SOLID MECHANICS
DYNAMICS
GYROSCOPES

On completion of this tutorial you should be able to

- Describe a gyroscope.
- Define angular momentum.
- Derive the formula for gyroscopic torque.
- Solve problems involving gyroscopic torque
- Define precession.

Note that a rigorous mathematical analysis of a gyroscopic is difficult and required very advanced techniques. The following work does not reveal the real depth of study required for accurate analysis of problems involving spinning bodies.

Contents

1. *Corkscrew Rule*

This is the rule that allows us to represent angular quantities such as velocity, momentum and torque as vectors. Point the index finger of the right hand so that rotating it clockwise is the same direction as the rotating body. The direction you are pointing is the direction of the vector. The motion is the same as doing up a corkscrew hence the name.

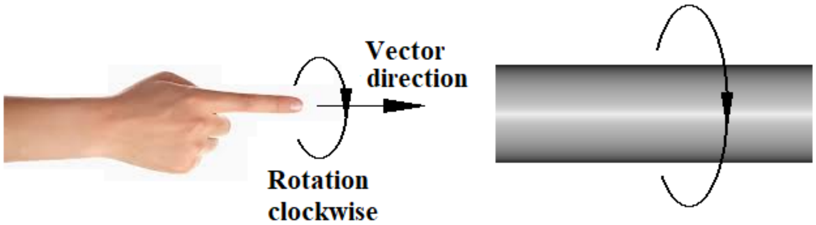

Figure 1

2. *Gyroscopes*

A Gyroscope is a spinning disc mounted in gimbals so that it may pivot in the x, y and z axis.

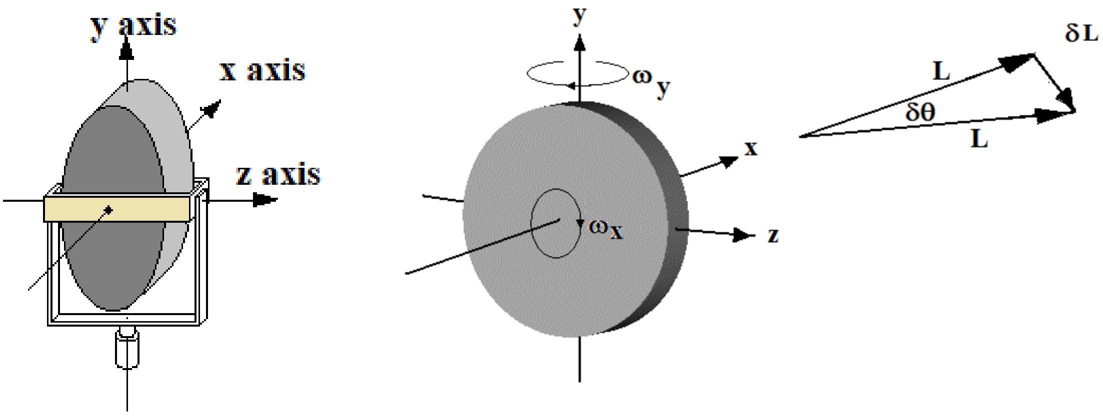

Figure 2

Now consider a disc spinning about the x axis with velocity ω_x as shown. The angular momentum of the disc is $L = I\omega_X$. This is a vector quantity and the vector is drawn with a direction conforming to the corkscrew rule. Suppose the disc also rotates about the y axis as shown through a small angle $\delta\theta$. The vector for L changes direction but not magnitude. This produces a change in the angular moment of

$$\delta L = \delta(I\omega_X)$$

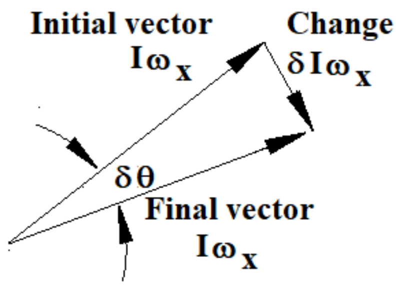

Figure 3

The vector diagram conforms to the vector rule, first vector + change = final vector. The change is the arrow going from the tip of the first to the tip of the second so the direction is as shown.

The vector representing the change is almost an arc of radius $I\omega_X$ and angle $\delta\theta$. The length of the arc is the product of radius and angle. Taking the radius as $I\omega_X$ and the angle as $\delta\theta$ the change is

$$\delta(I\omega_X) = I\omega_X\,\delta\theta = L\,\delta\theta$$

Newton's second law of motion applied to rotating bodies tells us that the change in momentum can only be brought about by applying a torque.

Torque = rate of change of angular momentum.

If the rotation occurred in time δt seconds, the rate of change of momentum is

$$L\frac{\delta\theta}{\delta t} = L\omega_y \qquad \frac{\delta\theta}{\delta t} \text{ is the angular velocity } \omega_y \text{ hence}$$

$$\mathbf{T = L\omega_y = I\omega_x\omega_y}$$

This is the torque that must be applied to produce the change in angle and the direction of the vector is the same as the change in momentum. The applied torque may hence be deduced in magnitude and direction. On the diagram the rotation $\delta\theta$ is about the y axis. The change $\delta I\omega_x$ is in the direction shown. The applied torque is clockwise when this vector is viewed from the back of the arrow and this is a rotation about the z axis. It follows that the vector for the applied torque is about the z axis.

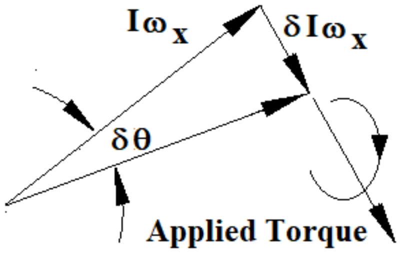

Figure 4

If the torque is applied about the z axis, the result will be rotation about the y axis and this is called ***precession***.

If the torque is not applied and the rotation is made to happen (applied), a reaction torque will be produced (Newton's 3 rd. law) and the disc will respond to the reaction torque. The reaction torque is equal and opposite.

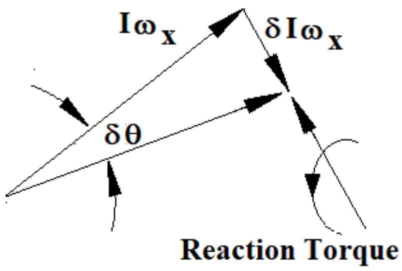

Figure 5

A gyroscopic torque may occur in any machine with rotating parts if a change in the direction of the x axis occurs. Examples are aeroplanes, ships and vehicles where a gyroscopic torque is produced by the engines when a change is made in the course. You can see video of this phenomenon at the following link - http://physics.nad.ru/Physics/English/gyro_tmp.htm

3. Spinning Tops

A top is a spinning body that is symmetrical about its axis of spin. This could be a cylinder or a cone rotating about its axis.

If the axis is inclined at angle ϕ as shown and the top is spinning at ω rad/s about its axis, it will have angular momentum $L = I\omega$ in the direction shown. The diagram shows the top precessing about the vertical axis that intercepts with the axis of spin at O. Point O is a fixed point so the axis of spin forms a surface of a cone as it precesses.

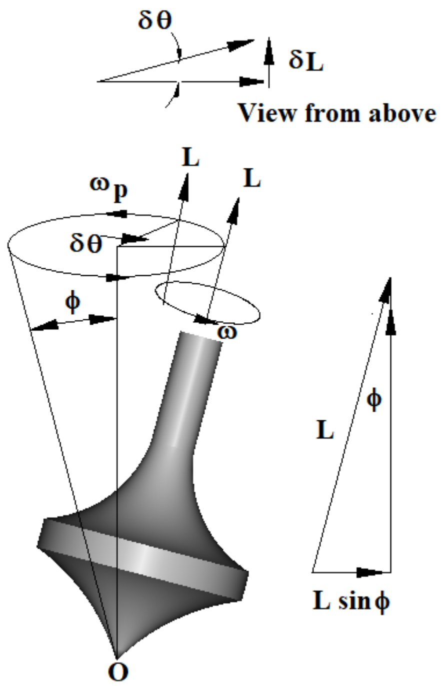

Figure 6

Suppose the axis of spin rotates through an angle $\delta\theta$ in time δt.

The change in angular momentum is about the axis of spin is

$$\delta L = L\delta\theta$$

The change in the horizontal plane is

$$\delta L = L \sin\phi \ \delta\theta$$

Divide through by the time it takes to happen

$$\frac{\delta L}{\delta t} = L\sin\phi \ \frac{\delta\theta}{\delta t}$$

In the limit when the change is infinitesimally small

$$\frac{dL}{dt} = L\sin\phi \ \frac{d\theta}{dt} = L\sin\phi \ \omega_p$$

The rate of change of angular momentum must be produced by a torque equal to T so

$$T = L \sin\phi \ \omega_p$$

The direction may be deduced from the corkscrew rule from the direction of the change. If the axis of spin is vertical $\phi = 0$ and $T = 0$ and $\omega_p = 0$

At any other angle, there must be an applied torque and this could be due to the weight.

WORKED EXAMPLE No. 1

A top consists of a spinning disc of radius 50 mm and mass 0.8 kg mounted at the end of a light rod as shown. If the disc rests on a pivot with its axis of spin horizontal as shown, and the distance X is 30 mm, calculate the velocity of the precession when its spins at 40 rev/min.

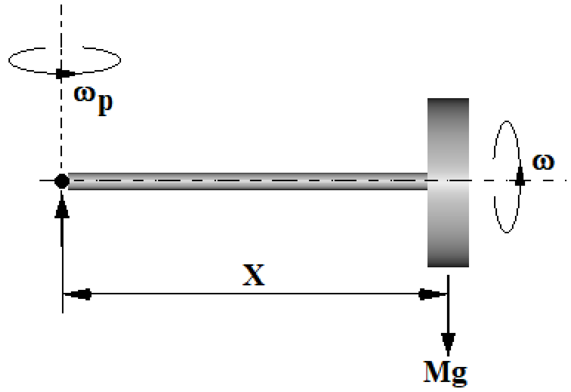

Figure 7

SOLUTION

The angle ϕ is 90^o so $T = L \sin\phi\ \omega_p = L\ \omega_p$

For a plain disc

$$I = M\frac{R^2}{2} = 0.8 \times \frac{0.05^2}{2} = 0.004 \text{ kg m}^2$$

$$\omega = 2\pi \times 50 = 100\pi \text{ rad/s}$$

$$L = I\omega = 0.004 \times 100\pi = 1.256 \text{ kg m}^2/\text{s}$$

T is due to the weight acting at 30 mm from the rest point.

$$T = MgX = 0.8 \times 9.81 \times 0.03 = 0.235 \text{ N m}$$

$$\omega_p = \frac{T}{L} = \frac{0.235}{1.256} = 0.187 \text{ rad/s}$$

WORKED EXAMPLE No. 2

A top consists of a spinning disc of radius 40 mm and mass 0.5 kg mounted at the end of a light rod as shown. The distance from the tip to the centre of gravity is 100 mm. Calculate the velocity of the precession when its spins at 30 rev/s.

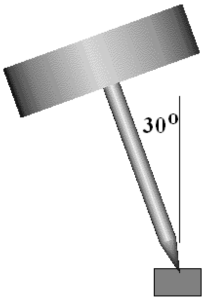

Figure 8

SOLUTION

The angle ϕ is 30^o so $T = L \sin\phi\ \omega_p = 0.5L\ \omega_p$

For a plain disc

$$I = M\frac{R^2}{2} = 0.5 \times \frac{0.04^2}{2} = 0.0004 \text{ kg m}^2$$

$$\omega = 2\pi \times 30 = 60\pi \text{ rad/s}$$

$$L = I\omega = 0.0004 \times 60\pi = 0.0754 \text{ kg m}^2\text{/s}$$

The moment arm X is 100 cos 30^o = 86.6 mm

$$T = MgX = 0.5 \times 9.81 \times 0.0866 = 0.424 \text{ N m}$$

$$\omega_p = \frac{T}{L} = \frac{0.424}{0.0754} = 5.63 \text{ rad/s}$$

WORKED EXAMPLE No. 3

A cycle takes a right hand bend at a velocity of v m/s and radius R. Show that the cyclist must lean into the bend in order to go around it.

SOLUTION

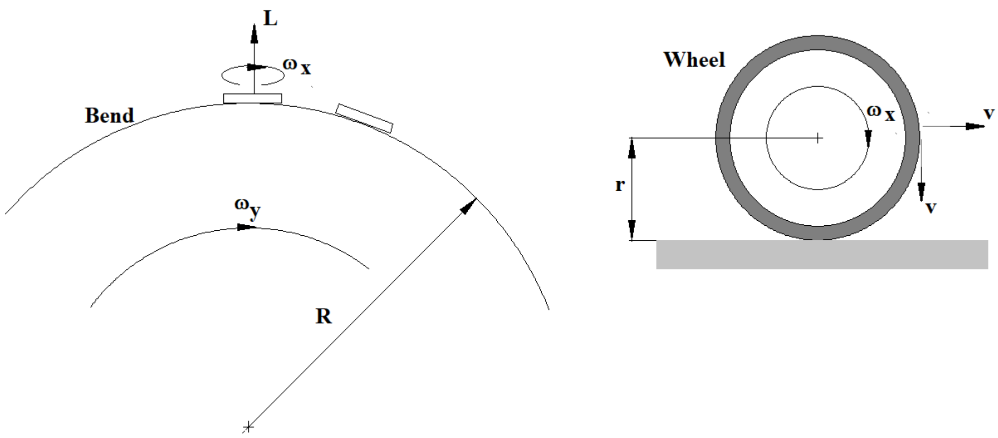

Figure 9

First remember that the velocity of the edge of the wheel must be the same as the velocity of the bike so

$$\omega_x = \frac{v}{r} \qquad \text{r is the radius of the wheel}$$

$$\omega_y = \frac{v}{R} \qquad \text{R is the radius of the bend}$$

$$L = I\omega_x$$

Now sketch the vector diagrams to determine the change. As the wheel goes around a right hand bend the direction of change in momentum is δL as shown. This is in the tangential direction at all times. The applied torque is a vector in the same direction as the change so we deduce that the applied torque must act clockwise viewed from behind the cyclist. This torque may be produced be leaning into the bend and letting gravity do the job. By applying this torque, the wheels must precess in the correct direction to go round the bend.

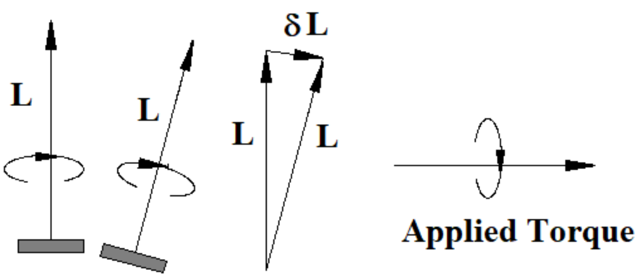

Figure 10

If the cyclist steers around the bend by turning the handlebars, then the reaction torque would throw him over outwards (anticlockwise as viewed from the back).

WORKED EXAMPLE No. 4

A ship has its turbine engine mounted with its axis of rotation lengthways in the ship. The engine rotates clockwise at 6 000 rev/min when viewed from the back. The effective rotating mass of the engine is 900 kg with a radius of gyration of 0.5 m.

Calculate the magnitude of the gyroscopic couple produced when the ship turns right on a radius of 300 m with a velocity of 2.2 m/s. Explain clearly the effect of the couple on the ships motion.

SOLUTION

The essential quantities are N = 6 000 rev/min
M = 900 kg k = 0.5 m v = 2.2 m/s R = 300 m
First calculate the angular velocity about the x axis.

$$\omega_x = \frac{2\pi N}{60} = \frac{2\pi \times 6\,000}{60} = 628.3 \text{ rad/s}$$

Next calculate the angular velocity of precession from the linear velocity

$$\omega_y = \frac{v}{R} = \frac{2.2}{300} = 7.333 \times 10^{-3} \text{ rad/s}$$

Next find the moment of inertia

$$I = Mk^2 = 900 \times 0.5^2 = 225 \text{ kg m}^2$$

Calculate the gyroscopic torque

$$T = I\omega_x\omega_y = 225 \times 628.3 \times 7.333 \times 10^{-3} = 1\,036.7 \text{ N m}$$

Now sketch the motion of the ship taking a right turn. Draw the vectors for the angular momentum using the corkscrew rule. The rotation is clockwise viewed from the back so the vector must point forward as shown.

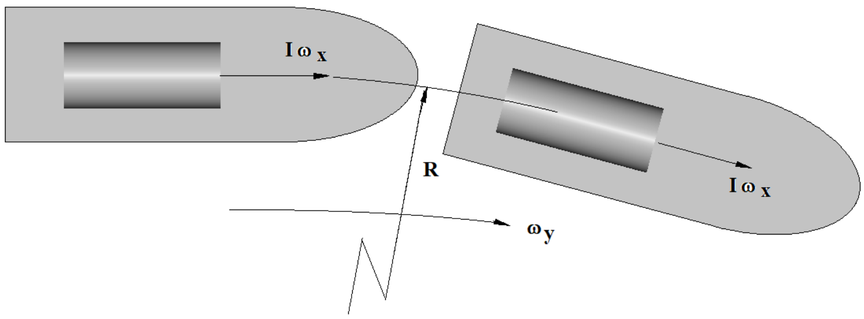

Figure 11

Now deduce the effect of the torque.

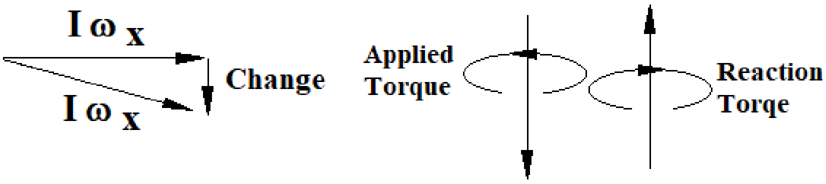

Figure 12

The vectors for the angular momentum are drawn as shown and the direction of the change is deduced (from the tip of the first to the tip of the second). The reaction torque is the opposite direction as shown (up on the diagram). From this we apply the corkscrew rule and deduce that the torque produces a rotation such that the bow of the ship dips down and the stern comes up. The magnitude of the torque is 1 037 Nm.

WORKED EXAMPLE No. 5

The engine of an aeroplane rotates clockwise when viewed from the front. The moment of inertia of all the rotating parts is 300 kg m^2. The engine rotates at 1 200 rev/min.

Determine the magnitude and effect of the gyroscopic action resulting when the aeroplane makes a right hand bend of radius 5000m at a speed of 1500 km/h.

SOLUTION

The essential data is as follows.

$I = 300 \text{ kg m}^2$ $\quad R = 5\,000 \text{ m}$ $\quad N = 1\,200/60 = 20 \text{ rev/s}$

$v = 1\,500\,000/3\,600 = 416.7 \text{ m/s}$

$$\omega_x = 2\pi N = 2\pi \times 20 = 125.67 \text{ rad/s}$$

$$\omega_y = \frac{v}{R} = \frac{416.7}{5\,000} = 0.08334 \text{ rad/s}$$

$$T = I\,\omega_x\,\omega_y = 300 \times 125.67 \times 0.08334 = 3\,142 \text{ Nm}$$

Now sketch the motion of the aeroplane making a right turn. Draw the vectors for the angular momentum using the corkscrew rule. The rotation is clockwise viewed from the front so the vector must point backwards as shown.

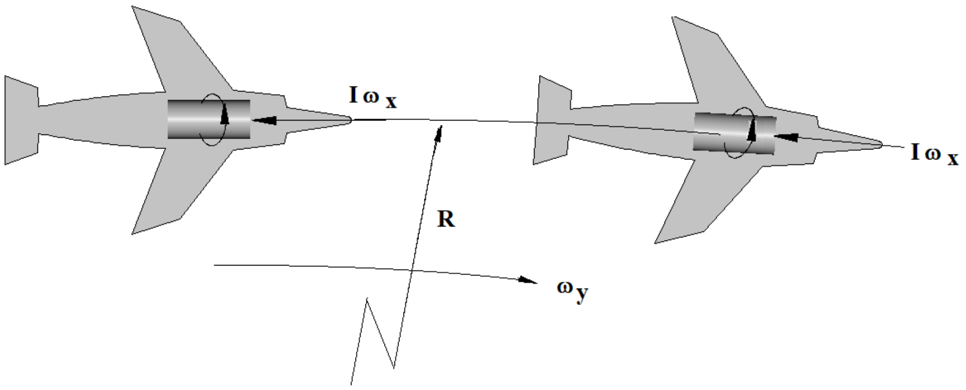

Figure 13

Now deduce the effect of the gyroscopic torque.

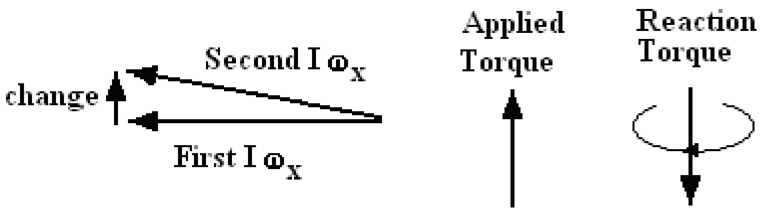

Figure 14

The change in angular momentum is a vector vertical as drawn above. The applied torque is a vector in the same direction and the reaction torque is the opposite (down on the diagram). Applying the corkscrew rule, the reaction torque would tend to lift the nose and depress the tail (pitch). The magnitude is 3 142 N m.

Note that if a torque was applied so that the nose pitched down, the aeroplane would precess and make a right turn.

4. Gyro Compasses and Gyro Stabilisers

A gyroscope with three degrees of freedom will rotate with the earth in a direction East to West. Due to the precession produces it will always settle with its axis of spin on the North South line or true meridian. This is a gyro compass.

Any rotating body has a tendency to preserve its plane of rotation. Think of a spinning top with no gravitational pull. If it its axis of spin changed due to a disturbance, in the absence of a gravitational torque, the reaction would make it return to its original axis.

A spinning bullet will be more stable in flight than one not spinning. Missiles and torpedoes fitted with gyroscopes will maintain their trajectory better than ones without. These are gyro stabilisers and usually contain more than one gyroscope mounted at right angles.

SELF ASSESSMENT EXERCISE No. 1

1. A ship has an engine that rotates clockwise when viewed from the front. It has an effective moment of inertia of 250 kg m^2 and rotates at 400 rev/min. Calculate the magnitude and effect of the gyroscopic torque when the ship pitches bow (front) down at a rate of 0.02 rad/s.
   (Answer 209.4 Nm causes the ship to yaw to the right.)

2. A motorcycle travels at 80 km/h around a left bend of radius 30 m. The wheels have a mass of 2.8 kg. an outer diameter of 0.5 m and a radius of gyration of 240 mm. Calculate the following.
   i. The angular velocity of the wheels. (88.89 rad/s)
   ii. The moment of inertia of each wheel. (0.161 kg m^2)
   iii. The magnitude of the gyroscopic torque produced on the bike. (21.22 Nm)

   Predict the affect of this torque.
   (It tends to tip the bike over to the outside of the bend)

3. Explain with the aid of vector diagrams why a motorcyclist going around a left bend must lean into it in order to go round it.

4. A spinning top has a radius of gyration of 5 mm. The centre of gravity is 30 mm from the tip about which it spins. Calculate the angular velocity required to make it precess at 0.2 rad/s and lean at 20^o to the vertical. (55 311 rad/s)