SOLID MECHANICS

BALANCING

TUTORIAL – BALANCING OF ROTATING BODIES

This work covers elements of the syllabus for the Edexcel module 21722P HNC/D Mechanical Principles OUTCOME 4.

On completion of this tutorial you should be able to do the following.

- Explain the importance of balancing.
- Explain static and dynamic balance.
- Solve problems involving coplanar balance.
- Solve problems involving balancing in four planes.
- Construct vector diagrams.
- Use vector resolution to solve problems.

It is assumed that the student is already familiar with the following concepts.

- Angular motion.
- Centrifugal force.
- Basic vector diagram construction.
- Basic trigonometry.

All these above may be found in the pre-requisite tutorials.
1. **INTRODUCTION**

The balancing of rotating bodies is important to avoid vibrations. In heavy industrial machines such as steam turbines and electric generators, vibration could cause catastrophic failure. Vibrations are noisy and uncomfortable and when a car wheel is out of balance, the ride is quite unpleasant. In the case of a simple wheel, balancing simply involves moving the centre of gravity to the centre of rotation but as we shall see, for longer and more complex bodies, there is more to it. For a body to be completely balanced it must have two things.

1. **Static Balance.** This occurs when there is no resultant centrifugal force and the centre of gravity is on the axis of rotation.

2. **Dynamic Balance.** This occurs when there is no resulting turning moment along the axis.

2. **BALANCING IN ONE PLANE**

If the system is a simple disc then static balance is all that is needed. Consider a thin disc or wheel on which the centre of gravity is not the same as the centre of rotation. A simple test for static balance is to place the wheel in frictionless bearings. The centre of gravity will always come to rest below the centre of rotation (like a pendulum). If it is balanced it will remain stationary no matter which position it is turned to.

If the centre of gravity is distance \( r \) from the centre of rotation then when it spins at \( \omega \) rad/s, centrifugal force is produced. This has a formula \( C.F. = M \omega^2 r \) where \( M \) is the mass of the disc. This is the out of balance force. In order to cancel it out an equal and opposite force is needed. This is simply done by adding a mass \( M_2 \) at a radius \( r_2 \) as shown. The two forces must have the same magnitudes.

\[
M \omega^2 r = M_2 \omega^2 r_2 \\
M r = M_2 r_2
\]

Placing a suitable mass at a suitable radius moves the centre of gravity to the centre of rotation. This balance holds true at all speeds down to zero hence it is balanced so long as the products of \( M \) and \( r \) are equal and opposite.
Now consider that our disc is out of balance because there are three masses attached to it as shown. The 3 masses are said to be coplanar and they rotate about a common centre.

The centrifugal force acting on each mass is \( F = M r \omega^2 \). The radius of rotation is \( r \) and the angular velocity is \( \omega \) in radians/second. The force acting on each one is hence

\[
F_1 = M_1 r_1 \omega^2 \quad F_2 = M_2 r_2 \omega^2 \quad F_3 = M_3 r_3 \omega^2
\]

These are vector quantities and we can add them up to find the resultant force as shown.

If the system was balanced, there would be no resultant force so the force needed to balance the system must be equal and opposite of the resultant (the vector that closes the polygon). The balancing mass \( M_4 \) is then added at a suitable radius and angle such that the product \( M r \) is correct.
The result obtained would be the same whatever the value of $\omega$ and when $\omega = 0$ we have static balance. In order to make the solution easier, we may make $\omega = 1$ and calculate $M_r$ for each vector. This is called the $M_r$ polygon or vector diagram.

Note that angles will be given in normal mathematical terms with anticlockwise being positive from the x axis as shown.

![Figure 6](image)

**WORKED EXAMPLE No.1**

Three masses A, B and C are placed on a balanced disc as shown at radii of 120 mm, 100 mm and 80 mm respectively. The masses are 1 kg, 0.5 kg and 0.7 kg respectively. Find the 4th mass which should be added at a radius of 60 mm in order to statically balance the system.

![Figure 7](image)

**SOLUTION**

First draw up a table to calculate the value of $M_r$ for each mass.

<table>
<thead>
<tr>
<th>Mass</th>
<th>radius</th>
<th>$M_r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>120</td>
</tr>
<tr>
<td>B</td>
<td>0.5</td>
<td>100</td>
</tr>
<tr>
<td>C</td>
<td>0.7</td>
<td>80</td>
</tr>
<tr>
<td>D</td>
<td>$M_D$</td>
<td>60</td>
</tr>
</tbody>
</table>

Draw the $M_r$ polygon to find the value of $M_r$ for the 4th mass.
The resultant is 144.3 kg mm and is equal to 60M_0. The mass required is $\frac{144.3}{60} = 2.4$ kg 208° anticlockwise of A as shown.

**SELF ASSESSMENT EXERCISE No. 1**

Find the 4\textsuperscript{th} mass that should be added at a radius of 50 mm in order to statically balance the system shown.

Mass A is 1 kg at 100 mm radius.
Mass B is 1.5 kg at 75 mm radius
Mass C is 2.0 kg at 90 mm radius.

Answer 0.52 kg at 156° clockwise from A.
3. **MASSES NOT IN THE SAME PLANE**

Consider 2 masses statically balanced as shown but acting at different places along the axis.

![Figure 10](image)

For static balance $M_{A} r_A = M_{B} r_B$

It is clear that even with static balance, centrifugal force will produce a turning moment about the centre of gravity for the system. In this simple case, the problem is solved by adding equal and opposite forces at the two points as shown.

![Figure 11](image)

Consider the turning moment due to a single mass.

![Figure 12](image)
The centrifugal force produced is \( F = Mr^2 \)
The turning moment about the reference plane = \( T.M. = Fx = Mr^2x \)
For dynamic and static balance we must work out the resultant turning moment and add masses at appropriate points to cancel it out. The appropriate points will be on two planes not coplanar with any of the original masses. This involves drawing two vector diagrams and since \( \omega \) is common to all vectors we can again take \( \omega = 1 \) and draw vectors representing \( Mr \) and \( Mrx \). This is best explained with a worked example.

**WORKED EXAMPLE No.2**

Find the mass and the angle at which it should be positioned in planes A and D at a radius of 60 mm in order to produce complete balance of the system shown.

![Figure 13](image)

**SOLUTION**

Note that the diagram has been drawn with B vertical. It is a good idea to always start by making one of the known masses horizontal or vertical to make the construction of the vector diagrams easier. All angles should be expressed in absolute terms.

Plane A is the reference plane. All values of \( x \) are measured from plane A thus making \( Mrx \) for A equal to zero. It follows that it does not appear in the vector diagram. Make up a table as follows leaving unknowns as symbols.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>r</th>
<th>Mr</th>
<th>x</th>
<th>Mrx</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>MA</td>
<td>60</td>
<td>60MA</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>5</td>
<td>75</td>
<td>375</td>
<td>200</td>
<td>75 000</td>
</tr>
<tr>
<td>C</td>
<td>2</td>
<td>50</td>
<td>100</td>
<td>300</td>
<td>30 000</td>
</tr>
<tr>
<td>D</td>
<td>MD</td>
<td>60</td>
<td>60MD</td>
<td>375</td>
<td>22 500MD</td>
</tr>
</tbody>
</table>

Radius B is 75 mm
Radius C is 50 mm
Mass of B is 5 kg
Mass of C is 2 kg
Now draw a polygon of Mrx vectors in order to find the value of Mrx at D. Start with B in this case because it is vertical.

Scaling the vector D which closes the triangle we find
Mrx for D = 94 000 = 22 500 Md
Hence Md = 94000/22500 = 4.17 kg and it is positioned 254°

Now we calculate Mr for D.
Mr for D = 4.178 x 60 = 250.68.

Next we draw the polygon for the Mr values as shown.

The vector which closes the polygon represents Mr for point A
Mr for A is 195.
60Ma = 195  Ma = 195/60 = 3.25 kg at 6° to the vertical.

The answer is best shown with an end view.
Figure 16
SELF ASSESSMENT EXERCISE No. 2

1. A shaft has 4 discs A, B, C and D along its length 100 mm apart. A mass of 0.8 kg is placed on B at a radius of 20 mm. A mass of 2 kg is placed on C at a radius of 30 mm and rotated 120° from the mass on B. Find the masses to be placed on A and D at a radius of 25 mm that will produce total balance. 

   (Answers 0.696 kg and 1.52 kg)

2. The diagram below shows masses on two rotors in planes B and C. Determine the masses to be added on the rotors in planes A and D at radius 40 mm which will produce static and dynamic balance. (1.9 kg at 177° and 2.2 kg at 141°)

![Figure 17](image-url)
4 SOLUTION BY USING VECTOR RESOLUTION

A more accurate approach to solving the vector diagrams in the preceding work is to resolve each vector into vertical and horizontal components. The resultant vector is then found by adding these components. Consider worked example No.1 again.

Each vector has a component in the x direction given by Mr cosθ and in the y direction it is MRsinθ

Work out these for each vector and include it in the table.

<table>
<thead>
<tr>
<th></th>
<th>Mass</th>
<th>radius</th>
<th>Mr</th>
<th>θ</th>
<th>Mr cosθ</th>
<th>MRsinθ</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>120</td>
<td>120</td>
<td>0°</td>
<td>120</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>0.5</td>
<td>100</td>
<td>50</td>
<td>30°</td>
<td>43.3</td>
<td>25</td>
</tr>
<tr>
<td>C</td>
<td>0.7</td>
<td>80</td>
<td>56</td>
<td>130°</td>
<td>-36</td>
<td>42.9</td>
</tr>
<tr>
<td></td>
<td>Totals</td>
<td></td>
<td>127.3</td>
<td>67.9</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The resultant vector has x and y components of 127.3 and 67.9. This can be solved with Pythagoras. Resultant Mr = (127.3² + 67.9²)½ = 144.3 kg mm as before. The mass required is 144.3/60 = 2.4 kg.

The angle ϕ = atan (67.9/127.3) or tan⁻¹(67.9/127.3) = 28°

The balancing force is 180° anticlockwise of this so the balancing mass must be placed at an angle of 208°.
Here is the solution to worked example No.2 using the same method.

<table>
<thead>
<tr>
<th>Mass</th>
<th>radius</th>
<th>Mr</th>
<th>x</th>
<th>Mrx</th>
<th>θ</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>M_A</td>
<td>60</td>
<td>60</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>5</td>
<td>75</td>
<td>375</td>
<td>200</td>
<td>75000</td>
</tr>
<tr>
<td>C</td>
<td>2</td>
<td>50</td>
<td>100</td>
<td>300</td>
<td>30000</td>
</tr>
<tr>
<td>D</td>
<td>M_D</td>
<td>60</td>
<td>60</td>
<td>375</td>
<td>22500</td>
</tr>
</tbody>
</table>

\[ MR_x \cos \theta \quad MR_x \sin \theta \]   | If balanced, the totals must be zero |
| A    | 0     | 0   | so the unknown values may be |
| B    | 0     | 75000 | deduced. |
| C    | 25980 | 15000 |
| D    | -25980| -90000 |
| Totals | 0     | 0   |

Resultant \( Mr_x \) for D = \( (25980^2 + 90000^2)^{\frac{1}{2}} = 93670 \)
This is the value for D so \( 22500 \cdot M_D = 93670 \) \( M_D = \frac{9360}{22500} = 4.16 \) kg
\( \phi = \tan^{-1}\left(\frac{90000}{25980}\right) = 73.98^\circ \) so the true angle \( \theta \) is \( 180^\circ + 73.98^\circ = 253.8^\circ \)

Now fill in the table for the \( Mr \) values. Again the totals must be zero so we can deduce the values for A.

<table>
<thead>
<tr>
<th>M</th>
<th>r</th>
<th>Mr</th>
<th>θ</th>
<th>Mr cosθ</th>
<th>MRsinθ</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>M_A</td>
<td>60</td>
<td>60</td>
<td>M_A</td>
<td>-17.3</td>
</tr>
<tr>
<td>B</td>
<td>5</td>
<td>75</td>
<td>375</td>
<td>90</td>
<td>0</td>
</tr>
<tr>
<td>C</td>
<td>2</td>
<td>50</td>
<td>100</td>
<td>30</td>
<td>86.6</td>
</tr>
<tr>
<td>D</td>
<td>4.167</td>
<td>60</td>
<td>250</td>
<td>253.8^\circ</td>
<td>-69.3</td>
</tr>
<tr>
<td>Totals</td>
<td></td>
<td>0</td>
<td>0</td>
<td></td>
<td>0</td>
</tr>
</tbody>
</table>

Resultant Mr vector is \( (17.3^2 + 185^2)^{\frac{1}{2}} = 185.8 \) and this is equal to 60 MA.

\( M_A = 185.8/60 = 3.1 \) kg
\( \phi = \tan^{-1}\left(\frac{185}{17.3}\right) = 84.6^\circ \) so the true angle \( \theta \) is \( 180^\circ + 84.6^\circ = 264.6^\circ \)

This gives a more accurate answer than that produced by drawing and scaling.