SOLID MECHANICS

DYNAMICS

TUTORIAL – LINEAR IMPULSE AND MOMENTUM

This work covers elements of the following syllabi.

- Parts of the Engineering Council Graduate Diploma Exam D225 – Dynamics of Mechanical Systems
- Parts of the Engineering Council Certificate Exam C105 – Mechanical and Structural Engineering.
- Part of the Engineering Council Certificate Exam C103 - Engineering Science.

On completion of this tutorial you should be able to do the following.

- State Newton's laws of motion.
- Define linear momentum and impulse.
- State the law of conservation of momentum.
- Define the coefficient of restitution.
- Solve problems involving the collision of bodies.
- Solve problems involving pile drivers.

1. NEWTON'S LAWS OF MOTION

1. A body at rest or with uniform motion will remain at rest or continue with uniform motion until it is acted on by an external force.

2. An external force will cause the body to accelerate or decelerate.

3. Every force has an equal and opposite reaction.

1.1 EXPLANATION

Imagine a person on an ice rink with absolutely no friction between the skates and the ice. If he was moving, he would be unable to neither slow down nor speed up. The person could only change his motion if an external force was applied to him. This is an example of the first law.

In reality the external force is obtained by finding friction with the ice by digging the skates into the ice and pushing or braking. This force produces changes in the motion of the skater. Using friction to enable him to either accelerate or decelerate is an application of the second law.

Next, imagine the person stationary on the ice. In his hands he has a heavy ball. If he threw the ball away, he would move on the ice. In order to throw the ball away he must exert a force on the ball. In return, the ball exerts an equal and opposite force on the person so he moves away in the opposite direction to the ball. This is an example of the third law.

The same principles apply to a space vehicle. There is no friction in space and the only way to change the motion of a space vehicle is to eject matter from a rocket so that the reaction force acts on the vehicle and changes its motion.

The law which has the greatest significance for us is the 2nd. law so let's look at this in detail.

2. NEWTON'S 2nd LAW OF MOTION

We usually think of the second law as stating **Force = mass x acceleration**. In fact it should be stated in a more fundamental form as follows.

The IMPULSE given to a body is equal to the change in MOMENTUM.

This requires us to make a few definitions as follows.

IMPULSE

IMPULSE is defined as the product of force and the time for which it is applied.

Impulse = Force x Time = Ft

WORKED EXAMPLE No.1

A vehicle has a force of 400 N applied to it for 20 seconds. Calculate the impulse?

SOLUTION

IMPULSE = Ft = 400 x 20 = 8000 N s

MOMENTUM

MOMENTUM is defined as the product of mass and velocity.

Momentum = Mass x Velocity = m u kg m/s

WORKED EXAMPLE No.2

A vehicle of mass 5 000 kg changes velocity from 2 m/s to 6 m/s. Calculate the change in momentum.

SOLUTION

Initial momentum = mu_1 = 5 000 x 2 = 10 000 kg m/s
Final momentum = mu_2 = 5 000 x 8 = 40 000 kg m/s

Change in momentum = 40 000 - 10 000 = 30 000 kg m/s

REWRITING THE LAW

From the statement Impulse = change in momentum, the second law can be written as

$$\mathbf{Ft = \Delta mu}$$

This equation may be rearranged into other forms as follows. $F = \Delta mu/t$

If the mass is constant and since acceleration = $\Delta u/t$ this becomes $F = m\ \Delta u/t = m\ a$

This is the most familiar form but if the mass is not constant then we may use

$$F = u\ \Delta m/t = \text{velocity x mass flow rate}$$

This form is used in fluid flow to solve forces on pipe bends and turbine blades.

The form we are going to use is **$F = \Delta mu/t$ = rate of change of momentum**

This form of the law is used to determine the changes in motion to solid bodies.

Let's use this to study what happens to bodies when they collide.

3. COLLISIONS

When bodies collide they must exert equal and opposite forces on each other for the same period of time so the impulse given to each is equal and opposite. Since the impulse is equal to the rate of change of momentum, it follows that each body will receive equal and opposite changes in their momentum. It further follows that the total momentum before the collision is equal to the total momentum after the collision. This results in the law of conservation of momentum.

3.1 THE LAW OF CONSERVATION OF MOMENTUM.

The total momentum before a collision is equal to the total momentum after the collision.

Consider two bodies of mass m_1 and m_2 moving at velocities u_1 and u_2 in the same direction. After collision the velocities change to v_1 and v_2 respectively.

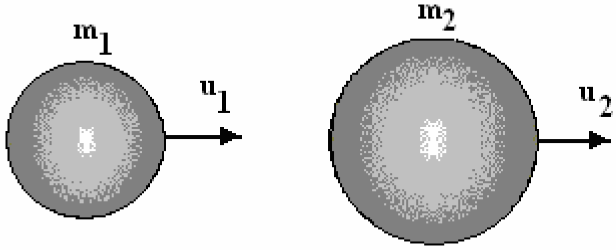

Figure 1

The initial momentum = $m_1\ u_1 + m_2\ u_2$

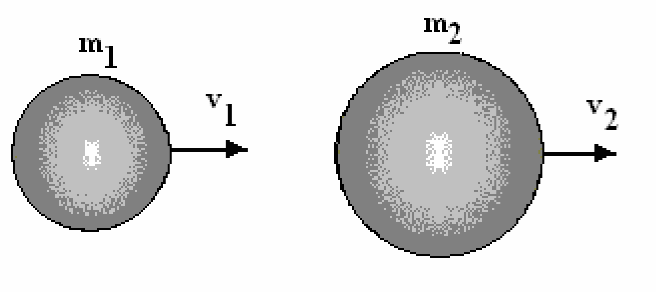

Figure 2

The Final momentum = $m_1 v_1 + m_2 v_2$

By the law of conservation we have

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2 \text{ divide by each term by } m_1$$

$$u_1 + \frac{m_2}{m_1}u_2 = v_1 + \frac{m_2}{m_1}v_2 \text{ rearrange } \frac{m_2}{m_1}(u_2 - v_2) = v_1 - u_1$$

$$\frac{m_2}{m_1} = \frac{(v_1 - u_1)}{(u_2 - v_2)} \text{ multiply by } \frac{-1}{-1} \qquad \frac{m_2}{m_1} = \frac{(u_1 - v_1)}{(v_2 - u_2)} \text{...............(1)}$$

3.2 ENERGY CONSIDERATIONS

The law of conservation of momentum is true regardless of any energy changes that may occur. However, in order to solve the velocities, we must consider the energy changes and the easiest case is when no energy is lost at all. The only energy form to be considered is kinetic energy.

$$\text{Total K.E. before the collision} = \frac{m_1u_1^2}{2} + \frac{m_2u_2^2}{2}$$

$$\text{Total K.E. AFTER the collision} = \frac{m_1v_1^2}{2} + \frac{m_2v_2^2}{2}$$

$$\text{If no K.E. is lost then } \frac{m_1u_1^2}{2} + \frac{m_2u_2^2}{2} = \frac{m_1v_1^2}{2} + \frac{m_2v_2^2}{2} \text{ multiply every term by } \frac{2}{m_1}$$

$$u_1^2 + \left(\frac{m_2}{m_1}\right)u_2^2 = v_1^2 + \left(\frac{m_2}{m_1}\right)v_2^2 \text{ ....(2)}$$

Substitute equation (1) into (2)

$$u_1^2 + \frac{(u_1 - v_1)u_2^2}{v_2 - u_2} = v_1^2 + \frac{(u_1 - v_1)v_2^2}{v_2 - u_2} \text{ multiply each term by } (v_2 - u_2)$$

$$(v_2 - u_2)u_1^2 + (u_1 - v_1)u_2^2 = (v_2 - u_2)v_1^2 + (u_1 - v_1)v_2^2$$

$$(v_2 - u_2)(u_1^2 - v_1^2) = (u_1 - v_1)(v_2^2 - u_2^2)$$

Factorising gives

$$(v_2 - u_2)(u_1 - v_1)(u_1 + v_1) = (u_1 - v_1)(v_2 - u_2)(v_2 + u_2)$$

$$(u_1 - u_2) = -(v_1 - v_2) \text{ ....(3)}$$

(u_1- u_2) is the relative velocity between the two bodies before they collide.

($v_1 - v_2$) is the relative velocity between them after the collision.

It follows that if no energy is lost then the relative velocity before and after the collision is equal and opposite. This means that the velocity with which they approach each other is equal to the velocity at which they separate.

If energy is lost in the collision then it follows that **$(u_1 - u_2) > -(v_1 - v_2)$**

In order to solve numerical problems when energy is lost we use the ***COEFFICIENT OF RESTITUTION*** which is defined as

$$e = -\frac{v_1 - v_2}{u_1 - u_2} \text{.....................(4)}$$

These equations must be remembered.

WORKED EXAMPLE No.3

A mass of 100 kg moves along a straight line at 1 m/s. It collides with a mass of 150 kg moving the opposite way along the same straight line at 0.6 m/s. The two masses join together on colliding to form one mass. Determine the velocity of the joint mass.

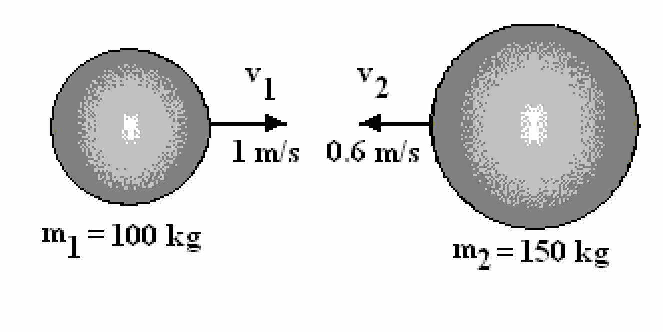

Figure 3

SOLUTION

The normal sign convention must be used namely that motion from left to right is positive and from right to left is negative.

$m_1 = 100$ kg $m_2 = 150$ kg
$u_1 = 1$ m/s $u_2 = -0.6$ m/s

Initial momentum = (100 x1) + {150 x (-0.6)} = 10 kg m/s
Final momentum = 10 kg m/s (conserved)

After collision the mass is 250 kg and the velocity is v.

Final momentum = 250 v = 10 v = 10/250 = 0.04 m/s

The combined mass ends up moving to the left at 0.04 m/s.

Notice that because the masses joined together, equation 3 was not needed.

WORKED EXAMPLE No.4

A mass of 5 kg moves from left to right with a velocity of 2 m/s and collides with a mass of 3 kg moving along the same line in the opposite direction at 4 m/s. Assuming no energy is lost, determine the velocities of each mass after they bounce.

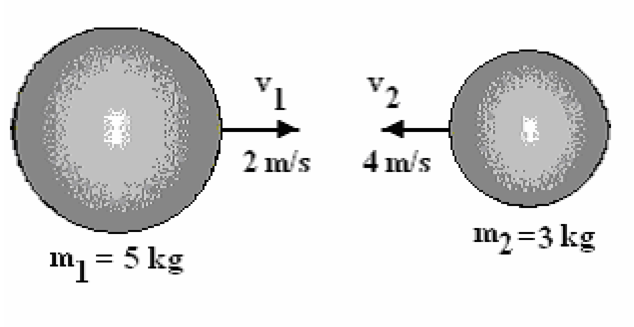

Figure 4

SOLUTION

m_1= 5 kg m_2= 3 kg
u_1= 2 m/s u_2= - 4 m/s

Initial momentum = 5 x 2 + 3 x (-4) = - 2 kg m/s
Final momentum = $5v_1 + 3\,v_2 = -2$ ....(a)

Relative velocity before collision = $(u_1 - u_2) = 2 - (-4) = 6$ m/s (coming together).

Since no energy is lost the coefficient of restitution is 1.0.

Relative velocity after collision = $(v_1 - v_2) = -1(6) = -6$ m/s (parting)..........(b)

The velocities may be solved by combining the simultaneous equation a and b. One way is as follows. Equation (b) x 3 + equation (a) yields

$3v_1 - 3v_2 = -18$
$5v_1 + 3v_2 = -2$
$8v_1 + 0 = -20$ hence $v_1 = -2.5$ m/s and $v_2 = +3.5$ m/s

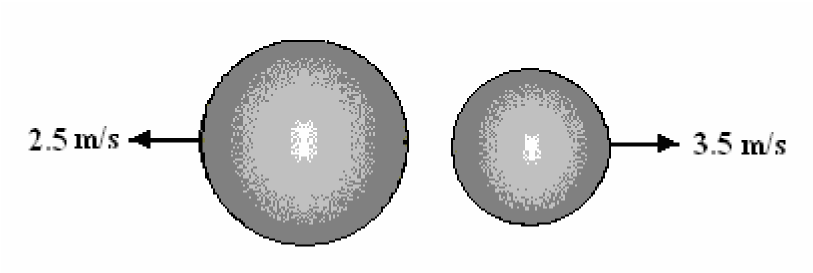

Figure 5

WORKED EXAMPLE No.5

Repeat example 2 but this time there is energy lost such the coefficient of restitution is 0.6.

SOLUTION

This time we have

$m_1 = 5$ kg $\quad$ $m_2 = 3$ kg

$u_1 = 2$ m/s $\quad$ $u_2 = -4$ m/s

Initial momentum = 5 x 2 + 3 x (-4) = - 2 kg m/s

Final momentum = $5v_1 + 3v_2 = -2$ ....(a)

Relative velocity before collision = $(u_1 - u_2) = 2 - (-4) = 6$ m/s (coming together).

Relative velocity after collision = $(v_1 - v_2) = -(0.6)(6) = -3.6$ m/s (parting)......(b)

The velocities may be solved by combining the simultaneous equation a and b. One way is as follows.

$5v_1 + 3v_2 = -2$ hence $v_1 = -0.4 - 0.6v_2$

$v_1 + v_2 = -3.6$ $\quad$ $v_1 = -3.6 + v_2 = -0.4 - 0.6v_2$

hence $v_1 = -1.6$ m/s $\quad v_2 = +2.0$ m/s

SELF ASSESSMENT EXERCISE No.1

1. A mass of 20 kg travels at 7 m/s and collides with a mass of 12 kg travelling at 20 m/s in the opposite direction along the same line. The coefficient of restitution is 0.7. Determine the velocities after collision.

   (Ans. -10.21 m/s and 8.687 m/s)

2. A mass of 10 kg travels at 8 m/s and collides with a mass of 20 kg travelling along the same line in the same direction at 4 m/s. The coefficient of restitution is 0.8. Determine the velocities after collision.

   (Ans. 3.2 m/s and 6.4 m/s)

3. A railway wagon of mass 3 000 kg travelling at 0.42 m/s collides with a stationary wagon of mass 3 500 kg and becomes coupled. Determine the common velocity after collision.

   (Ans. 0.194 m/s)

4. A mass of 4 kg moves at 5 m/s along a straight line and collides with a mass of 2 kg moving at 5 m/s in the opposite direction. The coefficient of restitution is 0.7. Calculate the final velocities.

   (Ans. -0.67 m/s and 6.33 m/s)

4. PILE DRIVERS

A particular application of the preceding section is to pile drivers. These are devices which drop large masses onto a pile in order to drive the pile into the ground. The pile has no initial momentum and the motion given to it is quickly decelerated by the frictional resistance as it moves into the ground. The velocity of the driver is obtained by gravitational acceleration and conversion of potential energy into kinetic energy.

WORKED EXAMPLE No.6

A pile driver has a mass of 100 kg and falls 3 m onto the pile which has a mass of 200 kg. The coefficient of restitution is 0.7. Calculate the velocity of the pile and the driver immediately after impact.

SOLUTION

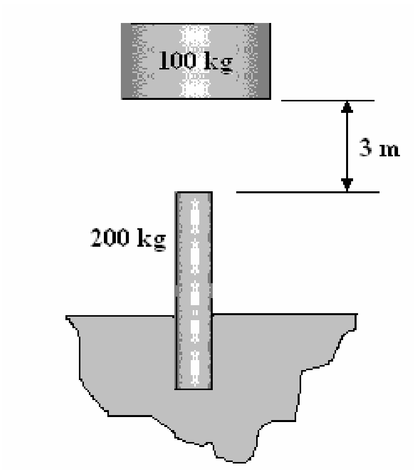

Figure 6

Initial potential energy of driver = mgz
Kinetic energy at impact = $mu_1^2/2$
Equating $mgz = mu_1^2/2$
$u_1 = (2gz)^{½} = (2 \times 9.81 \times 3)^{1/2} = 7.672$ m/s
Since this is down, by normal convention it is negative.
$u_1 = -7.672$ m/s
Initial velocity of the pile $u_2 = 0$
Initial momentum = -100 x 7.672 = -767.2 kg m/s
Final momentum = $m_1 v_1 + m_2 v_2$

$100v_1 + 200v_2 = -767.2$ kg m/s ....(1)

Initial relative velocity = $u_1 - u_2 = -7.672$ m/s coming together.
Final relative velocity = $v_1 - v_2 = -0.7(-7.672) = 5.37$ m/s parting.
$v_1 - v_2 = 5.37$ m/s ....(2)

Solving equations (1) and (2) we have
$v_1 = 1.023$ m/s (upwards)
$v_2 = -4.35$ m/s (downwards)

WORKED EXAMPLE No.7

Using the same data as for problem No.6, determine the height to which the driver rebounds. The pile is driven into the ground 0.08 m. Determine the average ground resistance assuming uniform deceleration.

SOLUTION

The K.E. of the driver is reconverted into P.E. so the height to which it rebounds is
$z_2 = v_1^2/2g = (1.023)/(2^2 \times 9.81) = 0.053$ m

From the laws of motion covered in earlier tutorials, for uniform deceleration

$x = at^2/2 = v^2/2a$

$0.08 = (-4.35)^2/2a$ hence $a = 118.26$ m/s^2

Since the pile is decelerating, then strictly $a = -118.26$ m/s^2

From Newton's 2nd Law of motion we have

$F = ma = 200(-118.26) = -26\,650$ N

The negative sign indicates F is a resistance and not a help to motion.

SELF ASSESSMENT EXERCISE No.2

1. a) A mass m_1 moves along a straight line with velocity v before colliding with a mass m_2. Following the impact, the two masses continue to move along the same line with velocities u_1 and u_2 respectively. Show that

$$u_1 = v(m_1 - em_2)/(m_1 + m_2)$$
$$u_2 = vm_1 (1 + e)/(m_1 + m_2)$$

The coefficient of restitution is defined as $e = (u_2 - u_1)/v$

b) A pile of mass 500 kg is driven into the ground by a ram of mass 1000 kg. On the final stroke, the ram descends 5 m freely under gravity and the pile is driven 0.025 m into the ground. The coefficient of restitution is 0.5. Determine

i. the velocity of the ram just before impact. (Ans. 9.9 m/s)
ii. the velocities of the ram and pile immediately after impact.
(Ans. 4.95 and 9.9 m/s both downwards)
iii. the resisting force of the ground assuming uniform deceleration. (Ans. 981 kN)
iv. the energy lost in the impact. (Ans. 12.3 kJ)

2. A projectile of mass 30 grammes travels horizontally at velocity v. It embeds itself in a stationary sand bag of mass 10 kg which is suspended on a rope. The centre of mass of the bag is 1.2 m below the point of suspension. After being struck, the bag swings freely through an arc of 30^o. Determine the velocity v and kinetic energy lost in the impact. (Answers 593.8 m/s and 5 273 J)

Hint... the easiest way to do this is by use of potential and kinetic energy as well as momentum theory).

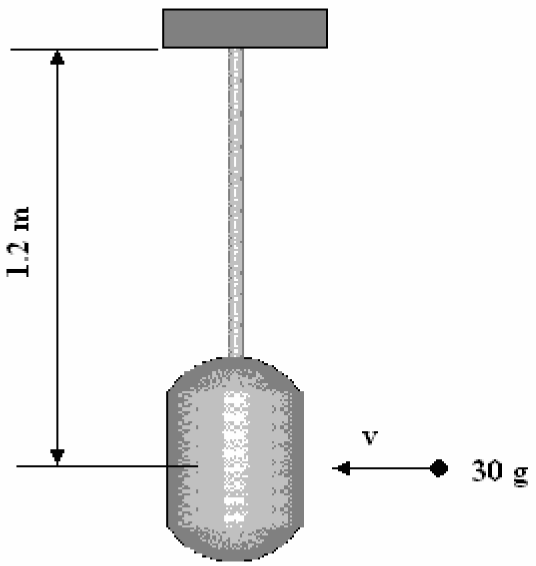

Figure 7

3. a) A mass m_1 moves along a straight line with velocity u_1 before colliding with a mass m_2 moving in the same direction at velocity u_2 (hence $u_1>u_2$). Following the impact, the two masses continue to move along the same line with velocities v_1 and v_2 respectively. Show that

$$(v_2 - v_1) < -(u_2 - u_1)$$

b) A pile of mass 600 kg is driven into the ground by a ram of mass 300 kg. On the final stroke, the ram descends 3 m freely under gravity and the pile is driven into the ground. The coefficient of restitution is 0.8 and the resisting force of the ground is a constant value of 40 kN. Determine

i. The depth of penetration of the pile into the ground. (Ans. 0.0159 m)
ii. The height to which the ram rebounds. (Ans. 0.12 m)
iii. The energy lost in the impact. (Ans. 2114 J)

Note that the coefficient of restitution is defined as

$$(v_2 - v_1) = -e(u_2 - u_1)$$