

# **SOLID MECHANICS**

## **DYNAMICS**

### **TUTORIAL – MOMENT OF INERTIA**

This work covers elements of the following syllabi.

- Parts of the Engineering Council Graduate Diploma Exam D225 – Dynamics of Mechanical Systems
- Parts of the Engineering Council Certificate Exam C105 – Mechanical and Structural Engineering.
- Part of the Engineering Council Certificate Exam C103 - Engineering Science.

On completion of this tutorial you should be able to

- Revise angular motion.
- Define and derive the moment of inertia of a body.
- Define radius of gyration.
- Examine Newton's second law in relation to rotating bodies.
- Define and use inertia torque.
- Define and use angular kinetic energy.
- Solve problems involving conversion of potential energy into kinetic energy.

It is assumed that the student is already familiar with the following concepts.

- Newton's Laws of Motion.
- The laws relating angular displacement, velocity and acceleration.
- The laws relating angular and linear motion.
- The forms of mechanical energy.

All the above may be found in the pre-requisite tutorials.

## 1. REVISION OF ANGULAR QUANTITIES

Angular motion is covered in the pre-requisite tutorial. The following is a summary needed for this tutorial.

### 1.1 ANGLE $\theta$

Angle may be measured in revolutions, degrees, grads (used in France) or radians. In engineering we normally use radians. The links between them are

$$1 \text{ revolution} = 360^\circ = 400 \text{ grads} = 2\pi \text{ radian}$$

### 1.2 ANGULAR VELOCITY $\omega$

Angular velocity is the rate of change of angle with time and may be expressed in calculus terms as the differential coefficient  $\omega = \frac{d\theta}{dt}$

### 1.3 ANGULAR ACCELERATION $\alpha$

Angular acceleration is the rate of change of angular velocity with time and in calculus terms may be expressed by the differential coefficient  $\alpha = \frac{d\omega}{dt}$  or the second differential

$$\text{coefficient } \alpha = \frac{d^2\theta}{dt^2}$$

### 1.4 LINK BETWEEN LINEAR AND ANGULAR QUANTITIES.

Any angular quantity multiplied by the radius of the rotation is converted into the equivalent linear quantity as measured long the circular path.

Hence angle is converted into the length of an arc by  $x = \theta R$

Angular velocity is converted into tangential velocity by  $v = \omega R$

Angular acceleration is converted into tangential acceleration by  $a = \alpha R$

### 1.5 TORQUE

When we rotate a wheel, we must apply torque to overcome the inertia and speed it up or slow it down. You should know that torque is a moment of force. A force applied to the axle of a wheel will not make it rotate (figure 1A). A force applied at a radius will (figure 1B). The torque is  $F r$  (N m).

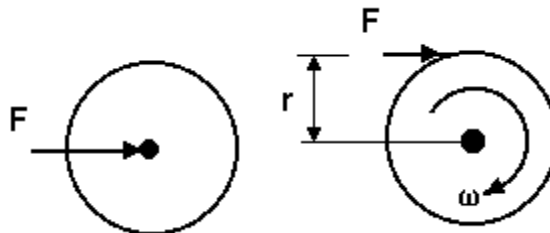


Figure 1 A

Figure 1 B

## 2. MOMENT OF INERTIA I

The moment of inertia is that property of a body which makes it reluctant to speed up or slow down in a rotational manner. Clearly it is linked with mass (inertia) and in fact moment of inertia means second moment of mass. It is not only the mass that governs this reluctance but also the location of the mass. You should appreciate that a wheel with all the mass near the axle (fig. 2A) is easier to speed up than one with an equal mass spread over a larger diameter (fig.2B).

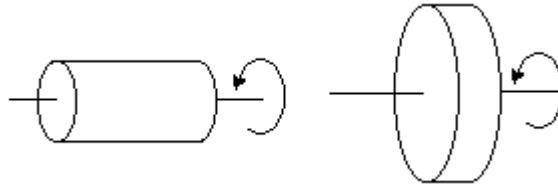


Figure 2A      Figure 2 B

### 2.1 DERIVATION

Consider the case where all the mass is rotating at one radius. This might be a small ball or a rim (like a bicycle wheel) as shown with mass  $\delta m$  at radius  $r$ . The angular velocity is  $\omega$  rad/s.

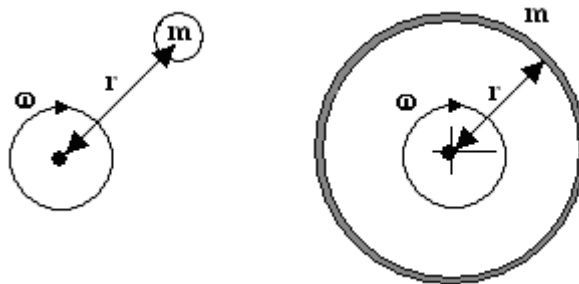


Figure 3

If we multiply the mass by the radius we get the first moment of mass  $r m$   
If we multiply by the radius again we get the second moment of mass  $r^2 m$   
This second moment is commonly called the **moment of inertia** and has a symbol **I**.

Unfortunately most rotating bodies do not have the mass concentrated at one radius and the moment of inertia is not calculated as easily as this.

#### 2.1.1 RADIUS OF GYRATION k

All rotating machinery such as pumps, engines and turbines have a moment of inertia. The radius of gyration is the radius at which we consider the mass to rotate such that the moment of inertia is given by

$$I = M k^2$$

M is the total mass and k is the radius of gyration.

The only problem with this approach is that the radius of gyration must be known and often this is deduced from tests on the machine.

### 2.1.2 PLAIN DISC

Consider a plain disc and suppose it to be made up from many concentric rings or cylinders. Each cylinder is so thin that it may be considered as being at one radius  $r$  and the radial thickness is a tiny part of the radius  $\delta r$ . These are called elementary rings or cylinders.

If the mass of one ring is a small part of the total we denote it  $\delta m$ . The moment of inertia is a small part of the total and we denote it  $\delta I$  and this is given by  $\delta I = r^2 \delta m$ . The total moment of inertia is the sum of all the separate small parts so we can write

$$I = \sum \delta I = \sum r^2 \delta m$$

The disc is  $b$  metres deep. Establish the formula for the mass of one ring.

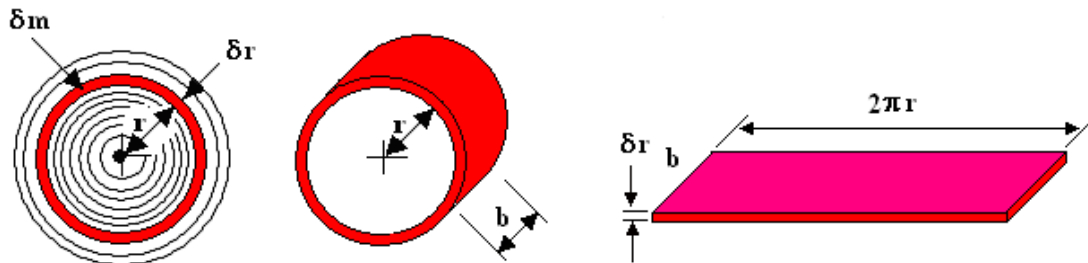


Figure 4

The elementary thin cylinder if cut and unrolled would form a flat sheet as shown.

Length = circumference =  $2\pi r$       depth =  $b$       thickness =  $\delta r$

Volume = length x depth x thickness =  $2\pi r b \delta r$

Change this to mass  $\delta m$  by multiplying by the density of the material  $\rho$ .

$$\text{Mass} = \delta m = \rho b 2\pi r \delta r$$

If the mass is multiplied by the radius twice we get the moment of inertia  $\delta I$ .

$$\delta I = \rho b 2\pi r^2 \delta r = \rho b 2\pi r^3 \delta r$$

As the radial thickness  $\delta r$  gets thinner and tends to zero, the equation becomes precise and we may replace the finite dimensions  $\delta$  with the differential  $d$ .

$$dI = \rho b 2\pi r^3 dr$$

The total moment of inertia is found by integration which is a way of summing all the rings that make up the disc.

$$I = \int_0^R 2\pi b \rho r^3 dr \text{ and taking the constants outside the integral sign we have}$$

$$I = 2\pi b \rho \int_0^R r^3 dr$$

Completing the integration and substituting the limits of  $r = 0$  (the middle) and  $r = R$  (the outer radius) we get the following.

$$I = 2\pi b \rho \int_0^R r^3 dr = 2\pi b \rho \frac{[r^4]_0^R}{4} = 2\pi \rho b \frac{[R^4 - 0^4]}{4}$$

$$I = \pi \rho b \frac{R^4}{2}$$

Now consider the volume and mass of the disc. The volume of the plain disc is the area of a circle radius  $R$  times the depth  $b$ .

$$\text{Volume} = \pi R^2 b$$

$$\text{Mass} = \text{volume} \times \text{density} = \rho \pi R^2 b$$

Examine the formula for  $I$  again.

$$I = \pi \rho b \frac{R^4}{2} = \pi \rho b R^2 \frac{R^2}{2} = \text{Mass} \times \frac{R^2}{2}$$

$$I = \frac{MR^2}{2}$$

For a plain disc the moment of inertia is  $MR^2/2$

If we compare this to  $I = Mk^2$  we deduce that the radius of gyration for a plan disc is

$$k = \sqrt{\frac{R^2}{2}} = \frac{R}{\sqrt{2}} = 0.707R$$

The effective radius at which the mass rotates is clearly not at the mid point between the middle and the outside but nearer the edge.

### WORKED EXAMPLE No.1

Show that the radius of gyration for a uniform rod of length  $L$  rotating about its end is  $0.577L$  from that end.

### SOLUTION

The rod has a uniform cross section  $A$ . Consider a small length  $\delta x$ . The mass of this element is  $\delta m = \rho A \delta x$ .

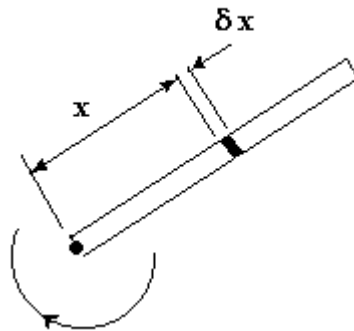


Figure 5

The moment of inertia is  $\delta I = \delta m x^2 = \rho A x^2 \delta x$ .

In the limit as  $\delta x$  gets smaller and smaller we may use the differential  $dx$  and integrate to find  $I$ .

$$I = \int_0^L \rho A x^2 dx = \rho A \int_0^L x^2 dx$$

$$I = \rho A \left[ \frac{x^3}{3} \right]_0^L = \rho A \frac{L^3}{3}$$

The mass of the rod is  $M = \rho AL$  so  $I = M \frac{L^2}{3}$

Using the radius of gyration  $I = Mk^2$

It follows that  $k^2 = \frac{L^2}{3}$  and  $k = \frac{L}{\sqrt{3}} = 0.577L$

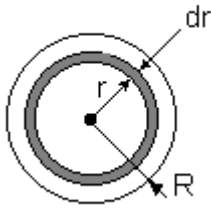
## 2.2 ANGULAR KINETIC ENERGY

You should already know that linear kinetic energy is given by the formula  $K.E. = \frac{mv^2}{2}$

It requires energy to accelerate a wheel up to speed so rotating bodies also possess kinetic energy and the formula is  $K.E. = \frac{m\omega^2}{2}$

### 2.2.1 DERIVATION

Consider again a disc and an elementary ring.



If a point rotates about a centre with angular velocity  $\omega$  rad/s, at radius  $r$ , the velocity of the point along the circle is  $v$  m/s and it is related to  $\omega$  by  $v = \omega r$ .

The mass of the ring is  $\delta m$ . The kinetic energy of the ring is  $\delta m v^2/2$ .  
If we convert  $v$  into  $\omega$  the kinetic energy becomes  $\delta m \omega^2 r^2/2$ . The total kinetic energy for the disc is found by integration so

Figure 6

$$\text{Kinetic energy of ring} = \frac{\delta m v^2}{2}$$

Convert linear velocity to angular velocity by substituting  $v = \omega r$

$$\text{Kinetic energy of ring} = \frac{\delta m \omega^2 r^2}{2}$$

The total K.E. may be found by integrating between the limits 0 and R.

$$\text{K.E. of disc} = \int_0^R \frac{\delta m \omega^2 r^2}{2} = \frac{\omega^2}{2} \int_0^R \delta m r^2$$

By definition, the term  $\int_0^R \delta m r^2$  is the moment of inertia  $I$  so we may write

$$\text{K.E.} = \frac{I\omega^2}{2}$$

### WORKED EXAMPLE No.2

Find the kinetic energy of a wheel rotating at 4 rad/s given the mass is 3 kg and the radius of gyration is 0.2 m.

### SOLUTION

$$I = M k^2 = 3 \times 0.2^2 = 0.12 \text{ kg m}^2$$

$$\text{K.E.} = I \omega^2/2 = 0.12 \times 4^2/2 = 0.96 \text{ Joules}$$

### WORKED EXAMPLE No.3

The wheel shown has a mass of 0.5 kg and a radius of gyration of 0.02 m. The radius of the drum is 0.01 m. Calculate the linear and angular velocity of the wheel after falls 0.6 m.

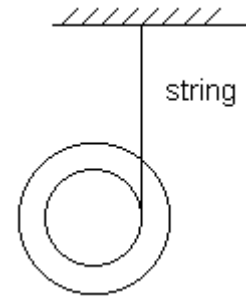


Figure No.7

### SOLUTION

As the wheel falls it will lose potential energy and gain two forms of kinetic energy because it has a velocity down and the string makes it spin. First calculate the change in potential Energy when it falls 0.6 m.

$$\text{P.E.} = mgz = 0.5 \times 9.81 \times 0.6 = 2.943 \text{ J}$$

Next formulate the kinetic energy gained.

$$\text{Linear K.E.} = \frac{mv^2}{2}$$

$$\text{Angular kinetic energy} = \frac{I\omega^2}{2}$$

$$\text{Calculate the moment of inertia} \quad I = Mk^2 = 0.5 \times 0.02^2 = 200 \times 10^{-6} \text{ kg m}^2$$

Equate the P.E. to the K.E.

$$2.943 = \frac{mv^2}{2} + \frac{I\omega^2}{2} = \frac{0.5 v^2}{2} + \frac{200 \times 10^{-6} \omega^2}{2}$$

Substitute  $v = \omega r$

$$2.943 = \frac{0.5 \omega^2 r^2}{2} + \frac{200 \times 10^{-6} \omega^2}{2} = \frac{0.5 \times \omega^2 \times 0.01^2}{2} + \frac{200 \times 10^{-6} \omega^2}{2}$$

$$2.943 = 25 \times 10^{-6} \omega^2 + 100 \omega^2 = 125 \times 10^{-6} \omega^2$$

$$\omega^2 = 23544 \quad \omega = 153.44 \text{ rad/s}$$

$$\text{Now calculate } v = \omega r = 153.44 \times 0.01 = 1.53 \text{ m/s}$$



**SELF ASSESSMENT EXERCISE No.1**

1. A cylinder has a mass of 1 kg, outer radius of 0.05 m and radius of gyration 0.03 m. It is allowed to roll down an inclined plane until it has changed its height by 0.6 m. Assuming it rolls with no energy loss, calculate its linear and angular velocity at this point.
2. A cylinder has a mass of 3 kg, outer radius of 0.2 m and radius of gyration 0.15 m. It is allowed to roll down an inclined plane until it has changed its height by 2 m. Assuming it rolls with no energy loss, calculate its linear and angular velocity at this point.

(Answers 25.06 rad/s and 5.01 m/s)

### 3. INERTIA TORQUE

Consider a plain disc again and an elementary ring with a mass  $\delta m$  at radius  $r$ . Let a force  $\delta F$  be applied tangential to accelerate the ring.

The acceleration of the ring along a tangent is 'a'

Newton's 2nd Law for linear motion states that  
Force = mass x acceleration.

In this case  $\delta F = \delta m a$

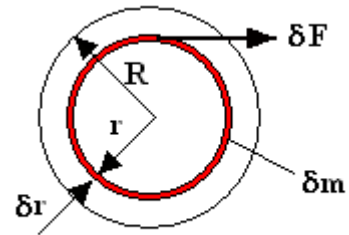


Figure 8

Change to angular acceleration using the relationship  $a = \alpha r$

$\delta F = \delta m \alpha r = r \delta m \alpha$  Multiply both sides by  $r$ .

$r \delta F = r^2 \delta m \alpha$

The expression  $r \delta F$  is the torque on the ring and we will denote it as  $\delta T$ .

The term  $r^2 \delta m$  is the moment of inertia for the elementary ring and this is a small part of the total moment of inertia for the disc so denote it  $\delta I$

The expression becomes  $\delta T = \delta I \alpha$

The plain disc is made up of many such rings so the torque needed to accelerate the whole disc at  $\alpha \text{ rad/s}^2$  is found by summing them.

$$T = \sum_0^T \delta T = \alpha \sum_0^I \delta I$$

$$T = I \alpha$$

Hence in order to accelerate a disc with angular acceleration  $\alpha$ , a torque  $T$  is required and the relationship is

$$\mathbf{T = I\alpha}$$

This is Newton's 2nd law written in angular terms and it applies to any shape, not just plain discs. Remember that the formula for the moment of inertia is as follows.

For any rotating body in general

$$I = Mk^2$$

$k$  is the radius of gyration.

For a plain disc radius  $R$ ,

$$I = MR^2/2$$

For an annular ring of inside radius  $R_i$  and outside radius  $R_o$

$$I = m (R_o^2 + R_i^2)/2$$

#### **WORKED EXAMPLE No.4**

A solid wheel has a mass of 5 kg and outer radius of 200 mm. Calculate the torque required to accelerate it from rest to 1500 rev/min in 10 seconds.

#### **SOLUTION**

The initial angular velocity is zero  $\omega_1 = 0$

The final angular velocity is found by converting 1500 rev/min

$$\omega_2 = \frac{1500}{60} \times 2\pi = 157.1 \text{ rad/s}$$

$$\alpha = \frac{\omega_2 - \omega_1}{t} = \frac{15.71}{10} = 1.571 \text{ rad/s}^2$$

$$I = \frac{MR^2}{2} = \frac{5 \times 0.2^2}{2} = 0.1 \text{ kg m}^2$$

$$T = I\alpha = 0.1 \times 15.71 = 1.571 \text{ Nm}$$

#### **WORKED EXAMPLE No.5**

A simple impact tester consists of a hammer of mass 0.6 kg on a pivoted light rod 0.5 m long as shown. The hammer is raised to the horizontal position and allowed to swing down hitting the test sample in a vice as shown. The hammer continues in its swing to a height of 0.2 m on the other side. Determine

- The velocity of the hammer when it strikes the sample.
- The energy absorbed in the impact.
- The angular velocity at the bottom of the swing.

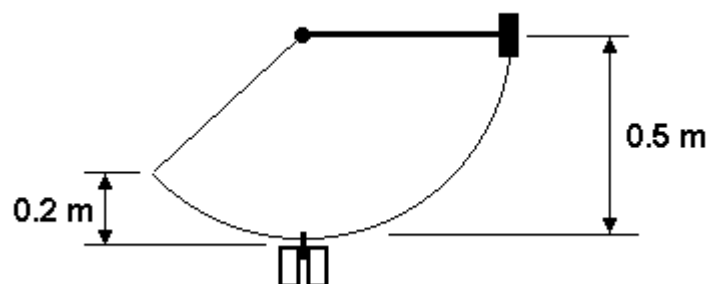


Figure 9

### SOLUTION

The initial potential energy of the hammer =  $mg z_1 = 0.6 \times 9.81 \times 0.5 = 2.943 \text{ J}$   
The Kinetic energy at the lowest point will be the same if none is lost so equate them.

$$\text{KE} = mv^2/2 = 2.943 \text{ J}$$

$$0.6 v^2/2 = 2.943$$

$$v = 3.13 \text{ m/s}$$

The potential energy at the end of the swing =  $mg z_2$

$$\text{P.E.} = 0.6 \times 9.81 \times 0.2 = 1.177 \text{ J}$$

$$\text{Energy lost in the impact} = 2.943 - 1.177 = 1.766 \text{ J}$$

The angular velocity of the hammer at the bottom of the swing =  $\omega = v/r$

$$\omega = 3.13/0.5 = 6.26 \text{ rad/s}$$

### WORKED EXAMPLE No.6

A drum has a mass of 200 kg and radius of gyration of 0.8 m. It is rotated by a chain drive. The sprocket on the drum has an effective diameter of 0.3 m. The force in the chain is a constant value 120 N.

Calculate the time required to accelerate it up to 20 rev/min.

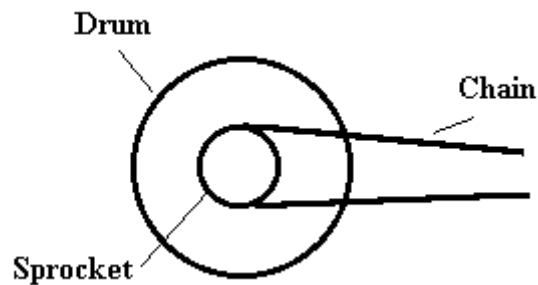


Figure 10

### SOLUTION

$$I = Mk^2 = 200 \times 0.8^2 = 128 \text{ kg m}^2.$$

$$T = F R = 120 \times 0.15 = 18 \text{ N m}$$

$$T = I \alpha$$

$$\alpha = 18/128 = 0.1406 \text{ rad/s}^2$$

$$\omega_1 = 0 \text{ rad/s} \quad \omega_2 = (20/60) \times 2\pi = 2.094 \text{ rad/s}$$

$$\alpha = (\omega_2 - \omega_1)/t = 2.094/t = 0.1406 \text{ rad/s}^2$$

$$t = 2.094/0.1406 = 14.9 \text{ seconds}$$

### WORKED EXAMPLE No.7

A rod of mass 2 kg has a uniform cross section and is 0.8 m long. It is pivoted at one end. The rod is turned in the vertical plane until it makes an angle of  $45^\circ$  with the vertical as shown.

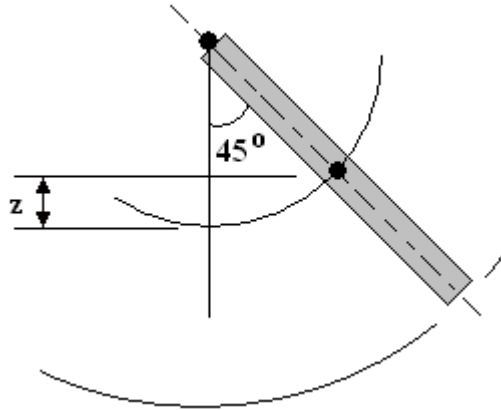


Figure 11

The rod is released and allowed to swing freely. Determine the velocity of the tip as it passes through the vertical

### SOLUTION

The rod is uniform so the centre of gravity is at the centre. Calculate the change in height of the centre of gravity  $z$ . The geometry is shown below.

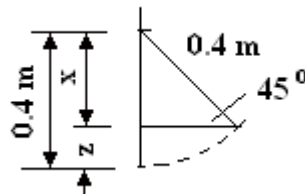


Figure 12

$$x = 0.4 \sin 45^\circ = 0.2828 \text{ m. } z = 0.4 - 0.2828 = 0.117 \text{ m}$$

The potential energy that will be changed into kinetic energy is  $mgz$ .

$$\text{P.E.} = 2 \times 9.81 \times 0.117 = 2.299 \text{ Joules.}$$

$$\text{Kinetic Energy gained} = I\omega^2/2$$

For a plain rod  $I = Mk^2$  and  $k = 0.577 L$  (worked example No.1)

$$k = 0.577 \times 0.8 = 0.4616 \text{ m}$$

$$I = 2 \times 0.4616^2 = 0.426 \text{ kg m}^2$$

Equate KE and PE

$$2.299 = I\omega^2/2 = 0.426 \times \omega^2/2 \quad \omega^2 = 2.299 \times 2/0.426 = 10.79$$

$$\omega = \sqrt{10.79} = 3.28 \text{ rad/s}$$

$$\text{The velocity at the tip of the rod will be } v = \omega r = 3.28 \times 0.8 = 2.63 \text{ m/s}$$

## SELF ASSESSMENT EXERCISE No.2

1. A pendulum similar to that shown on figure 9 has a mass of 20 kg. It is fixed on the end of a rod of mass 10 kg and 1.8 m long. The pendulum is raised through  $45^\circ$  and allowed to swing down and strike the specimen. After impact it swings up  $20^\circ$  on the other side. Determine the following.

- i. The velocity just before impact. (3.33 m/s)
- ii. The energy absorbed by the impact. (102.7 J)

Hints for solution.

Calculate the P.E. of the hammer and rod separately assuming the centre of gravity of the rod is at the middle.

Calculate the moment of inertia of the hammer with a  $k = 1.8$  and for the rod with  $k = 0.577L$ . Equate PE to angular KE and then convert angular velocity to linear velocity.

2. A lift has a mass of 1200 kg. It is raised by a rope passing around a winding drum and a counterbalance mass of 1000 kg hangs down on the other end. The drum has a radius of 1 m and a moment of inertia of  $500 \text{ kg m}^2$ .

During operation, the lift is accelerated upwards from rest at a rate of  $2 \text{ m/s}^2$  for 2 seconds. The lift then rises at constant velocity for another 10 seconds and then the drive torque is removed from the drum shaft and the lift coasts to a halt.

Determine the following.

- i. The maximum and minimum force in rope during this period.  
(14.17 kN and 9.81 kN)
- ii. The torque applied to the drum during the acceleration period.  
(7.362 kNm)
- iii. The rate of deceleration.  
( $0.727 \text{ m/s}^2$ )
- iv. The distance moved by the lift during the journey.  
(55 m)

3. An experiment is performed to find the moment of inertia of a flywheel as follows. A mass is attached by a string to the axle which has a radius of 37.5 mm. The mass is adjusted until its weight is just sufficient to overcome frictional resistance and rotate the flywheel without acceleration. This mass is 2 kg. Another 2.5 kg is added and the mass is allowed to fall under the action of gravity and measurements show that it takes 5 s to fall 1.5 m. Determine the following.

- i. The angular acceleration. (3.2 rad/s<sup>2</sup>)
- ii. The moment of inertia of the flywheel. (0.281 kg m<sup>2</sup>)

4. Figure 13 shows a schematic of a space module orbiting in a gravity free zone. Rockets are attached at points and exert forces  $F_1$ ,  $F_2$  and  $F_3$  as shown all acting in the same plane as the centre of mass G. The forces produce a linear acceleration  $a$  and angular acceleration  $\alpha$  at G. The module has a mass of 1000 kg and moment of inertia 800 kg m<sup>2</sup> about G. Determine the three forces  $F_1$ ,  $F_2$  and  $F_3$ .

(1573 N, 505 N and 1200 N)

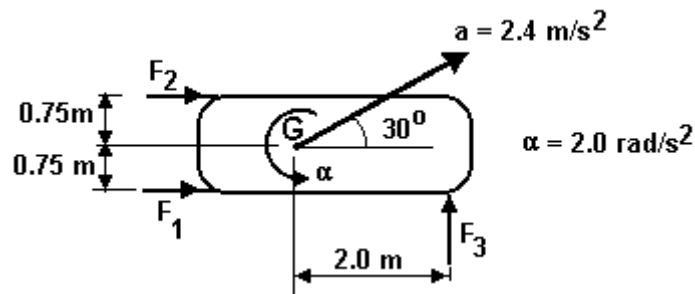


Figure 13

5. A drum which is 300 mm in diameter, has a moment of inertia of 200 kg m<sup>2</sup>. It revolves at 2 rev/s. A braking torque of 285.8 N m is applied to it producing uniform deceleration. Determine the following.

- i. The initial kinetic energy stored in the drum (15791 J)
- ii. The angular deceleration of the drum. (1.43 rad/s<sup>2</sup>).
- iii. The time taken to for the drum to stop. (8.8 s)