. (5.7 kW)

Calculate the initial tension in the belts. (622.7 N)

7. MAXIMUM POWER WITH CENTRIFUGAL FORCE INCLUDED

The equation for power in full is

$$P = v(F_1 - \rho A v^2)(1 - e^{-\mu\theta}) = (vF_1 - \rho A v^3)(1 - e^{-\mu\theta})$$

If we plot power against speed for a given set of parameters we get a graph as shown. Clearly the graph must start at zero power at zero speed. The power climbs as the speed is increased but a point is reached when the centrifugal force reduces the grip to such an extent that slippage reduces the power and further increase in speed reduces the power as the belt slips more. At very high speeds there will be no grip at all and the power drops back towards zero.

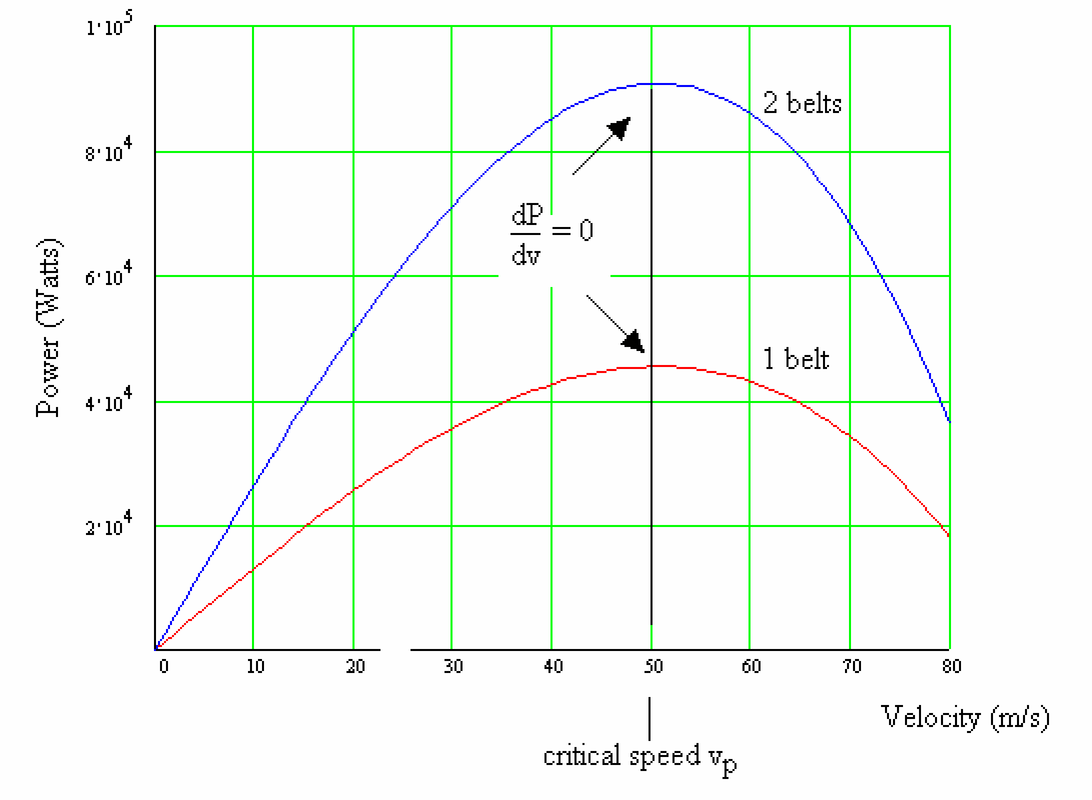

Figure 11

The velocity at which the power peaks is found from max and min theory. At the peak point, the gradient is zero so the differential coefficient is zero. Differentiating w.r.t. velocity we have

$$dP/dv = (F_1 - 3\rho A v^2)(1 - e^{-\mu\theta}) = 0$$

Since $(1 - e^{-\mu\theta})$ is not zero then the bracket $(F_1 - 3\rho A v^2)$ must be zero so $F_1 = 3\rho A v^2$.

The velocity is the critical velocity which gives maximum power so this is given by

$$\mathbf{v_p = \{F_1/3\rho A\}^{1/2}}$$

Substituting this back into the power formula, the maximum power transmitted is

$P_{(max)} = (v_pF_1 - \rho A v_p^3)(1 - e^{-\mu\theta})$ substitute $v_p^2 = F_1/3\rho A$

$$\mathbf{P_{(max)} = (2/3)(v_pF_1)(1 - e^{-\mu\theta})}$$

If there are n belts this becomes

$$\mathbf{P_{(max)} = (2/3)(v_pF_1)(1 - e^{-\mu\theta})n}$$

Note that you must calculate v_p before calculating P_{max}

<u>WORKED EXAMPLE No.4</u>

A pulley system uses 4 flat belts each of c.s.a. 800 mm^2 and density 1300 kg/m^3. The angle of lap is 130^o on the smaller wheel. The coefficient of friction is 0.4. The maximum force allowed in the belt is 600 N. Calculate

a. the speed at which max power occurs.
b. the maximum power transmittable by varying the speed.

Calculate the initial belt tension.

<u>SOLUTION</u>

$$v_p = \left(\frac{F_1}{3\rho\rho}\right)^{\frac{1}{2}} = \left(\frac{600}{3 \times 1300 \times 800 \times 10^{-6}}\right)^{\frac{1}{2}} = 13.87 \text{ m/s}$$

$$\theta = 130 \times \frac{\pi}{180} = 2.269 \text{ radian}$$

$$P_{max} = \frac{2v_pF_1}{3}\left(1 - e^{-\mu\theta}\right) = \frac{2 \times 13.87 \times 600}{3}\left(1 - e^{-0.4 \times 2.269}\right) = 3309 \text{ Watts}$$

For 4 belts the maximum power is 3 3309 = 13 236 Watts

$F_c = \rho A v^2 = 1300 \times 800 \times 10^{-6} \times 13.87^2 = 200$ N

$F_2 = (F_1 - F_c)e^{-\mu\theta} + Fc = (800 - 200)\ e^{-0.4 \times 2.269} + 200 = 361.4$ N

Initial tension = $(F_1 + F_2)/2 = (800 + 361.4)/2 = 480.7$ N

SELF ASSESSMENT EXERCISE No. 4

1. A pulley system uses a flat belt of c.s.a. 800 mm^2 and density 1200 kg/m^3. The angle of lap is 160^o on the smaller wheel. The coefficient of friction is 0.3. The maximum stress allowed in the belt is 3 N/mm^2. Calculate

a. the maximum power transmittable by varying the speed. (26.233 kW)

b. the speed at which it occurs. (28.9 m/s)

c. the initial belt tensions (1947 N)

2. A pulley system uses a flat belt of c.s.a. 1000 mm^2 and density 1100 kg/m^3. The angle of lap is 120^o on the smaller wheel. The coefficient of friction is 0.3. The maximum force allowed in the belt is 500 N. Calculate

a. the speed at which max power occurs. (12.3 m/s)

b. the maximum power transmittable by varying the speed. (1.914 kW)

c. the initial belt tensions (422.3 N)