SOLID MECHANICS

DYNAMICS

TUTORIAL – FORCED VIBRATIONS

This work covers elements of the syllabus for the Engineering Council Exam D225 – Dynamics of Mechanical Systems and C105 Mechanical and Structural Engineering.

On completion of this tutorial you should be able to do the following.

- Define a forced vibration in general terms.
- Explain the whirling of shafts and solve problems.
- Solve problems involving mass – spring – damper systems.
- Analyse the case of a harmonic disturbing force.
- Analyse the case of a harmonic disturbance of the support.
- Analyse the frequency response for the same.
- Define and use phasors.

This tutorial covers the theory of forced vibrations. You should study the tutorial on free vibrations before commencing.

CONTENTS

1. INTRODUCTION

In the tutorial on damped oscillations, it was shown that a free vibration dies away with time because the energy trapped in the vibrating system is dissipated by the damping. The equation for the displacement in a damped oscillation was derived and given as

$$x = Ce^{-\delta\omega_n t}\cos(\omega t)$$

δ is the damping ratio and ω_n the natural angular frequency. The following cases were described.

When δ>1 we have an over damped system.
When δ =1 we have a critically damped oscillation.
When δ< 1we have a damped oscillation that dies away with time.
When δ = 0 we have a system with no damping and a steady oscillation occurred
It might be inferred from this pattern that if δ<0 we get an oscillation that grows with time. The diagram illustrates this pattern.

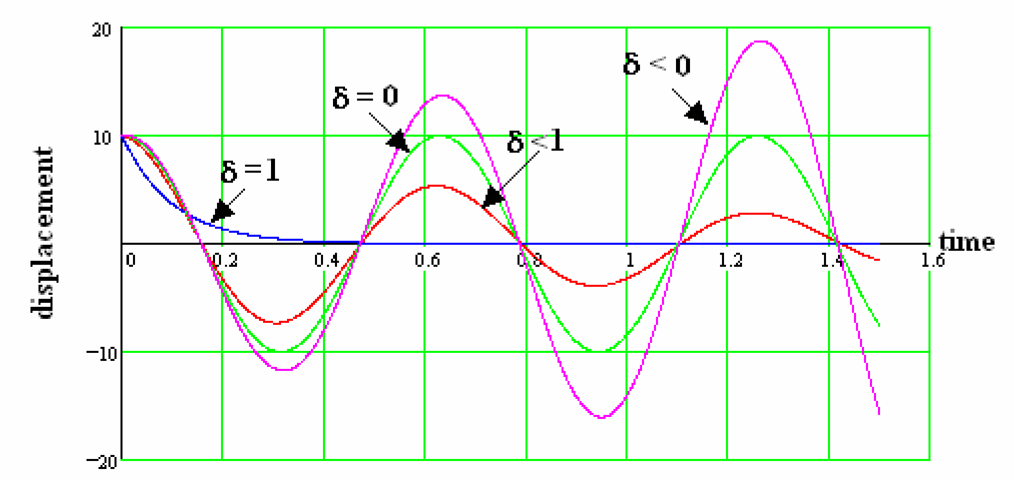

Figure 1

In order for the damping ratio δ to be less than zero, that is, to be negative, we would have to have the opposite of damping, something that puts energy into the system instead of taking it out. As the energy is added to the system the amplitude grows and grows. The energy is added by an outside source and such oscillations are called forced, (the object of this tutorial). A good example of such an oscillation is a child on a swing. If nothing is done, friction will make the swing come to a halt. If someone gives the swing a small push at he start of each swing, energy is added to the system and the swing goes higher and higher. This phenomenon is also known as excitation.

In engineering, many structures are prone to vibrate when excited at or near the natural frequency. A good example is what happens to a car when the wheels are out of balance or when you drive along a corrugated surface. If the disturbance is close to the natural frequency of the suspension system the vehicle might bounce out of control. Vehicles are fitted with dampers to prevent this. The wind blowing around chimney stacks, cooling towers and suspended cables can excite them into catastrophic oscillations.

The first case to be examined here is that of whirling shafts.

2. WHIRLING OF SHAFTS

When a shaft rotates, it may well go into transverse oscillations. If the shaft is out of balance, the resulting centrifugal force will induce the shaft to vibrate. When the shaft rotates at a speed equal to the natural frequency of transverse oscillations, this vibration becomes large and shows up as a whirling of the shaft. It also occurs at multiples of the resonant speed. This can be very damaging to heavy rotary machines such as turbine generator sets and the system must be carefully balanced to reduce this effect and designed to have a natural frequency different to the speed of rotation. When starting or stopping such machinery, the critical speeds must be avoided to prevent damage to the bearings and turbine blades. Consider a weightless shaft as shown with a mass M at the middle. Suppose the centre of the mass is not on the centre line.

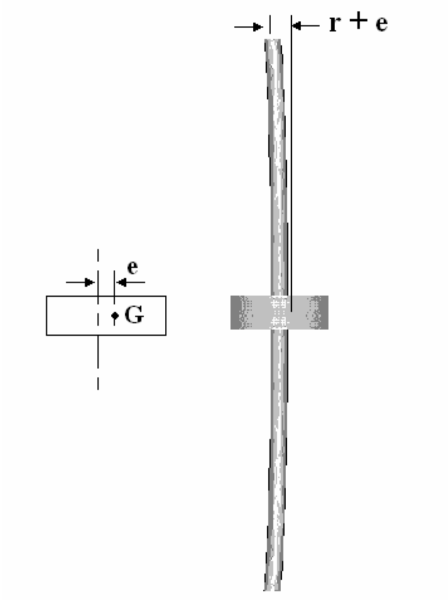

Figure 23
Figure 2

When the shaft rotates, centrifugal force will cause it to bend out. Let the deflection of the shaft be r. The distance to the centre of gravity is then r + e.
The shaft rotates at ω rad/s. The transverse stiffness is k_t N/m
The deflection force is hence $F = k_t r$
The centrifugal force is $M\omega^2(r + e)$
Equating forces we have

$$k_t r = M\omega^2(r+e) \quad \text{from which} \quad r = \frac{M\omega^2(r+e)}{k_t} = \frac{M\omega^2 r}{k_t} + \frac{M\omega^2 e}{k_t}$$

$$r = \frac{M\omega^2 e}{k_t\left(1 - \frac{M\omega^2}{k_t}\right)} \quad \text{It has already been shown that } \frac{k_t}{M} = \omega_n^2 \quad r = \frac{\omega^2 e}{\omega_n^2\left(1 - \frac{\omega^2}{\omega_n^2}\right)} = \frac{e}{\left(\frac{\omega_n}{\omega}\right) - 1}$$

From this we see that when $\omega_n = \omega$ r = e/0 which is infinity. This means that no matter how small the imbalance distance e is, the shaft will whirl at the natural frequency. Balancing does help but can never be perfect so whirling is to be avoided on the best of machines.

The frequencies at which whirling occurs are calculated by the same methods as for transverse vibrations of beams and the derivation is not given here.

SIMPLY SUPPORTED – The ends are free to rotate normal to the axis (e.g. self aligning bearings)

$$f = \frac{\pi}{2} n^2 \sqrt{\frac{gEI}{wL^4}}$$ where n is the mode and must be an integer 1, 2, 3 ......

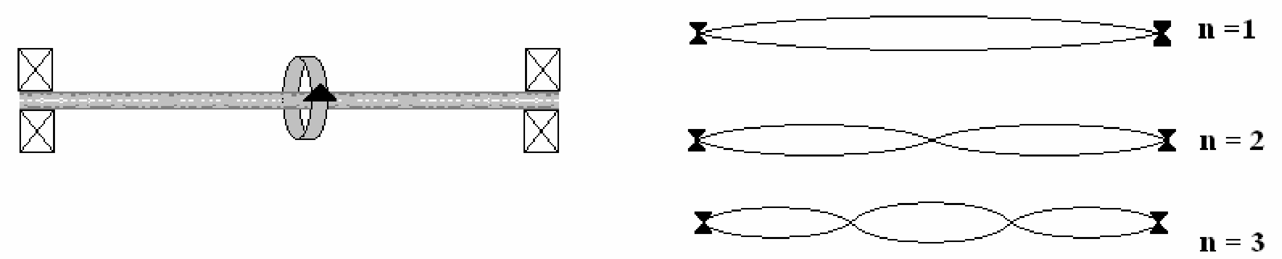

Figure 3

FIXED ENDS (e.g. fixed bearings or chucks)

The lowest critical speed is $f = 3.562\sqrt{\frac{gEI}{wL^4}}$ and the higher critical speeds are given by

$f = \frac{\pi}{2}\left(n + \frac{1}{2}\right)^2 \sqrt{\frac{gEI}{wL^4}}$ where n = 2, 3, ... Note that the lowest speed almost corresponds to n = 1

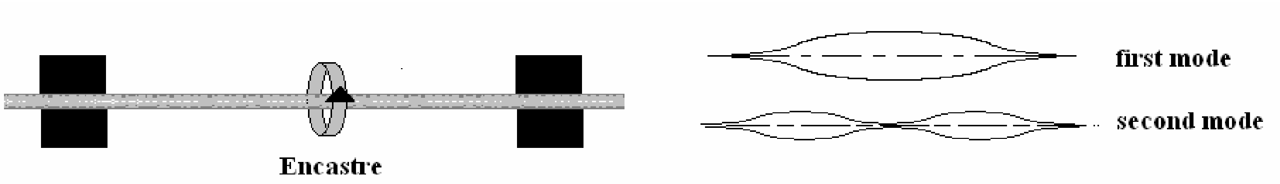

Figure 4

CANTILEVER (e.g. chuck at one end and free at the other)

The lowest critical speed is $f = 0.565\sqrt{\frac{gEI}{wL^4}}$

$f = \frac{\pi}{2}\left(n - \frac{1}{2}\right)^2 \sqrt{\frac{gEI}{wL^4}}$ where n = 2, 3 .... Note that the lowest speed almost corresponds to n = 1

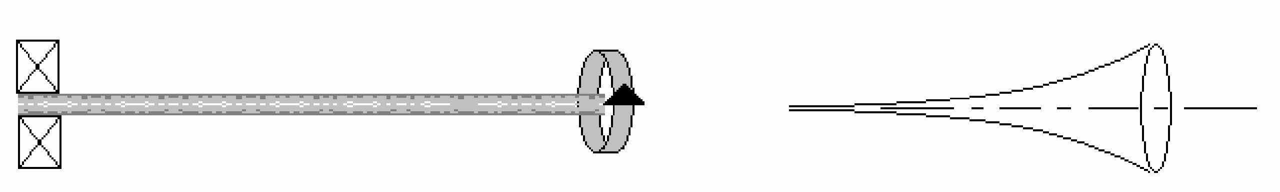

Figure 5

WORKED EXAMPLE No.1

A shaft is 30 mm diameter and 4 m long and may be regarded as simply supported. The density is 7 830 kg/m^3. E = 205 GPa. Calculate the first three critical frequencies.

SOLUTION

The essential information is d = 0.03 m L = 4 m ρ = 7830 kg/m^3 E = 205 GPa
First calculate the distributed weight w by calculating the weight of 1 m length.

$A = \frac{\pi d^2}{4} = \frac{\pi \text{ x } 0.03^2}{4} = 706.9 \text{ x } 10^{-6} \text{ m}^2 \quad \text{Volume} = A \text{ x } 1 = 706.9 \text{ x } 10^{-6} \text{ m}^3$

$\text{Weight} = \rho \text{ x A x g} = 7830 \text{ x } 706.9 \text{ x } 10^{-6} \text{ x } 9.81 = 54.3 \text{ N/m}$

$I = \frac{\pi d^4}{64} = \frac{\pi \text{ x } 0.03^4}{64} = 39.76 \text{ x } 10^{-9} \text{ m}^4$ Now calculate the funamental frequency.

$f_n = 1.572\sqrt{\frac{gEI}{wL^4}} = 1.572\sqrt{\frac{9.81 \text{ x } 205 \text{ x } 10^9 \text{ x } 39.76 \text{ x } 10^{-9}}{54.3 \text{ x } 3^4}} = 3.77 \text{ rev/s}$

If the shaft took up the second mode the frequncy would be $3.77 \text{ x } 2^2 = 15.1$ rev/s

If the shaft took up the third mode the frequncy would be $3.77 \text{ x } 3^2 = 33.9$ rev/s

WORKED EXAMPLE No.2

A light shaft carries a pulley at the centre with its centre of gravity on the centre line. The shaft is supported in self aligning bearing at the ends. The shaft deflects 0.5 mm under the static weight of the shaft. Determine the lowest critical speed.

SOLUTION

The frequency corresponds to that of a simply supported beam so

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{x_s}} = \frac{1}{2\pi}\sqrt{\frac{9.81}{0.0005}} = 22.3\text{ Hz}$$ The critical speed is 60 x 22.3 = 1338 rev/min

WORKED EXAMPLE No.3

The shaft in the previous example has a distributed weight of 40 N/m and the bearings are 1.2 m apart. The flexural stiffness is 4500 N m^2.
What is the critical speed taking into account the distributed mass?

SOLUTION

The frequency due to the distributed weight only is

$$f = \frac{\pi}{2}\sqrt{\frac{gEI}{wL^4}} = 1.572\sqrt{\frac{9.81 \text{ x } 4500}{40 \text{ x } 1.2^4}} = 36.3$$ N = 60f = 2176 rev/min

The critical speed is N is found from

$$\frac{1}{N^2} = \frac{1}{N_1^2} + \frac{1}{N_2^2} = \frac{1}{1338^2} + \frac{1}{2176^2}$$ Hence N = 1140 rev/min

WORKED EXAMPLE No.4

A steel wire 2 mm diameter is held between chucks 1m apart. The wire weighs 0.241 N/m. The flexural stiffness is 0.157 N m^2. Calculate the first two critical speeds critical speed.

SOLUTION

The lowest frequency due to the distributed weight only is

$$f = 3.562\sqrt{\frac{gEI}{wL^4}} = 3.562\sqrt{\frac{9.81 \text{ x } 0.157}{0.241 \text{ x } 1^4}} = 9$$ N = 60f = 540 rev/min

The second critical speed is $f = \frac{\pi}{2}\left(n + \frac{1}{2}\right)^2\sqrt{\frac{gEI}{wL^4}} = \frac{\pi}{2}\left(2 + \frac{1}{2}\right)^2\sqrt{\frac{9.81 \text{ x } 0.157}{0.241 \text{ x } 1^4}} = 24.8$

N = 60 x24.8 = 1489 rev/min

SELF ASSESSMENT EXERCISE No.1

1. A shaft 30 mm diameter is made from steel with density 7830 kg/m^3. Calculate the weight per metre (54.3 N/m)

   The shaft runs between self aligning bearings 2.2 m apart. The modulus of elasticity E is 200GN/m^2. Calculate the lowest critical speed. (738 rev/min)

   A pulley is mounted at the middle and this makes the shaft deflect 1.5 mm. Calculate the lowest critical speed. (760 rev/min)

2. A thin rod 5 mm diameter is held between two chucks 0.8 m apart. The wire weighs 1.508 N/m and the flexural stiffness is 6.136 N m^2. Calculate the first two critical speeds.
   (2110 rev/min and 5815 rev/min)

3. A shaft is 50 mm diameter and 8 m long and may be regarded as simply supported. The density is 7 830 kg/m^3. E = 205 GPa. Calculate the first three critical frequencies. (1.571 , 6.279 and 14.13 rev/s)

4. An aluminium rod is held in a chuck with the other end unsupported. It is 12 mm diameter and 400 mm long. The density of aluminium is 2710 kg/m^3 and the modulus of elasticity E is 71 GPa. Calculate the first two critical speeds. (3253 rev/min and 20350 rev/min)

3. OTHER FORCED VIBRATIONS

We must examine two common types of forced vibrations, first when a mass has a disturbing force acting on it and second when the spring support is disturbed harmonically.

3.1. HARMONIC DISTURBING FORCE

Consider an ideal system as shown. A mass M is suspended on a spring and a damper is placed between the spring and the support. The support does not move. Located on the mass is a small rotating machine that is out of balance. It has the equivalent of a small mass m rotating at radius r that produces an out of balance force due to the centripetal/centrifugal affect. The magnitude of this force is $F_o = mr\omega^2$. The main mass is constrained in guides so that it will only move up and down (one degree of freedom). At the position shown with rotation through angle θ, the component of F_o acting vertically is $F' = F_o \sin\theta$.

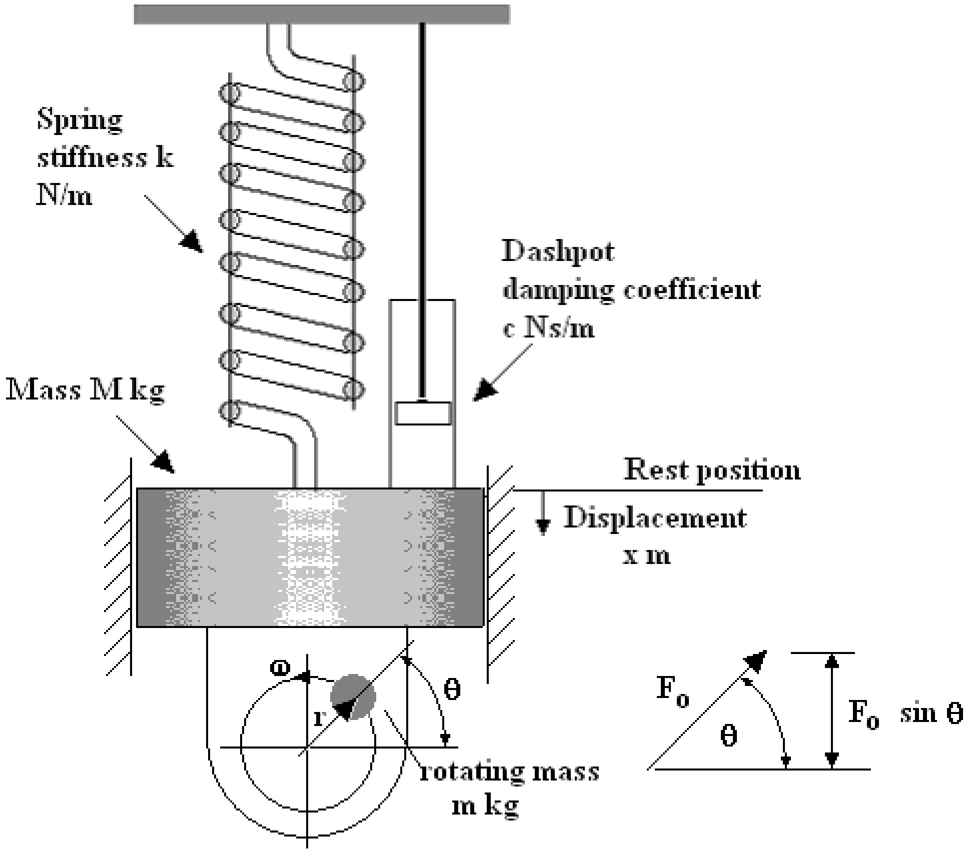

Figure 6

Any force applied to the mass to make it move must overcome the inertia, damping and spring force. The applied force is hence

$$F = F_i + F_d + F_s$$

$$F = M\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx$$

In this case the mass can only move vertically so the only force applied to it in this direction is the vertical component of the centrifugal force.

$$F_o \sin(\omega t) = M\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx$$

3.1.1 PHASOR REPRESENTATION

We may assume (and it is known from observations) that the mass is going to oscillate up and down with a sinusoidal oscillation of amplitude A. Let's assume that time starts when the oscillation passes through the rest position. The displacement is given by **$x = A \sin \omega t$**
where A is the amplitude.

The velocity is then **$v = dx/dt = A\omega\cos\omega t$** where $A\omega$ is the amplitude.

The acceleration is **$a = dv/dt = -A\omega^2\sin\omega t$** where $A\omega^2$ is the amplitude.

The displacement x, velocity v and acceleration are plotted against time in the diagram below. Each graph may be generated by a vector rotating at ω rad/s and with a length equal to the amplitude. Such vectors are called phasors. At a given moment in time, the tip of each vector is projected across to the appropriate point as shown.

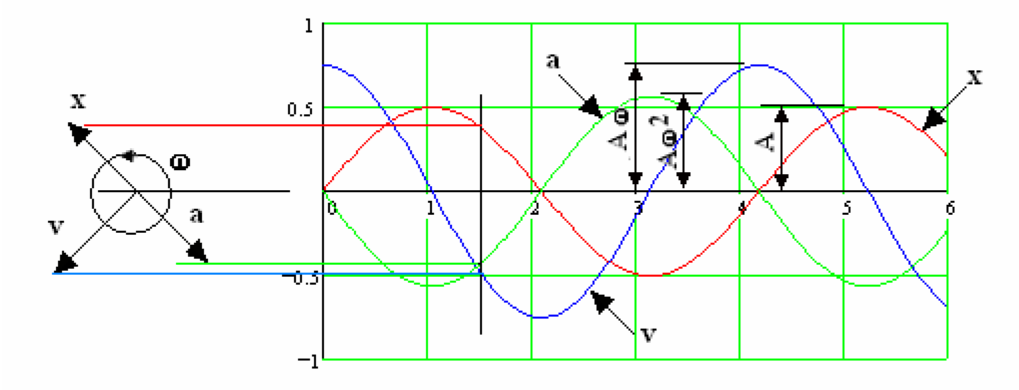

Figure 7

We can see that in order to produce the result, the velocity vector is 90° in front of the displacement and the acceleration is 90° in front of the velocity.

The spring force is directly proportional to displacement x so it must be in phase with x. The damping force is directly proportional to the velocity v so it must be in phase with v. The inertia force is directly proportional to the acceleration a so it must be in phase with a. It follows that the three forces can also be represented by phasors all rotating at angular velocity ω rad/s. We can choose a moment in time when the displacement is horizontal as shown.

The spring force is in phase with the movement so we draw the vector horizontally. The other vectors are 90° and 180° ahead respectively.

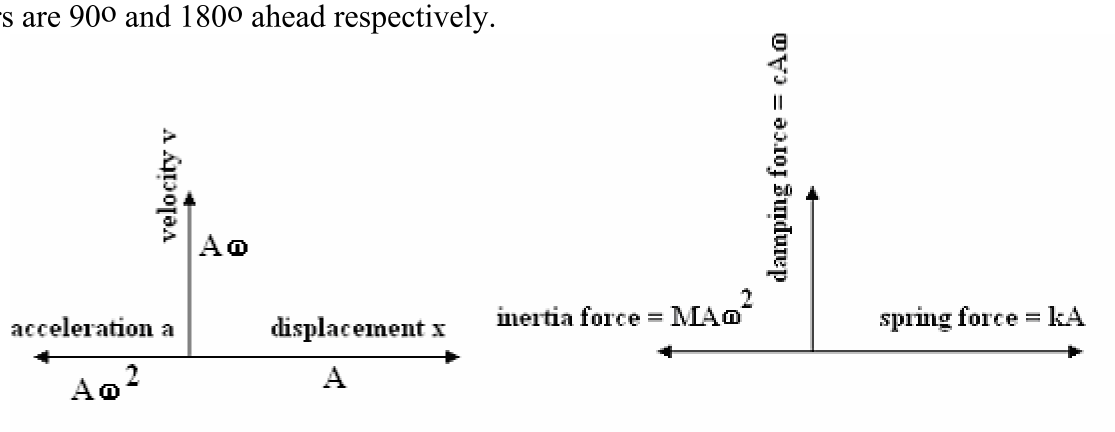

Figure 8

The sum of these three vectors is F_o so adding them we get a typical vector diagram as shown.

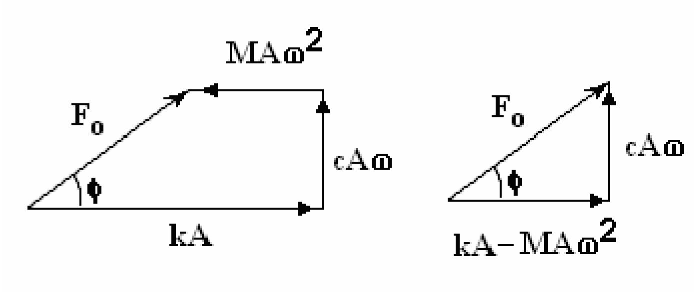

Figure 9

The diagram shows that the applied force F_o is at an angle ϕ to the horizontal so it must be displaced by a phase angle ϕ relative to x. Applying trigonometry we have

$$Fo^2 = \left(kA - MA\omega^2\right)^2 + (cA\omega)^2 \text{........Pythagoras}$$

$$Fo^2 = A^2\left(k - M\omega^2\right)^2 + A^2(c\omega)^2$$

$$Fo^2 = A^2\left\{\left(k - M\omega^2\right)^2 + (c\omega)^2\right\} \quad \text{Divide every term by } M^2$$

$$\frac{Fo^2}{M^2} = A^2\left\{\left(\frac{k}{M} - \omega^2\right)^2 + \left(\frac{c\omega}{M}\right)^2\right\}$$

It has been shown in the tutorial on damped vibrations that $\omega_n^2 = \dfrac{k}{M}$ and $\dfrac{c}{M} = 2\delta\ \omega_n$

$$A^2 = \left(\frac{Fo}{M}\right)^2\left\{\frac{1}{\left(\omega_n^2 - \omega^2\right)^2 + (2\delta\,\omega\,\omega_n)^2}\right\}$$

From the triangle we also get the phase angle.

$$\text{Tan}\ \phi = \frac{2\delta\,\omega\,\omega_n}{\omega_n^2 - \omega^2}$$

Plotting x and F_o for a given applied frequency against time produces a graph similar to below.

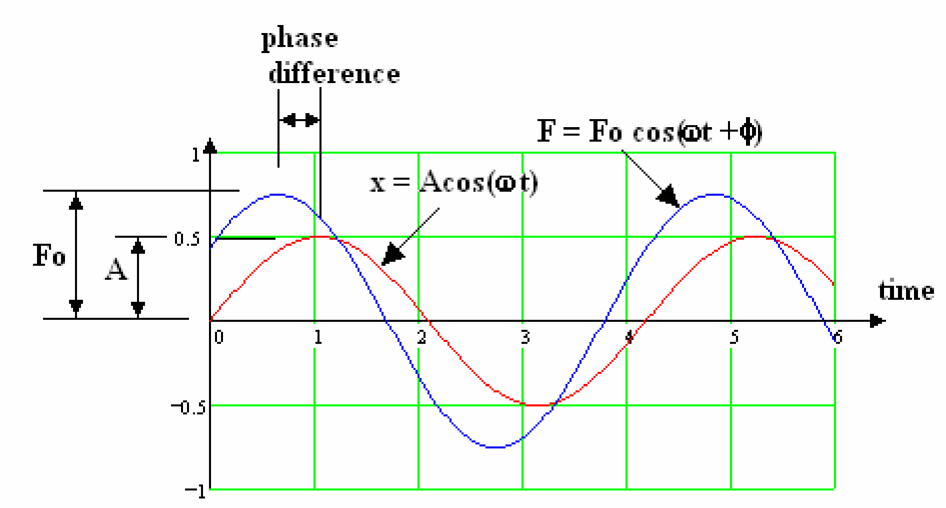

Figure 10

3.1.2 FREQUENCY RESPONSE DIAGRAMS

Suppose we start the out of balance machine and gradually increase the speed ω from zero. Taking a typical value of ω_n = 10 and plotting ϕ against ω for various values of δ produces the graph below.

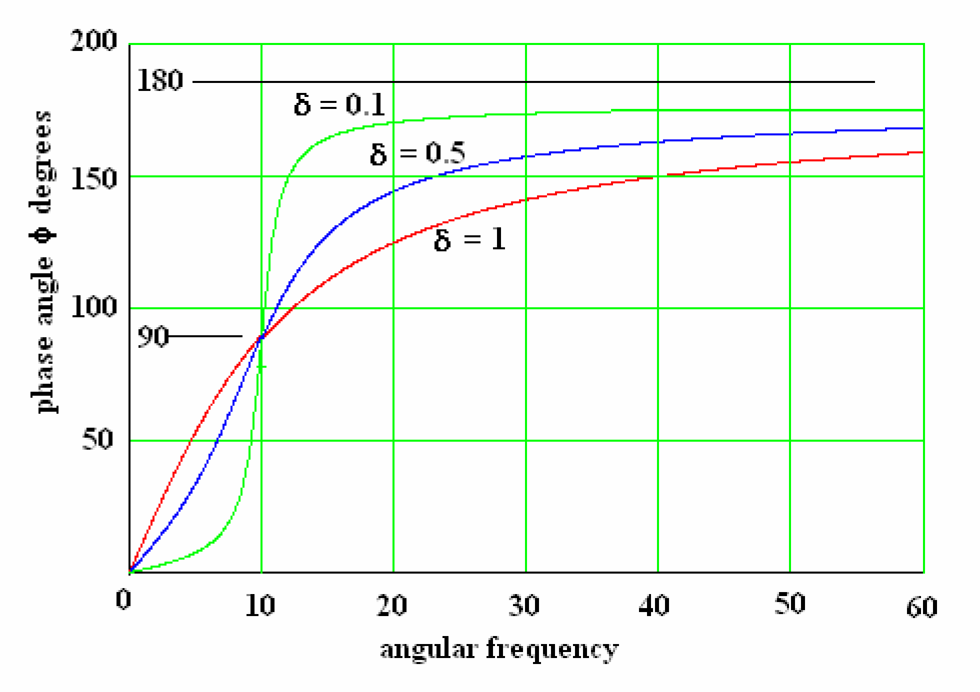

Figure 11

The plot shows that the phase angle ϕ starts at zero and reaches 90° when $\omega = \omega_n$. As the speed increases to large values, the phase angle approaches 180°.

Now consider what happens to the amplitude A. Plotting A against ω for various values of δ gives the graph below.

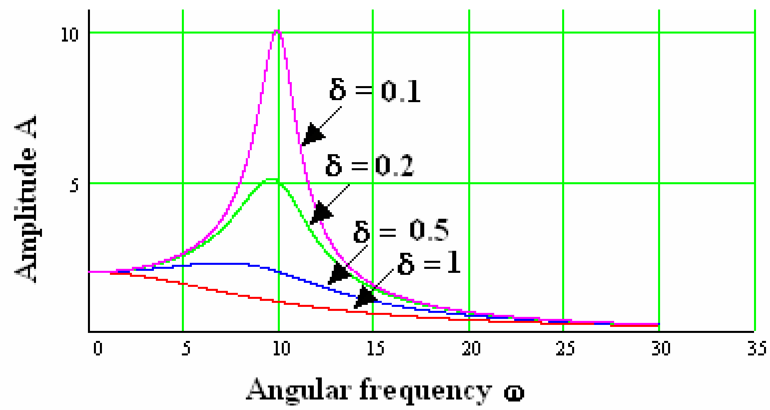

Figure 12

The following analyses the equation for A at the three obvious points

$$A^2 = \left(\frac{Fo}{M}\right)^2 \left\{\frac{1}{\left(\omega_n^2 - \omega^2\right)^2 + \left(2\delta\,\omega\,\omega_n\right)^2}\right\}$$ when ω = 0 this reduces to

$$A = \left(\frac{Fo}{M}\right)\left\{\frac{1}{\left(\omega_n^2\right)}\right\}$$

This has a finite value (2 on the diagram) and this is the same for all values of δ.

When $\omega = \omega_n$ the amplitude becomes

$$A = \left(\frac{Fo}{M}\right)\left\{\frac{1}{(2\delta\,\omega^2)}\right\}$$ and the value depends upon the value of δ.

The smaller the value of the damping ratio, the greater the peak value of A becomes. If $\delta = 0$ then in theory $A = \infty$

When the frequency becomes very large, the amplitude tends to die away to zero for all values of δ.

We may conclude from this, that when the out of balance machine rotates very fast, there is very little disturbance to the system but when it approaches the natural frequency of the system the amplitude might become very large depending on the damping. It should also be noted that the frequency at which the amplitude peaks is called the resonant frequency and this is not quite the same as the natural frequency.

<u>WORKED EXAMPLE No.5</u>

The diagram shows a mass-spring-dashpot system. The mass has a harmonic disturbing force applied to it given by the equation F' = 400 sin(30 t) Newton.
Determine the amplitude of the mass and the phase angle.

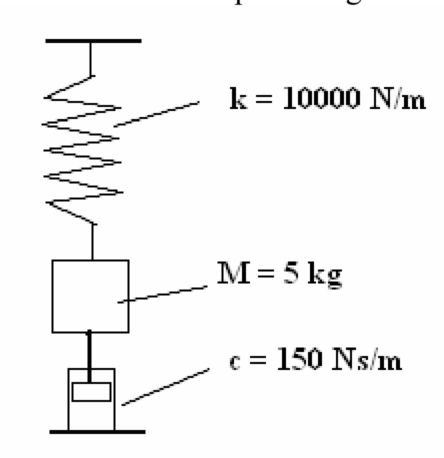

Figure 13

SOLUTION

From the question we know that k = 10000, M = 5 and c = 150

$$\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{10000}{5}} = 44.72 \text{ rad/s}$$

$$c_c = \sqrt{4Mk} = \sqrt{4 \times 5 \times 10000} = 447.21$$

$$\delta = \frac{c}{c_c} = \frac{150}{447.21} = 0.335$$

From the equation of motion Fo = 400 N and ω = 30 rad/s

$$A^2 = \left(\frac{Fo}{M}\right)^2 \left\{\frac{1}{\left(\omega_n^2 - \omega^2\right)^2 + \left(2\delta\,\omega\,\omega_n\right)^2}\right\}$$

$$A^2 = \left(\frac{400}{5}\right)^2 \left\{\frac{1}{\left(44.72^2 - 30^2\right)^2 + \left(2 \times 0.335 \times 30 \times 44.72\right)^2}\right\}$$

$A^2 = 0.00317$ A = 0.056 m or 56 mm

$$\tan\varphi = \frac{2\delta\,\omega\,\omega_n}{\omega_n^2 - \omega^2} = \frac{2 \times 0.335 \times 30 \times 44.72}{44.72^2 - 30^2} = 0.818$$

$\varphi = 39.3^o$

SELF ASSESSMENT EXERCISE No.2

1. A mass of 12 kg rests on a springy base of stiffness 8 kN/m. There is a damper between the mass and the support with a damping coefficient of 400 N s/m. The support is subjected to a harmonic disturbing force given by F = 200sin(30 t).
   Calculate the amplitude of the mass and the phase angle.
   (16 mm and 103.1 degrees).

2. A mass of 150 kg rests on a springy base of stiffness 60 kN/m. There is a damper between the mass and the support with a damping coefficient of 5000 N s/m. The support is subjected to a harmonic disturbing force given by F = 800sin(25 t).
   Calculate the amplitude of the mass and the phase angle.
   (6.2 mm and 105.1 degrees).

3.2. HARMONIC MOVEMENT OF THE SUPPORT

DAMPER BETWEEN MASS AND FIXED POINT

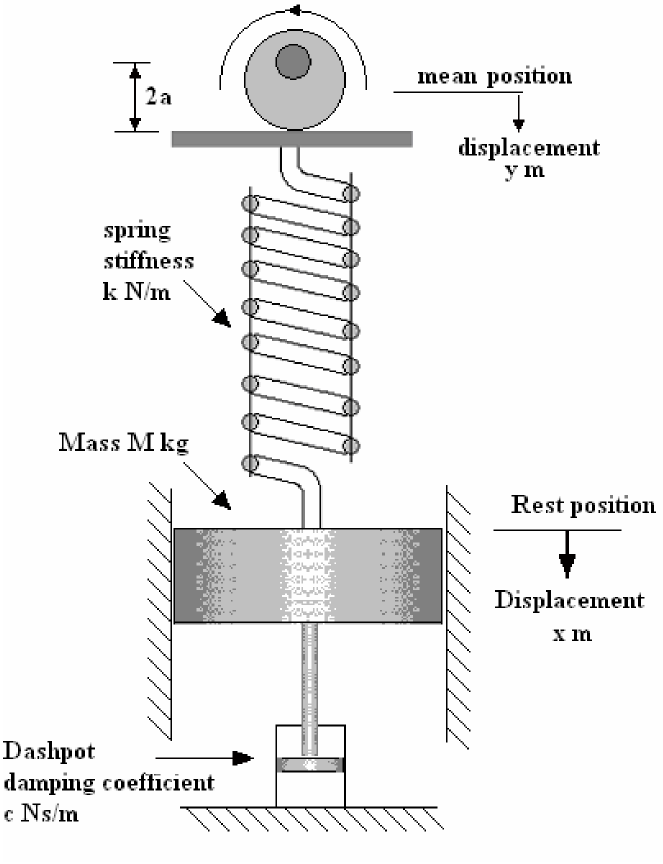

Figure 14

The diagram shows mass spring damper system. The mass can only move vertically. The support is made to move up and down by a cam that rotates at ω rad/s with amplitude a. If time starts when the support passes through the mean position, the motion of the support is described by the equation y = a sin(ωt).

We may assume that the mass is going to move up and down harmonically with an amplitude A but we cannot assume that the motion is in phase with the support so the equation of motion will be given by the equation x = A sin(ωt + ϕ) where ϕ is the phase angle.

The damper is attached between the mass and a fixed point so the velocity of the damper piston is dx/dt.

At any given moment in time the spring is stretched or shortened by an amount x- y at any time. The spring force is hence F = k(x-y).

The three forces are: Spring force k(x-y) Damping force c dx/dt Inertia force M d^2x/dt^2
In this case there is no applied so the force balance gives

$$0 = M\frac{d^2x}{dt^2} + c\frac{dx}{dt} + k(x - y)$$

$$0 = M\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx - ky$$

$$ky = M\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx$$

$$ka\sin(\omega t) = M\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx$$

Compare this with the previous case. $F_o\sin(\omega t) = M\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx$

This is the same except the term ka replaces the term F_o. It follows that the solutions are the same with this substitution.

$$A^2 = \left(\frac{ka}{M}\right)^2\left\{\frac{1}{\left(\omega_n^2 - \omega^2\right)^2 + \left(2\delta\,\omega\,\omega_n\right)^2}\right\}. \qquad \tan\phi = \frac{2\delta\,\omega\,\omega_n}{\omega_n^2 - \omega^2}$$

DAMPER BETWEEN THE MASS AND THE SUPPORT

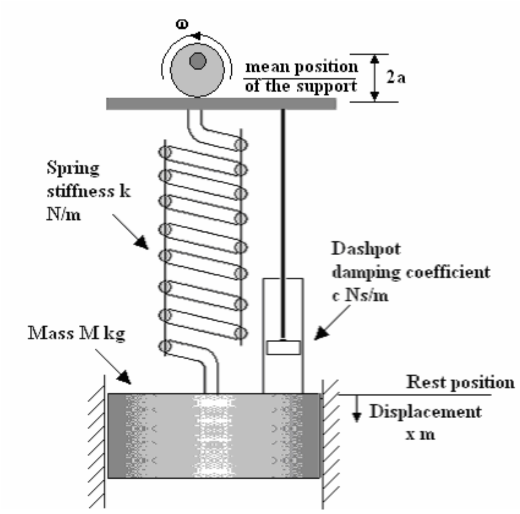

Figure 15

If the damper is fixed between the support and the mass then the velocity of the damper piston is dx/dt - dy/dt. The equation becomes

$$c\frac{dy}{dt} + ky = M\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx$$

$$c\,a\,\omega\cos(\omega t) + ka\sin\omega\,t = M\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx$$

This reduces to

$$\sqrt{(c\,a\,\omega)^2 + (ka)^2}\,\sin(\omega\,t + \varepsilon) = M\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx$$

Where $\varepsilon = \tan^{-1}(c\,\omega/k) = 2\delta(\omega/\omega_n)$

Compare this with the previous case. $F_o\sin(\omega\,t) = M\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx$

The steady state solution to this is $x = A\cos\{(\omega t + \varepsilon) + \phi\}$

As before, $A^2 = \left(\frac{ka}{M}\right)^2\left\{\frac{1}{(\omega_n^2 - \omega^2)^2 + (2\delta\,\omega\,\omega_n)^2}\right\}$. $\quad$ $\tan\phi = \frac{2\delta\,\omega\,\omega_n}{\omega_n^2 - \omega^2}$

3.2.1 MAGNIFICATION FACTOR

The magnification factor is the ratio A/a when the support is excited. The last equation may be arranged into the following form.

$$MF = \frac{A}{a} = \left(\frac{k}{M}\right)\sqrt{\left\{\frac{1}{\left(\omega_n^2 - \omega^2\right)^2 + \left(2\delta\,\omega\,\omega_n\right)^2}\right\}}$$

Since $k/M = \omega_n^2$ then

$$MF = \sqrt{\left\{\frac{\left(\omega_n^2\right)^2}{\left(\omega_n^2 - \omega^2\right)^2 + \left(2\,\delta\;\omega\,\omega_n\right)^2}\right\}} = \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\,\delta\,\frac{\omega}{\omega_n}\right)^2}}$$

This formula also applies to the case when a harmonic disturbing force is applied since $k_a = F_o$ and it follows that MF = Maximum force in spring/F_o

The response graph is shown below and could apply to either case. At low speeds the support and mass move up and down together. As ω approaches ω_n the amplitude of the mass grows and the phase angle approaches 90^o. As the speed passes resonance, the amplitude of the mass reduces and eventually becomes almost static. The phase angle tends to 180^o at high speeds. The magnification is greatest at resonance and as before, the resonant frequency is not quite the same as the natural frequency.

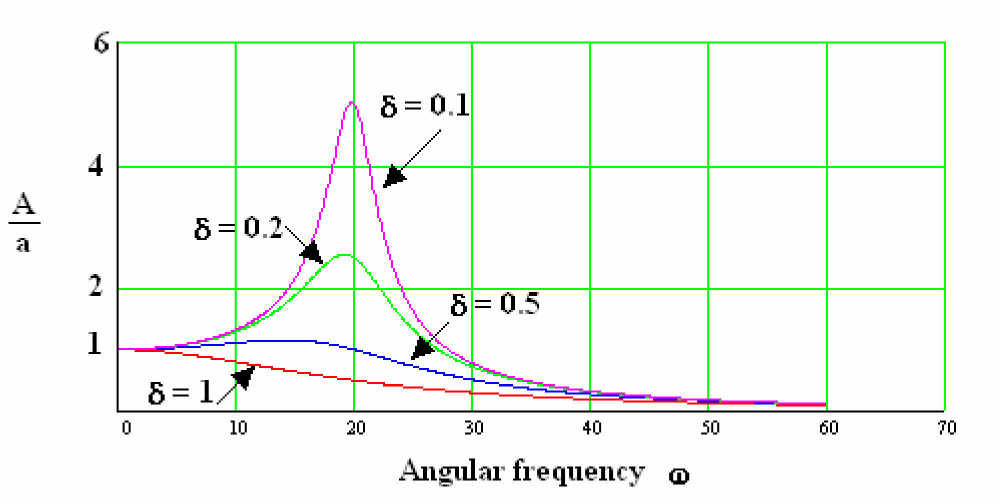

Figure 16

The PEAK MF occurs when $\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\,\delta\,\frac{\omega}{\omega_n}\right)^2$ is a maximum. This can be found by max and min theory. Simplify this to $\left(1 - r^2\right)^2 + \left(2\,\delta\,r\right)^2$

Differentiate w.r.t. r

$$\frac{d\left\{\left(1 - r^2\right)^2 + \left(2\,\delta\,r\right)^2\right\}}{dr} = 2(1 - r^2)(-2r) + 8\delta^2 r$$

Equate to zero and

$$2(1 - r^2)(-2r) + 8\delta^2 r = 0$$

$$r = \sqrt{1 - 2\delta^2}$$

Peak MF occurs when $\omega = \omega_n\sqrt{(1 - 2\delta^2)}$

WORKED EXAMPLE No.6

The diagram shows a mass-spring-dashpot system. The support is moved with a motion of y = 6 sin(40t) mm. Determine the amplitude of the mass and the phase angle.

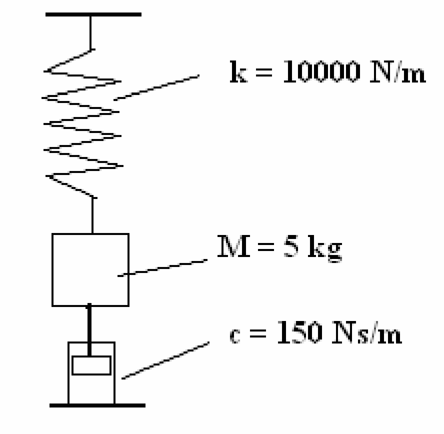

Figure 17

SOLUTION

From the question we know that k = 10000, M = 5 and c = 150

$$\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{10000}{5}} = 44.72 \text{ rad/s}$$

$$c_c = \sqrt{4Mk} = \sqrt{4 \text{ x } 5 \text{ x } 10000} = 447.21$$

$$\delta = \frac{c}{c_c} = \frac{150}{447.21} = 0.335$$

From the equation of motion a = 6 mm and ω = 40 rad/s

$$\frac{A}{a} = \left(\frac{k}{M}\right)\sqrt{\left\{\frac{1}{\left(\omega_n^2 - \omega^2\right)^2 + \left(2\delta\,\omega\,\omega_n\right)^2}\right\}}$$

$$\frac{A}{6} = \left(\frac{10000}{5}\right)\sqrt{\left\{\frac{1}{\left(44.72^2 - 40^2\right)^2 + \left(2 \text{ x } 0.335 \text{ x } 40 \text{ x } 44.72\right)^2}\right\}}$$

$$\frac{A}{a} = 1.581 \quad A = 1.581 \text{ x } 6 = 9.487 \text{ mm}$$

$$\text{Tan}\,\phi = \frac{2\delta\,\omega\,\omega_n}{\omega_n^2 - \omega^2} = \frac{2 \text{ x } 0.335 \text{ x } 40 \text{ x } 44.72}{44.72^2 - 40^2} = 3$$

$$\phi = 71.56^o$$

3.2.2 TRANSMISSIBILITY

When a mass vibrates on an elastic support, a force is transmitted through the spring and damper to the frame or ground. This is the sum of the spring and damping force. This may be illustrated with the vector diagram.

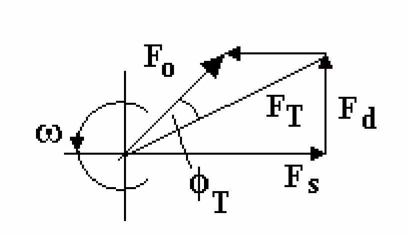

Figure 18

From the vector diagram we deduce that the transmitted force is $F_T = \sqrt{(F_s^2 + F_d^2)}$

$F_s = kA$ and $F_d = cA\omega$ $\quad\quad F_T = \sqrt{\{(kA)^2 + (cA\omega)\}}$

The ratio F_T/F_o is called the transmissibility ratio.

The phase angle between the transmitted fore and the applied force is $\phi_T = \phi - \tan^{-1}(F_d/F_s)$

The above work applies to both harmonic disturbing forces and harmonic motion of the support if the substitution $F_o = ka$

WORKED EXAMPLE No.7

Calculate the transmitted force and the phase angle of the transmitted force for example No.2.

SOLUTION

In example 2 we calculated A = 56 mm k = 10 kN/m and ω = 30 rad/s $\phi = 39.3^o$
$F_s = kA = 10\,000 \times 0.056 = 560$ N
$F_d = cA\,\omega = 150 \times 0.056 \times 30 = 252$ N

$F_T = \sqrt{(560^2 + 252^2)} = 614.1$ N

$\phi_T = 39.3 - \tan^{-1}(252/560) = 15.1^o$

SELF ASSESSMENT EXERCISE No.3

1. A mass of 500 kg rests on a springy base of stiffness 40 kN/m. The damping ratio is 0.25. The support is moved harmonically with an amplitude of 0.2 mm at 6 Hz. Calculate the amplitude of the mass and the phase angle.

   (0.0118 mm and -7.165 degrees).

   Calculate the transmitted force and the phase angle to the motion of the support.

   (1.104 N and -71.8^o)

2. A mass of 60 kg hangs from a spring of stiffness 100 kN/m. The damping ratio is 0.2. The support is moved harmonically with an amplitude of 3 mm at 6 Hz. Calculate the amplitude of the mass and the phase angle.

   (7.544 mm and 68.3 degrees).

   Calculate the transmitted force and the phase angle to the motion of the support.

   (80.4 N and 48^o)