SOLID MECHANICS

BALANCING

TUTORIAL 2 – BALANCING RECIPROCATING MACHINERY

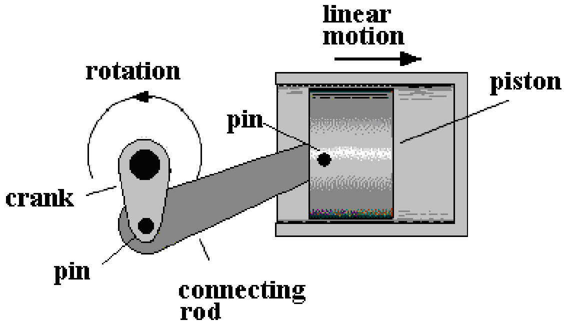
On completion of this tutorial you should be able to solve problems involving balancing primary and secondary forces and moments for reciprocating machines

Contents

1. *Introduction*

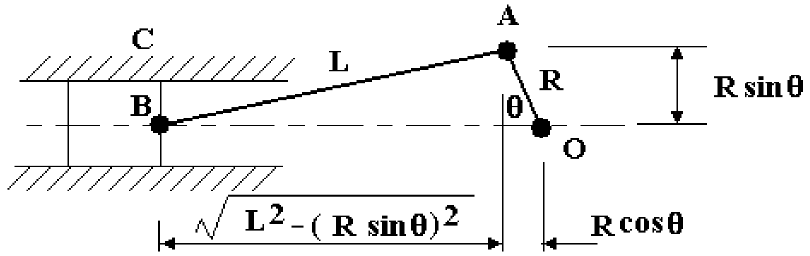
Reciprocating machines here means a piston reciprocating in a cylinder and connected to a crank shaft by a connecting rod. You can skip the derivation of the acceleration by going to the next page.

First let's establish the relationship between crank angle, and the displacement, velocity and acceleration of the piston.



2 *Derivation of Acceleration Equation*

A crank, con rod and piston mechanism is shown below.



When θ = 0 the piston will be furthest left at a distance of L+R from point O. Take this as the reference point and measure displacement x from there. Remember that $\theta = \omega t$ and $\omega = 2\pi \times N$. The displacement is then

$$x = (L + R) - R\cos\theta - \sqrt{L^2 - (R\sin\theta)^2} = R\left[\left(\frac{L}{R} + 1\right) - \cos\theta - \sqrt{\frac{L^2}{R^2} - (\sin\theta)^2}\right]$$

Let the ratio L/R = n

$$x = R\left[(n + 1) - \cos\theta - \sqrt{n^2 - (\cos\theta)^2}\right]$$

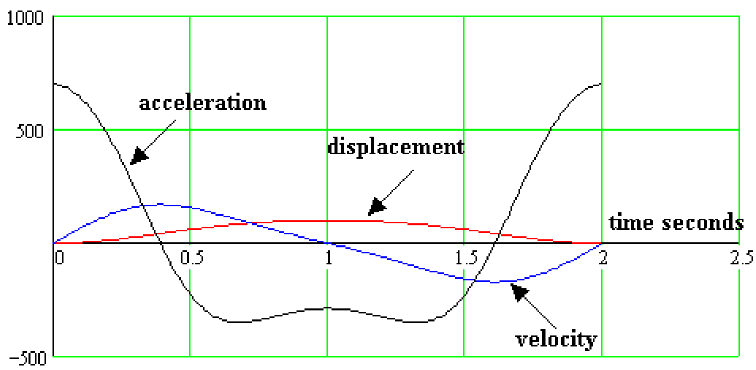
Differentiate to get the velocity.

$$v = \omega R\left[\sin\theta + -\frac{\sin\theta\cos\theta}{\sqrt{n^2 - (\sin^2\theta)}}\right] = \omega R\left[\sin\theta + \frac{\sin 2\theta}{2\sqrt{n^2 - (\sin^2\theta)}}\right]$$

Differentiate again and simplify to get the acceleration.

$$a = \omega^2 R\left[\cos\theta + \frac{\sin^2 2\theta}{4\left(n^2 - (\sin^2\theta)\right)^{3/2}} + \frac{\cos 2\theta}{(n^2 - \sin^2\theta)^{1/2}}\right]$$

The diagram shows a plot of displacement, velocity and acceleration against angle when L=120 mm, R=50 mm and ω = π rad/s. It should be noted that none of them are sinusoidal and not harmonic (in particular, the acceleration). The larger the value of n, the nearer it becomes to being harmonic.



The equation for acceleration may be expanded as a Fourier series into the form

$$a = \omega^2 R[\cos(\theta) + A_2\cos(2\theta) + A_4\cos(4\theta) + A_6\cos(6\theta)]$$

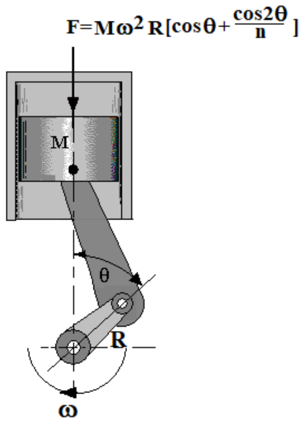
A is a constant involving n. The following gives a very good approximation except at very high speeds.

$$a = \omega^2 R\left[\cos(\theta) + \frac{\cos(2\theta)}{n}\right]$$

3. Force

Using the close approximation for acceleration, the inertia force required to accelerate the piston is given by

$$F = M\omega^2 R\left[\cos(\theta) + \frac{\cos(2\theta)}{n}\right]$$



This may be thought of as two separate forces, the primary force F_p and the secondary force F_s.

$$F_p = M\omega^2 R\cos(\theta) \quad \text{and} \quad F_s = M\omega^2 R\left[\frac{\cos(2\theta)}{n}\right]$$

3.1 Primary Force for a Single Cylinder

The primary force must be thought of as a force with a peak value $M\omega^2 R$ that varies cosinusoidally with angle θ.

$$F_p = M\omega^2 R\cos(\theta)$$

WORKED EXAMPLE No. 1

Determine the primary out of balance force for a single cylinder machine with a piston of mass 0.5 kg, with a connecting rod 120 mm long and a crank radius of 50 mm when the speed of rotation is 3 000 rev/min.

SOLUTION

$\omega = 2\pi \times 3\,000/60 = 100\pi$ rad/s $F_p = M\omega^2 R\cos(\theta) = 0.5 \times (100\pi)^2 \times 0.05\cos\theta = 2\,467\cos\theta$ N

3.2 Secondary Force for a Single Cylinder

The secondary force must be thought of as a force with peak value $M\omega^2 R/n$ that varies Cosinusoidally with the double angle 2θ.

$$F_s = M\omega^2 R\left[\frac{\cos(2\theta)}{n}\right]$$

WORKED EXAMPLE No. 2

Determine the secondary force produced in a single cylinder machine with a piston of mass 0.5 kg, with a connecting rod 120 mm long and a crank radius of 50 mm when the speed of rotation is 3 000 rev/min.

SOLUTION

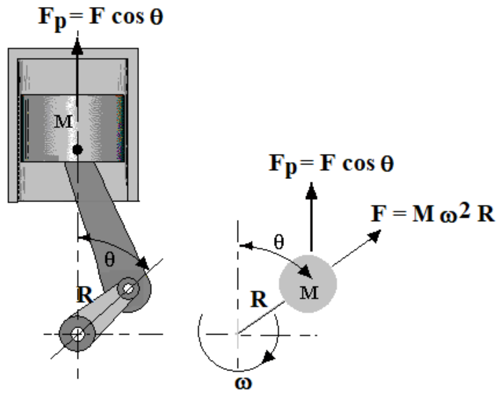
$\omega = 2\pi \times 3\,000/60 = 100\pi$ rad/s $\qquad n = 120/50 = 2.4$

$$F_s = M\omega^2 R\left[\frac{\cos(2\theta)}{n}\right] = 0.5 \times (100\pi)^2 \times 0.05 \times \frac{\cos(2\theta)}{2.4} = 1\,028\cos(2\theta)\ \text{N}$$

3.3 Representation with a Rotating Mass

The primary force is the same as the vertical component of the centrifugal force of a rotating mass M at the crank radius R and angular velocity ω.

$$F_p = M\omega^2 R \cos\theta$$



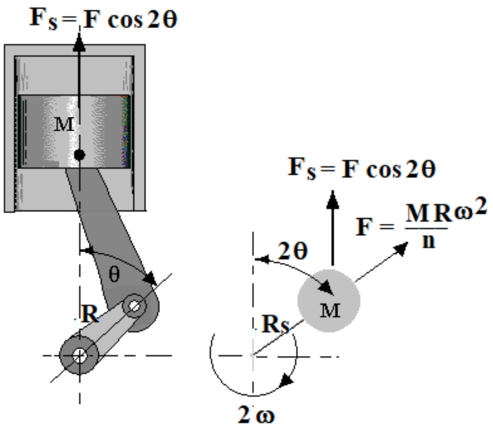
The secondary force is the same as the vertical component of the centrifugal force of a mass M rotating at 2ω at a radius of R_s where

$$R_s = \frac{R}{4n}$$

Hence

$$F_s = MR_s(2\omega)^2\cos(2\theta)$$

$$F_s = \frac{MR}{4n}(2\omega)^2\cos(2\theta) = \frac{MR}{n}\omega^2\cos(2\theta)$$



Note that this means that if the shaft is running at 2ω then the mass must be M/4 in order to produce the same force. This is important when balancing the system as covered later.

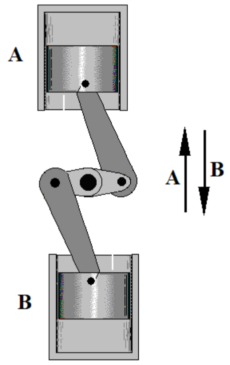
3.4 Primary and Secondary Forces for Multiple Cylinders

The primary and secondary inertia forces for multiple pistons are simply the resultant force of all the force vectors. Problems are easier to solve when the radii and masses of all the pistons are the same but the graphical method can be used quite easily with any combination.

In all the following it will be assumed that the reciprocating masses are all moving in the same vertical direction with various crank angles. For simplicity the crank angles are referred to the axis of the pistons and this is made to be vertical.

It is usual to make one piston the reference crank (A) with θ = 0
The crank angles of the other are relative to A and are typically designated α, β and γ

WORKED EXAMPLE No. 3a



Two reciprocating pistons as shown have equal mass and crank radii and are placed 180° apart. Determine the primary force.

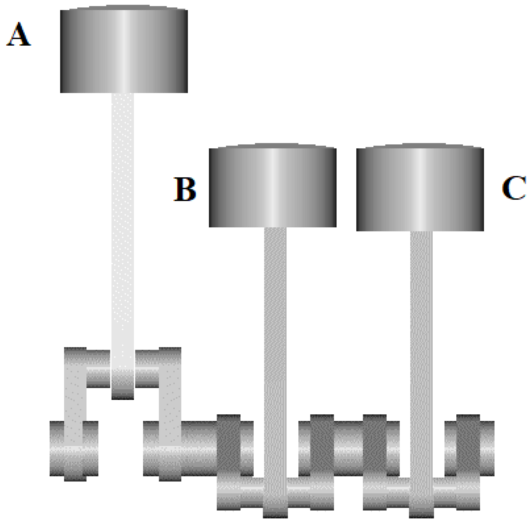
SOLUTION

The force for each piston is $F_p = M\omega^2 R\cos(\theta)$

What ever the angle of the crank, the vertical components of the forces will be equal and opposite so $F_p = 0$.

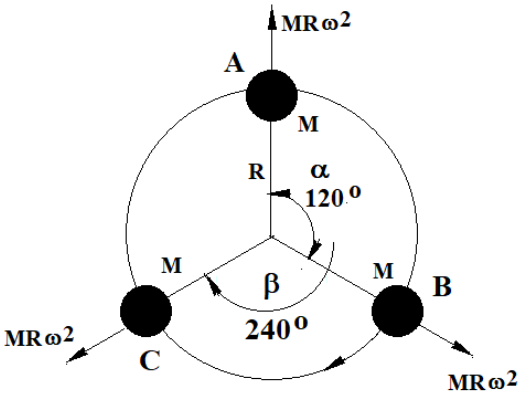
WORKED EXAMPLE No. 3b

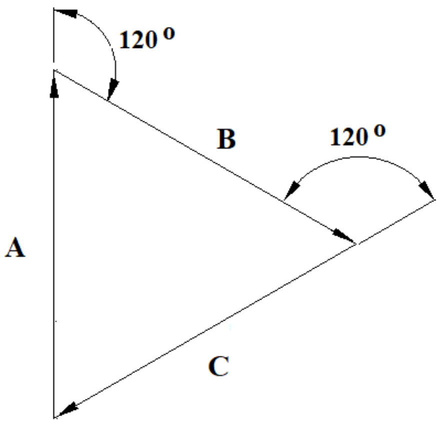
Three reciprocating pistons have equal mass and crank radii and are placed 120° apart from each other. Determine the primary force.



SOLUTION

Represent the three pistons with a rotating mass at radius R as shown.





The force for each piston is $F_p = MR\omega^2\cos(\theta)$

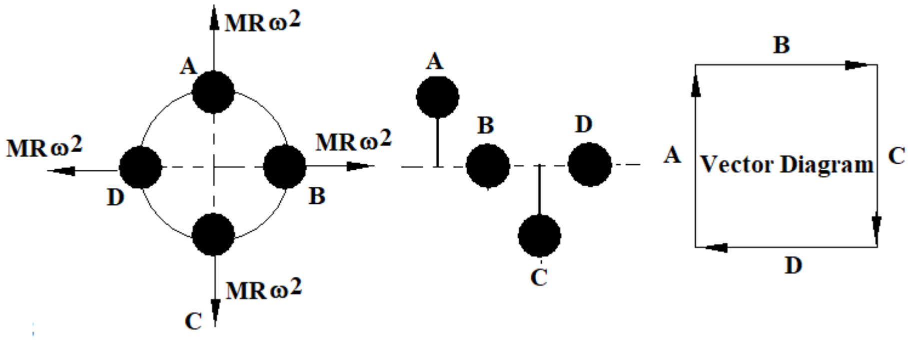
In order to draw the vectors choose that A is at zero degrees. Each vector has a value $MR\omega^2$ and adding them we see there is no resultant so there is no resultant vertical component ($MR\omega^2 \cos\theta$) either and so $F_p = 0$ and this will be true whatever the crank angle.

WORKED EXAMPLE No. 3c

Four reciprocating pistons in the same line have equal mass and crank radii and are placed 90^o apart from each other. Determine the primary force.

SOLUTION

Represent each mass as shown.



SOLUTION

The force for each piston is

$$F_p = MR\omega^2\cos(\theta)$$

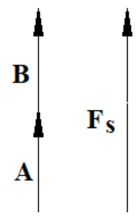
In order to draw the vectors choose that A is at zero degrees. Each vector has a value $M\omega^2R$ and adding them we see there is no resultant so there is no resultant vertical component either so

$F_p = 0$ and this will be true whatever the crank angle θ.

WORKED EXAMPLE No. 4

Establish the secondary force for the same cases as example 3a to 3c.

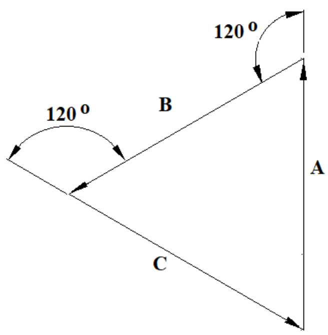
4a. **SOLUTION**



2 Pistons. The angle between the two cranks is 180^o so doubling we get 360^o.
The vector A may be drawn at any angle but is normally vertically up.
Vector B is drawn at 360^o to vector A and added. Each vector has a length $M\omega^2R/n$

The resulting vertical component is $F_s = 2\ M\omega^2(R/n)\cos 2\theta$ so there is a resultant force that needs to be balanced.

4b SOLUTION



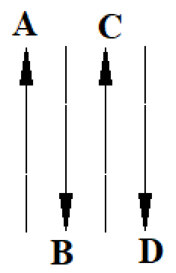
3 Pistons. The angle between each cranks is 120 ^{o} so doubling, vector B will be at 240 ^{o} and vector C will be at 480^o all relative to A. (A is drawn vertical for convenience).

Adding them we see there is no Resultant whatever the angle of vector A so $F_s = 0$ at all crank angles and the secondary force is balanced

4C SOLUTION

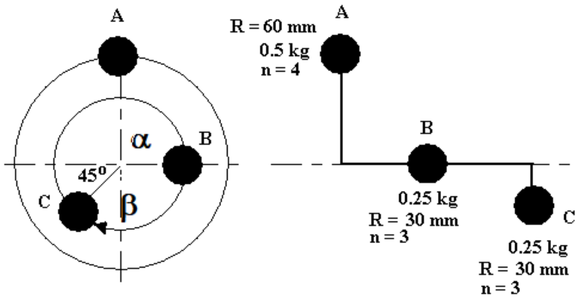
4 Pistons. The angle between each cranks is 90^o so doubling, vector B will be at 180^o, vector C will be at 360^o and vector D will be 540^o all relative to A.

Adding them we see there is no resultant force whatever the angle of vector A so $F_s = 0$ at all crank angles and the secondary forces are balanced.



WORKED EXAMPLE No. 5

A machine has three reciprocating masses A, B and C with cranks located as shown in the diagram. Determine the primary and secondary forces produced at a speed of 600 rev/min.



SOLUTION

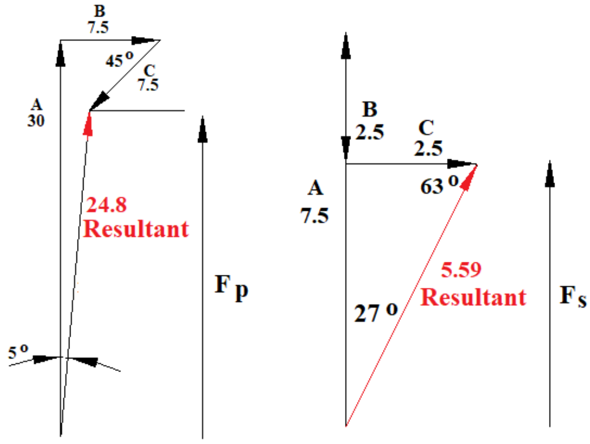
The angle of the crank A is $\theta = 0$ The angle of crank B relative to A is $\alpha = 90^o$
The angle of crank C relative to A is $\beta = 225^o$ $\omega = 600 \times 2\pi/60 = 20\pi$ rad/s

Crank A $F_p = MR\omega^2 \cos\theta$ $F_s = \omega^2(M\,R/n)\cos 2\theta$
Crank B $F_p = MR\omega^2 \cos\alpha$ $F_s = \omega^2(M\,R/n)\cos 2\alpha$
Crank C $F_p = MR\omega^2 \cos\beta$ $F_s = \omega^2(M\,R/n)\cos 2\beta$ (or $m_s\omega^2 \cos 2\beta$)

The solution is best done with a table.

	Mass (kg)	R (mm)	MR (kg mm)	Angle (deg)	n	MR/n (kg mm)	2×angle (deg)
A	0.5	60	30	0	4	7.5	0
B	0.25	30	7.5	α=90	3	2.5	2α=180
C	0.25	30	7.5	β=225	3	2.5	2β=450

Draw the MR and the MR/n polygons with A drawn at 0 degrees.

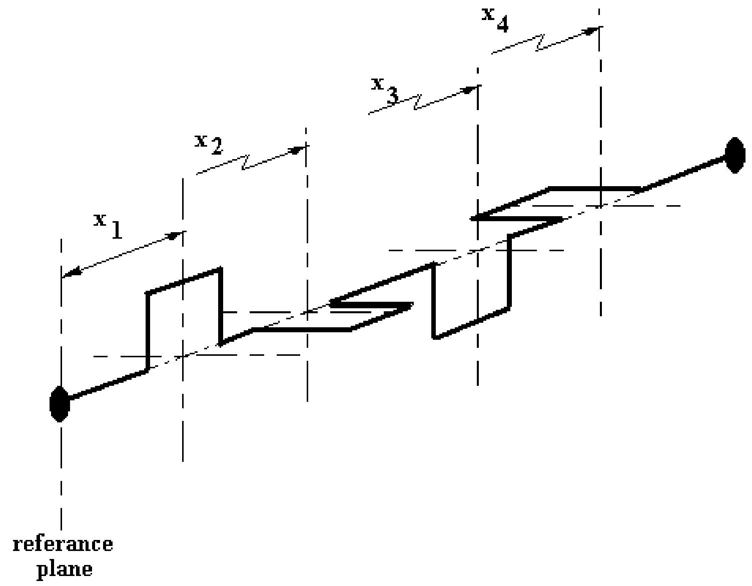


MR Polygon **MR/n Polygon**

The resultant for the MR/n polygon is 5.59 kg mm 27^o clockwise of A
The secondary force is $F_s = (M\,R/n)\,\omega^2 \cos27^o = 5.59 \times (20\pi)^2 \cos27^o = 19.6$ N 27^o clockwise of A
The resultant for the MR polygon is 24.8 kg mm at 5^o to A
$F_p = 24.8 \times 10^{-3} \times (20\pi)^2 \cos5^o = 97.5$ N (the force in line with cylinders

4. Moments

Each force produces a moment about any point distance x from the centre line of the cylinder along the axis of the crank shaft. Consider the crank below. The distance from the reference plane to the centreline of each crank is x_1, x_2 and so on.



The turning moment about the reference plane is

$$TM = M\omega^2R\left[x_1\left\{\cos(\theta) + \frac{\cos(2\theta)}{n}\right\} + x_2\left\{\cos(\theta+\alpha) + \frac{\cos2(\theta+\alpha)}{n}\right\} + x_3\cos(\theta+\beta) + \frac{\cos2(\theta+\beta)}{n} + \cdots\right]$$

α, β, γ …. are the angles each crank has relative to crank A. This can be separated into primary and secondary moments.

4.1 Primary Moment

$$TM = M\omega^2R[x_1\{\cos(\theta)\} + x_2\{\cos(\theta+\alpha)\} + x_3\cos(\theta+\beta) + \cdots]$$

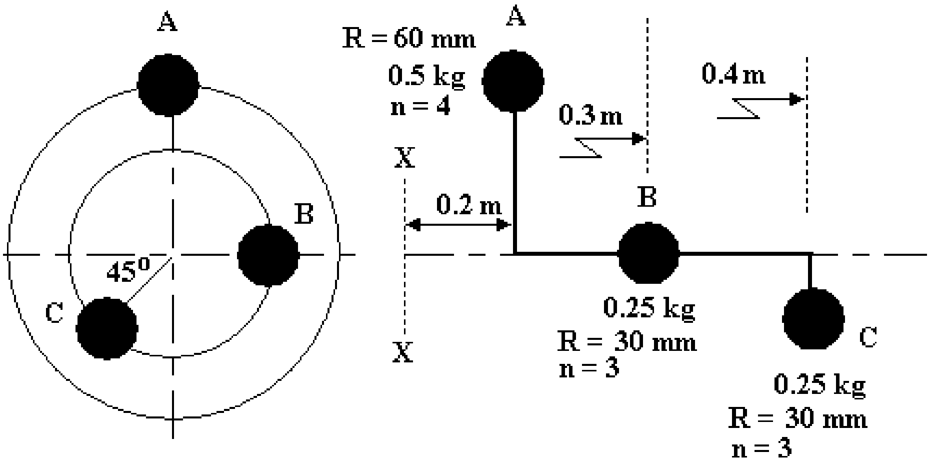
4.2 Secondary Moment

$$TM = M\omega^2\frac{R}{n}[x_1\{\cos(2\theta)\} + x_2\{\cos2(\theta+\alpha)\} + x_3\cos2(\theta+\beta) + \cdots]$$

Both may solved with vectors but this time it is MRx and MRx/n that we plot and evaluate.

WORKED EXAMPLE No. 6

A machine has three reciprocating masses A, B and C with cranks located as shown in the diagram. Determine the primary and secondary moments produced at 600 rev/min about plane X – X.

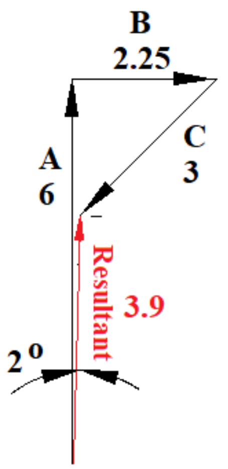


SOLUTION – PRIMARY MOMENTS

$\omega = 600 \times 2\pi/60 = 20\pi$ rad/s

	Mass (kg)	R (m)	x (m)	MRx (kg m^2)	angle (deg)
A	0.5	0.06	0.2	6×10^{-3}	$\theta = 0$
B	0.25	0.03	0.3	2.25×10^{-3}	$\alpha = 90$
C	0.25	0.03	0.4	3×10^{-3}	$\beta = 225$

Draw the MRx polygon

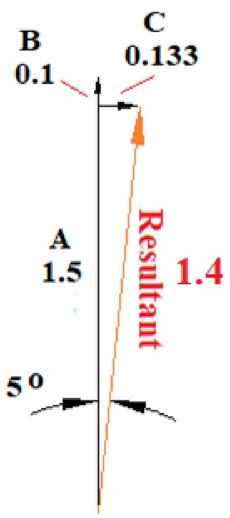


The resultant MRx vector is 3.9×10^{-3} kg m^2 at 2° clockwise of A

The moment produced in plane XX is $3.9 \times 10^{-3} \times \omega^2 \cos 2^\circ = 15.4$ Nm

SOLUTION – SECONDARY MOMENTS

	Mass (kg)	R (m)	n	x (m)	MRx/n (kg m^2)	angle (deg)
A	0.5	0.06	4	0.2	1.5×10^{-3}	$2\theta = 0$
B	0.25	0.03	3	0.3	0.1×10^{-3}	$2\alpha = 180$
C	0.25	0.033	3	0.4	0.133×10^{-3}	$2\beta = 450$

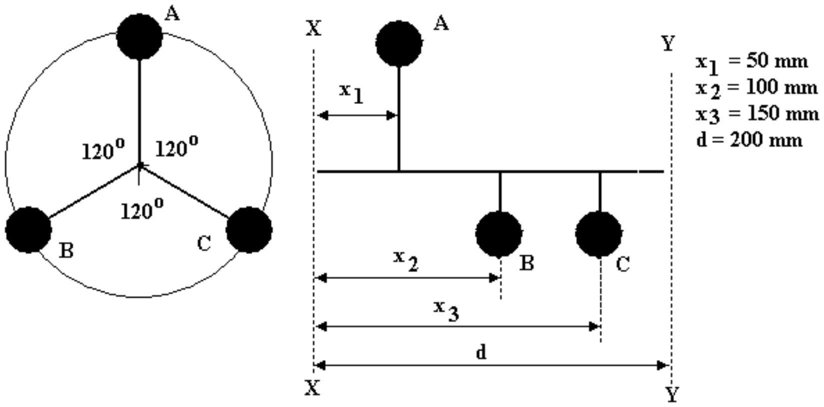


Draw the MRx/n polygon with A drawn at 0 degrees.

The resulting MRx/n vector is 1.4 kg m^2 at 5^o clockwise of A
The moment about XX is $1.4 \times 10^{-3} \times \omega^2 \cos 5^o = 5.51$ Nm

WORKED EXAMPLE No. 7

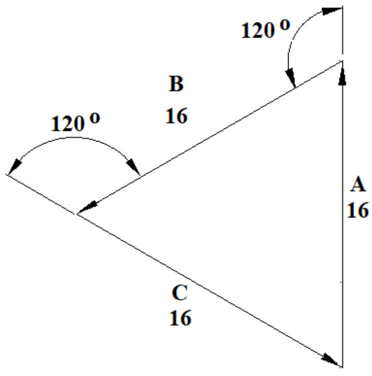
A compressor has three inline pistons of mass 0.4 kg with a crank radius of 40 mm and ratio n of 3. The cranks are equally spaced in angle and positioned as shown. Determine the primary and secondary force and turning moment about the reference plane X when it revolves at 30 rad/s.



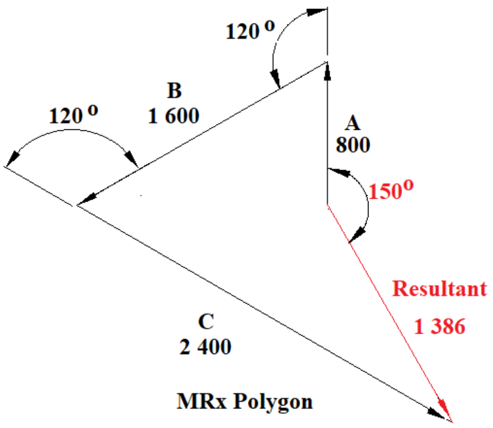
SOLUTION - PRIMARY

	Mass (kg)	Radius (m)	MR (kg m)	x (m)	MRx (kg m^2)	angle (degrees)
Piston A	0.4	0.04	16×10^{-3}	0.050	800 x 10^{-6}	$\theta = 0$
Piston B	0.4	0.04	16×10^{-3}	0.1	1 600 x 10^{-6}	α =120
Piston C	0.4	0.04	16×10^{-3}	0.15	2 400 x 10^{-6}	$\beta = 240$

Drawing the MR polygon with A vertical we see the resultant force is zero as expected.



MR Polygon



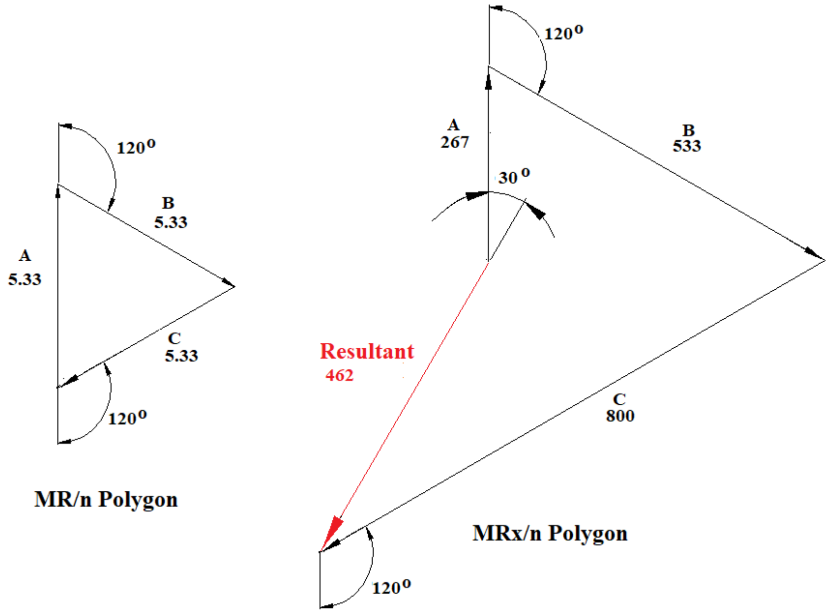
MRx Polygon

Draw the MRx polygon and by scaling or calculation the resultant is $1\,386 \times 10^{-6}$ kg m^2
The resulting moment about plane XX is $M_x = \omega^2 \times 1\,386 \times 10^{-6} \cos 30^o$
$M_x = 30^2 \times 1\,386 \times 10^{-6} \cos 30^o = 1.08$ Nm

SOLUTION - SECONDARY

	MR (kg m)	n	MR/n (kg m)	x (m)	MRx/n (kg m^2)	angle (degrees)
A	16×10^{-3}	3	5.33×10^{-3}	0.05	266.7×10^{-6}	$2\theta = 0$
B	16×10^{-3}	3	5.33×10^{-3}	0.1	533.3×10^{-6}	$2\alpha = 240$
C	16×10^{-3}	3	5.33×10^{-3}	0.15	800×10^{-6}	$2\beta = 480$

Draw the MR/n polygon with double angles and we again get a closed triangle showing that the secondary forces are balanced. Draw the MRx/n polygon with double angles and the resultant vector is $R = 462 \times 10^{-6}$ kg m^2



The resultant vector is 462×10^{-6} kg m^2

The moment produced is $462 \times 10^{-6} \times 30^2 \cos 30^o = 0.36$ N m about plane (XX)

(Note the moment to balance this is equal and opposite)

5. Balancing

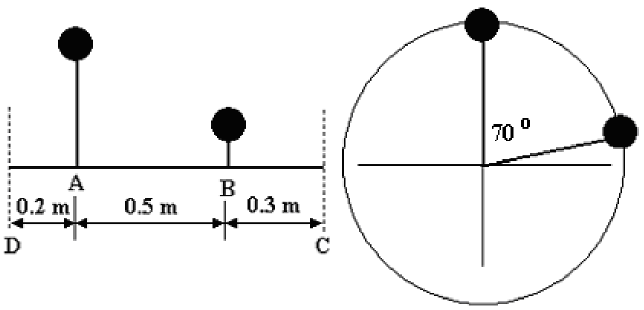
5.1 Reciprocating Balance

We know from the first balancing tutorial that in order to balance rotors we need to place balancing masses on two planes making one of them a reference plane. Reciprocating machines can be balanced in this way by placing reciprocating masses on two planes. To balance primary components the planes can be placed on the crank shaft to rotate at the crank speed. To balance secondary components we would place them on a second parallel shaft running at twice the speed. This gives us the double angles required for the MRx/n and MR/n polygons.

For the solution we first we draw the MRx polygon and deduce the primary balancing component for the moment about the reference plane. Adding this component we then draw the MR polygon to deduce the balancing component needed for all the forces. This is placed on the other reference plane where it will not add to the moment. We then repeat the process for the secondary forces and moments using the MRx/n and MR/n polygons. These are placed on the parallel shaft.

WORKED EXAMPLE No. 8

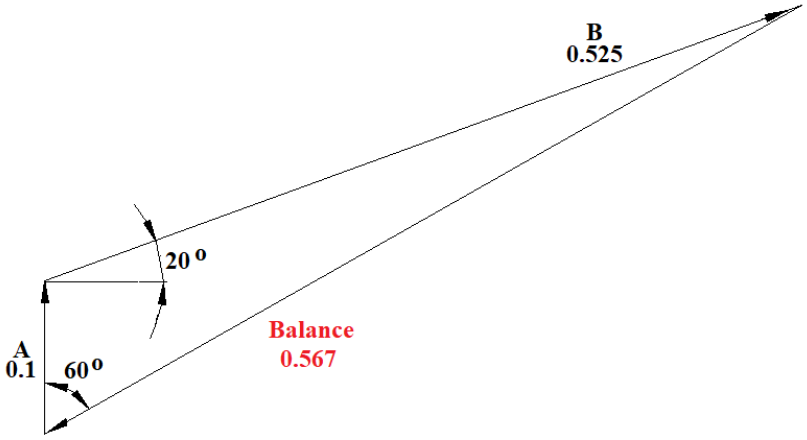
Two lines of reciprocating masses at A and B are to be balanced for ***Primary*** forces and couples by two lines of reciprocating pistons at C and D. Given $M_A = 0.5$ kg and $M_B = 0.75$ kg and that crank B is rotated 70^o relative to A, determine the masses M_C and M_D and the angle of their cranks. All crank radii are the same.



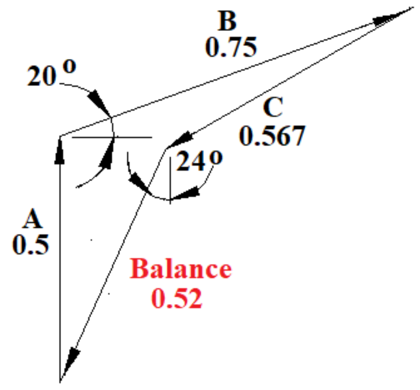
Make D the reference plane.

Mass	M kg	R m	MR kg m	x m	MRx kg m^2
A	0.5	R	0.5R	0.2	0.1R
B	0.75	R	0.75R	0.7	0.525R
C	M_C	R	M_C R	1.0	M_C R
D	M_D	R	M_D R	0	0

Draw the MRx polygon and using calculation or scaling find that for balance we need 0.567 kg m^2 120^o anticlockwise of A. For the same radius the mass will be 0.567 kg. This would be placed at A



Now draw the MR polygon with M_C R = 0.567 at the same radius.



Using trigonometry or scaling from the diagram reveals that for balance we need 0.52 R so M_D = 0.52 kg at the same radius and it must be placed at 204^o to crank A at located at D.

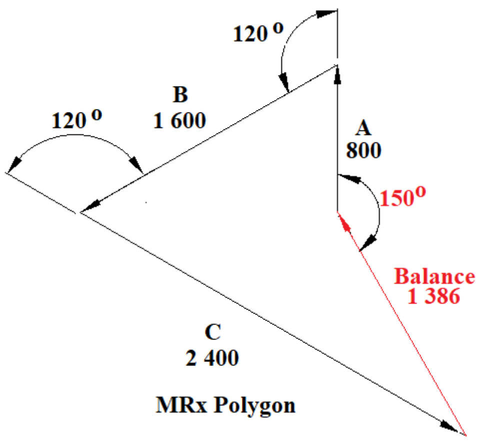
WORKED EXAMPLE No. 9

The system described in example 7 is to be balanced for primary forces and moments by placing a reciprocating mass in planes X and Y with the same crank radius and ratio n. The secondary forces and moments are to be balanced by using a parallel shaft running at double speed. Determine the masses and angles of the cranks for primary and secondary balance.

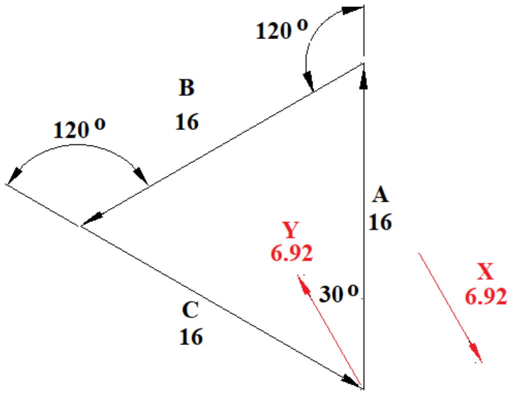
SOLUTION PRIMARY BALANCING

	Mass kg	Radius m	MR kg m	n	x m	MRx kg m^2	angle
X	M_x	0.04	0.04 M_x	3	0	0	
A	0.4	0.04	16×10^{-3}	3	0.05	800×10^{-6}	$\theta = 0$
B	0.4	0.04	16×10^{-3}	3	0.1	$1\,600 \times 10^{-6}$	$\alpha = 120$
C	0.4	0.04	16×10^{-3}	3	0.15	$2\,400 \times 10^{-6}$	$\beta = 240$
Y	M_y	0.04	0.04 M_y	3	0.2	$8\,M_y \times 10^{-3}$	

Draw the MRx polygon. The resultant is $1\,386 \times 10^{-6}$ kg m^2 at 30^o as shown.
$1\,386 \times 10^{-6} = 8 \times 10^{-3} M_y$ $M_y = 0.173$ kg located at Y.



Evaluate $0.04M_y = 6.92 \times 10^{-3}$ and draw the MR polygon.

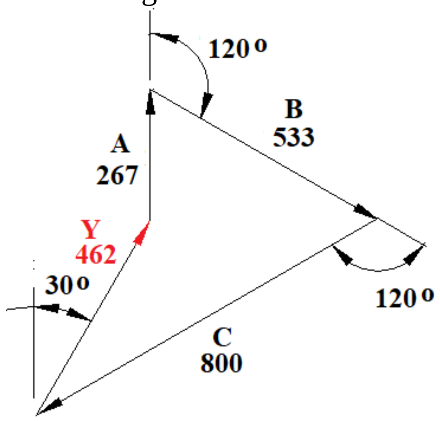


The closing vector is equal and opposite of Y
$0.04M_x = 6.92 \times 10^{-3}$ at 30^o as shown.
$M_x = 0.173$ kg located at X
For primary balance we need a piston mass of 173 g placed 30^o to A at Y and another at 150^o clockwise to A at X.

SECONDARY COMPONENTS

Mass kg		Radius m	n	MR/n kg m	x m	MRx/n kg m^2	angle
X	Mx	0.04	3	13.33×10^{-3}Mx	0	0	0
A	0.4	0.04	3	5.33×10^{-3}	0.05	266.7×10^{-6}	$2\theta = 0$
B	0.4	0.04	3	5.33×10^{-3}	0.1	533.3×10^{-6}	$2\alpha = 240$
C	0.4	0.04	3	5.33×10^{-3}	0.15	800×10^{-6}	$2\beta = 480$
Y	My	0.04	3	13.33×10^{-3}My	0.2	2.66×10^{-3}My	

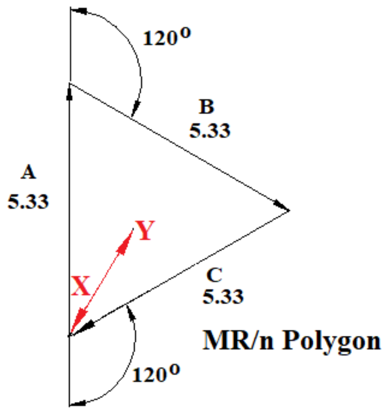
Draw the MRx/n polygon with double angles



From the MRx/n polygon we get a closing vector Y = 462 x 10^{-6} kg m^2 at 30^o as shown.

Equate $462 \times 10^{-6} = 2.66\, M_y \times 10^{-3}$ $\quad$ $M_y = 0.173$ kg located at Y

Now evaluate $13.33 \times 10^{-3}M_y = 2.306 \times 10^{-3}$ and draw the MR/n polygon at double angles.

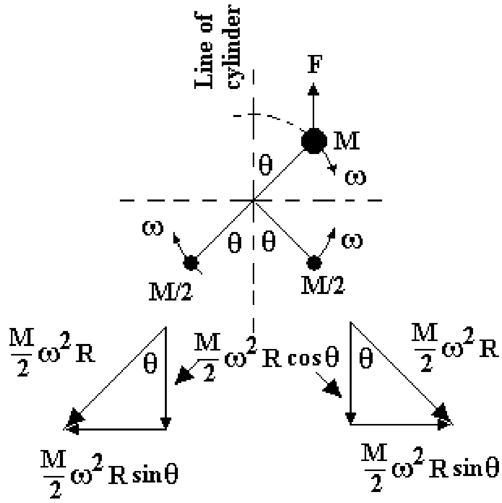


The closing vector is equal and opposite to Y so $13.33 \times 10^{-3} M_x = 2.3 \times 10^{-3}$ $M_x = 0.173$ kg.

Earlier it was argued that since the crank will revolve at double speed the mass to be used is M/4 so secondary balance will be produced by a piston of mass 173/4 = 43.3 g. The crank will have an angle of 30^o clockwise of A on Y and 43.3 g at 150^o anticlockwise to A on X.
These cranks to revolve at double speed.

5.2 Contra-Rotating Masses

A better method for balancing is to use equal contra-rotating masses. With these, the centrifugal force produces two components, one in line with the cylinder and one normal to it.



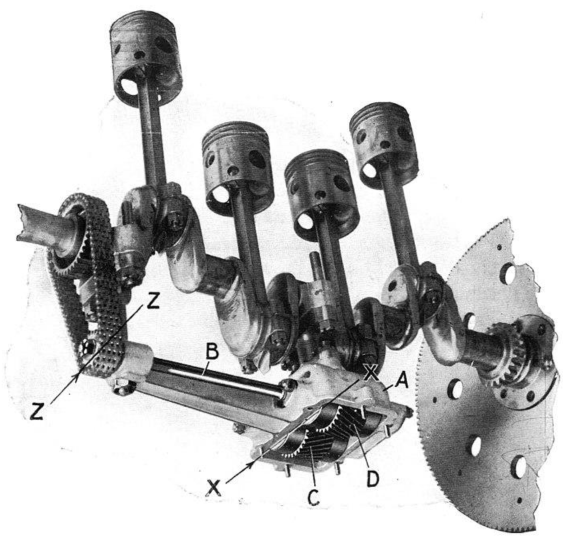
The centrifugal force produced by each is $(M/2)\omega^2R$ and resolving horizontally and vertically we see the horizontal components cancel and the vertical components add up to $M\omega^2R\cos\theta$ and so cancel the force produced by the piston. The mass and radius can be changed so long as the total product of (MR)/2 is the same.

For the balance of primary components, the contra-rotating masses revolve at the crank speed.
For secondary components the contra-rotating cranks must rotate at twice the crank speed (2ω) in order to satisfy the double angle requirement. It was argued earlier that the secondary mass is hence M/4 so the masses on contra rotating wheels must be M/8.

If we balanced the compressor in example 9 in this way, the mass on X and Y would be 173/2 = 86.5 g for primary balance and 21.6 g for secondary balance.

5.3 Lanchester Balancing System

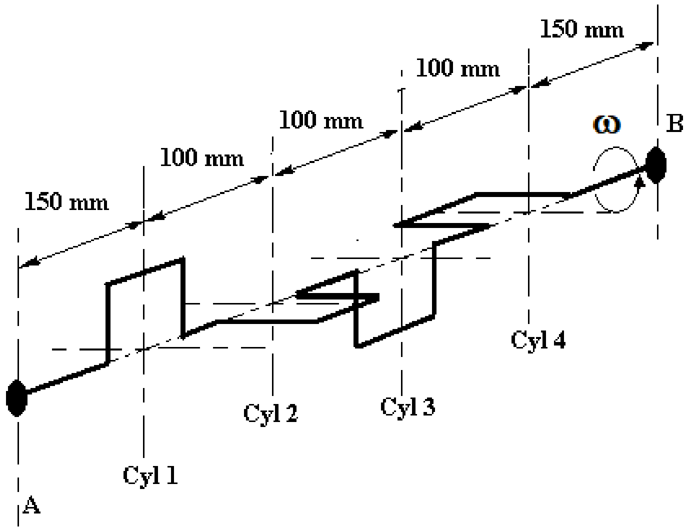
The balancing principles described in the previous section are embodied in the Lanchester balancer. Contra rotating parallel shafts are driven by the main shaft at double speed and have equal rotating weights arranged to eliminate the vibration. This is also known as harmonic balancing and is often the preferred balancing technique. The Lanchester balancer (inventor Frederick Lanchester 1907) is only used for machines where the pistons slide on a radial line through the centre of rotation. The picture below (unknown author) illustrates the principle. The weights are at C and D. There are many variations of this old design.



WORKED EXAMPLE No. 10

An air compressor has four cylinders in line with cranks as shown. The piston in each cylinder has a mass m of 400 g and each crank is 30 mm radius. The length L of the connecting-rod for each piston is 100 mm. The crankshaft is held in stiff bearings at ends A and B and rotates at ω rad/s.

In order to balance the primary and secondary components, two pairs of contra-rotating discs, to which balance weights can be attached, are fitted close to plane A and a further two pairs of contra-rotating wheels are fitted close to plane B. One pair at each end rotates at crankshaft speed, ω and the second pair at each end rotates at 2 ω. Determine the imbalance masses to be added given the radius is 30 mm. You may neglect the small distances between the discs and the bearings.



You may assume that the vertical acceleration of the pistons is given by

$$\omega^2 R\left[\cos(\theta) + \frac{\cos(2\theta)}{n}\right]$$

Where θ is the crankshaft angle and n = R/L

SOLUTION

The mass of the piston is M kg so the inertia force F produced is

$$F = M\omega^2 R\left[\cos(\theta) + \frac{\cos(2\theta)}{n}\right]$$

$M\omega^2 R$ is also the centrifugal force produced by a mass M rotating at radius R when ω is the angular velocity.

$\omega = 2\pi \times 500 = 100\,\pi$ rad/s R = 30 mm L = 100 mm n = 100/30 = 3.333 $\omega = 100\pi$ rad/s

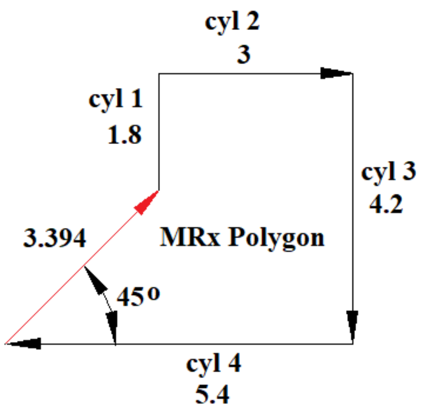
Both the primary and secondary forces are balanced as the value of MR is the same for each and the resultant is zero in both cases.

Primary Turning Moment

Making A the reference plane the primary turning moment is $M\,\omega^2 Rx \cos\theta$ where x is the distance from the reference plane. The table is:

Cylinder	M (kg)	$R \times 10^3$ (m)	$MR \times 10^3$ (kg m)	x (m)	$MRx \times 10^3$ (kg m^2)	Angle (deg)
A	M_A	30	$30\,M_A$	0	0	
1	0.4	30	12	0.15	1.8	$\theta = 0$
2	0.4	30	12	0.25	3.0	$\alpha = 90$
3	0.4	30	12	0.35	4.2	$\beta = 180$
4	0.4	30	12	0.45	5.4	$\gamma = 270$
B	M_B	30	$30\,M_B$	0.60	$18\,M_B$	

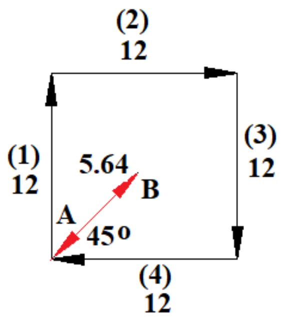
Draw the MRx polygon



The balancing MRx on plane B is $\sqrt{\{(2.4 \times 10^{-3})^2 + (2.4^2 \times 10^{-3})\}} = 3.394 \times 10^{-3}$ kg m^2 at 45^o as shown.

$18M_B \times 10^{-3} = 3.394 \times 10^{-3}$ $\quad M_B = 0.188$ kg or 188 g $\quad 30\,M_B = 30\,M_A = 30 \times 0.188 = 5.64$

For contra rotating masses at B this would be halved to 94.3 g and placed at 45^o either side (relative to crank 1). This would produce a force that has to be balanced with the same arrangement on plane A but rotated 180^o. This will not affect the moment balance. This can be shown by drawing the MR polygon



Draw the MR polygon. The closing vector A is equal and opposite to B.

$5.64 \times 10^{-3} = 30 \times 10^{-3}\,M_A$ $\qquad M_A = 0.188$ kg so 94.3 g would be placed at 45^o either side equal and opposite to those at B.

Secondary Turning Moment

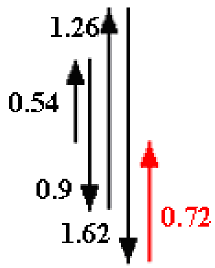
The secondary turning moment about any reference plane is

$$TM = M\omega^2 Rx\left[\frac{\cos(2\theta)}{n}\right]$$

Taking the reference plane as plane A for the turning moment for each cylinder (n = 3.333)

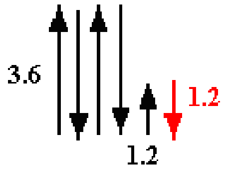
Cylinder	M (kg)	$R \times 10^3$ (m)	$MR/n \times 10^3$ (kg m)	x (m)	$MRx/n \times 10^3$ (kg m^2)	Angle (deg)
A	M_A	30	9 M_A	0		
1	0.4	30	3.6	0.15	0.54	$2\theta = 0$
2	0.4	30	3.6	0.25	0.9	$2\alpha = 180$
3	0.4	30	3.6	0.35	1.26	$2\beta = 360$
4	0.4	30	3.6	0.45	1.62	$2\gamma = 540$
B	M_B	30	9 M_B	0.60	5.4 M_B	

Draw the MRx/n polygon. All the vectors are vertical. The closing vector is hence 0.72×10^{-3} kg m^2 vertically up.



$5.4 \times 10^{-3} M_B = 0.72 \times 10^{-3}$ hence $M_B = 0.72/5.4 = 0.1333$ kg. The contra rotating mass on B will be 133/8 = 16.6 g and would be placed on the contra-rotating discs at 180^o to crank 1.

9 M_B = 1.2 kg m. Now draw the MR/n polygon

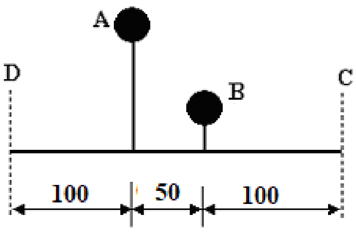


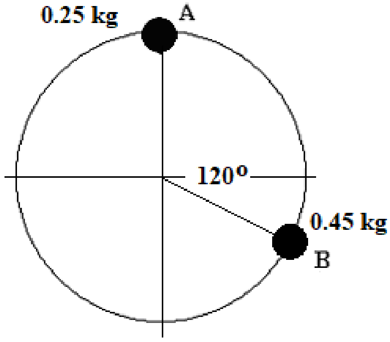
The closing vector is $1.2 \times 10^{-3} = 9\, M_A \times 10^{-3}$ $\quad M_A = 0.133$ kg

The contra-rotating mass will be 133/8 = 16.6 g and would be placed on the contra-rotating discs at 100^o to crank 1.

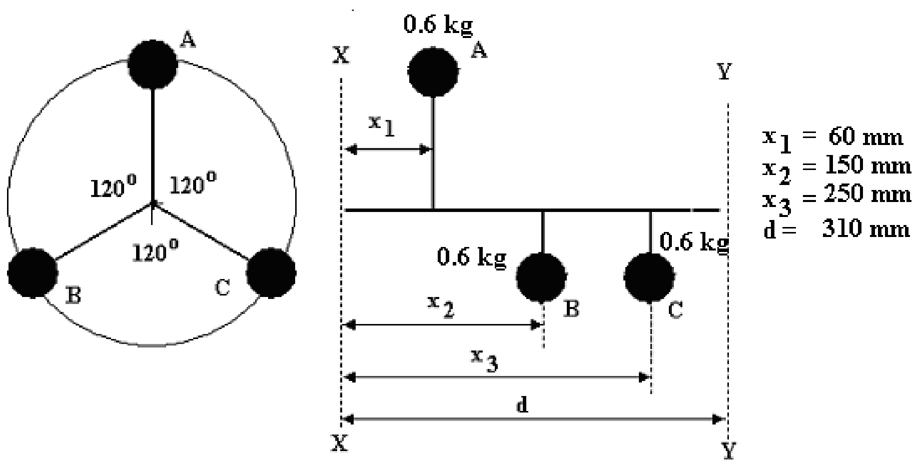
SELF ASSESSMENT EXERCISE No. 1

1. Two inline reciprocating masses at A and B are to be balanced for primary forces and couples by two reciprocating pistons at C and D in the same line as shown. A is 100 mm from D, B is 150 mm from D and C is 250 mm from D. Given M_A = 0.25 kg and M_B = 0.45 kg and that crank B is rotated 120^o relative to A, determine the masses M_C and M_D and the angle of their cranks. All crank radii are the same. Outline the procedure to balance the secondary forces and couples.
(0.236 kg at 81.5^o anticlockwise of A for C and 0.167 kg at 69^o clockwise of A)





2. A compressor has three inline pistons A, B and C positioned as shown with crank radii of 80 mm and connecting rods 240 mm long. The compressor is to be balanced for primary and secondary components by placing two sets of contra rotating masses at 50 mm radius at each bearing, one running at the crank speed for the primary balance and one at double the speed for secondary balance. Determine the masses and angles relative to crank A.



(For primary 255 g on Y at 31.7^o either side of A and 255 g at X at 211.7^o and 148.3^o. For secondary 63.7 g on X and Y at the same angles)

3. An engine has four cylinders in line with cranks equally spaced in order from 1 to 4. The piston in each cylinder has a mass m of 500 g and each crank is 40 mm radius. The length L of the connecting-rod for each piston is 120 mm. The crankshaft is held in stiff bearings at ends A and B and rotates at Ω rad/s. The bearings are 250 mm apart and the cranks are equally spaced at 50 mm intervals with a 50 mm space between the end cranks and the bearings.
In order to balance the primary and secondary components, two pairs of contra-rotating discs, to which balance weights can be attached, are fitted close to plane A and a further two pairs of contra-rotating wheels are fitted close to plane B. One pair at each end rotates at crankshaft speed, Ω and the second pair at each end rotates at 2Ω. Determine the imbalance masses to be added given the radius is 40 mm. You may neglect the small distances between the discs and the bearings.
(141.5 g 45^o either side of crank 1 and 25 g at 180^o to crank 1)

6. *An Analytical Approach*

The equations for force and moments developed earlier for multiple equal masses were

$$F_p = M\omega^2 R[\cos(\theta) + \cos(\theta+\alpha) + \cos(\theta+\beta) + \cdots] \quad \text{primary force}$$

$$F_s = M\omega^2 \frac{R}{n}[\cos 2(\theta) + \cos 2(\theta+\alpha) + \cos 2(\theta+\beta) + \cdots] \quad \text{secondary force}$$

$$TM_p = M\omega^2 R[x_1\cos(\theta) + x_2\{\cos(\theta+\alpha)\} + x_3\cos(\theta+\beta) + \cdots] \quad \text{primary moment}$$

$$TM_s = M\omega^2 \frac{R}{n}[x_1\{\cos(2\theta)\} + x_2\{\cos 2(\theta+\alpha)\} + x_3\cos 2(\theta+\beta) + \cdots] \quad \text{secondary moment}$$

All of these may be expanded using the trigonometry identity cos (A+B) = cos A cos B – sin A sin B
This gives us:

$$F_p = M\omega^2 R[\cos(\theta)\{1 + \cos(\alpha) + \cos(\beta) + \cdots\} - \sin(\theta)\{\sin(\alpha) + \sin(\beta) + \cdots\}] \quad \text{primary force}$$

$$F_s = M\omega^2 \frac{R}{n}[\cos(2\theta)\{1 + \cos(2\alpha) + \cos 2(\theta+\beta) + \cdots\} - \sin(2\theta)\{\sin(2\alpha) + \sin(2\beta) + \cdots\}] \quad \text{secondary force}$$

$$TM_p = M\omega^2 R[x_1\cos(\theta) + x_2\cos(\theta)\cos(\alpha) - x_2\sin(\theta)\sin(\alpha) + x_3\cos(\theta)\cos(\beta) - x_3\sin(\theta)\sin(\beta)] \quad \text{primary moment}$$

$$TM_s = M\omega^2 \frac{R}{n}[x_1\cos(2\theta) + x_2\cos(2\theta)\cos(2\alpha) - x_2\sin(2\theta)\sin(2\alpha) + x_3\cos(2\theta)\cos(2\beta) - x_3\sin(2\theta)\sin(2\beta)] \quad \text{secondary moment}$$

If the system is balanced, these would equate to zero. If the mass M and radius R are the same for all cylinders, we can split each into two expressions that must be equated to zero.

$\{1 + \cos\alpha + \cos\beta \ldots\} = 0$………………..…(1)	primary force
$\{\sin\alpha + \sin\beta + \ldots\} = 0$………………….....(2)	primary force
$1+\cos(2\alpha) + \cos(2\beta)+\ldots = 0$..…...………….(3)	secondary force
$\sin(2\alpha) + \sin(2\beta)+\ldots = 0$…………………..(4)	secondary force
$x_1 + x_2\cos\alpha + x_3\cos\beta + x_4\cos\gamma + \ldots = 0$ ...(5)	primary moment
$x_2\sin\alpha + x_3\sin\beta + x_4\sin\gamma + \ldots = 0$ ...…….(6)	primary moment
$x_1 + x_2\cos 2\alpha + x_3\cos 2\beta \ldots = 0$ ….....…....(7)	secondary moment
$x_2\sin 2\alpha + x_3\sin 2\beta \ldots = 0$ ……………….(8)	secondary moment

These may be used to determine how to balance a system.

WORKED EXAMPLE No. 11a

Using the criteria just developed determine the state of balance for the 2 crank system in worked example 3a with $x_1 = c$ $x_2 = 2c$

SOLUTION

We must satisfy equation 1, 2, 5 and 6 with $\alpha = 180^o$

(1)	$1 + \cos\alpha = 1 - 1 = 0$	primary force balanced
(2)	$\sin\alpha = 0$	primary force balanced
(3)	$1 + \cos(2\alpha) = 1 + 1 = 2$	secondary force not balanced
(4)	$\sin(2\alpha) = 0$	secondary force balanced
(5)	$x_1 + x_2\cos\alpha = x_1 - x_2 = -c$	primary moment not balanced
(6)	$x_2\sin\alpha = 0$	primary moment balanced
(7)	$x_1 + x_2 \cos 2\alpha = x_1 + x_2 = 3c$	secondary moment not balanced
(8)	$x_2 \sin 2\alpha = 0$	secondary moment balanced

Only the primary force is completely balanced.
There is a primary moment of $TM_p = -cM\omega^2R\cos\theta$
a secondary force of $F_s = 2M\omega^2(R/n)\cos(2\theta)$
a secondary moment of $F_s = 3cM\omega^2(R/n)\cos(2\theta)$

WORKED EXAMPLE No. 11b

Using the criteria just developed determine the state of balance for the 3 piston system in worked example 3b given $x_1 = c$ $x_2 = 2c$ $x_3 = 3c$

SOLUTION

$\alpha = 120^o$ and $\beta = 240^o$

(1)	$1 + \cos\alpha + \cos\beta = 1 - 0.5 - 0.5 = 0$	primary force balanced
(2)	$\sin\alpha + \sin\beta = 0.866 - 0.866 = 0$	primary force balanced
(3)	$1 + \cos(2\alpha) + \cos(2\beta) = 1 - 0.5 - 0.5 = 0$	secondary force balanced
(4)	$\sin(2\alpha) + \sin(2\beta) = -0.866 + 0.866 = 0$	secondary force balanced
(5)	$x_1 + x_2\cos\alpha + x_3\cos\beta = c - 0.5(2c) - 0.5(3c) = -1.5c$	primary moment not balanced
(6)	$x_2\sin\alpha + x_3\sin\beta = 0.866(2c) - 0.866(3c) = -0.866c$	primary moment not balanced
(7)	$x_1 + x_2\cos 2\alpha + x_3\cos 2\beta = c - 0.5(2c) - 0.5(3c) = -1.5c$	secondary moment not balanced
(8)	$x_2\sin 2\alpha + x_3\sin 2\beta = -0.866(2c) + 0.866(3c) = 0.866c$	secondary moment not balanced

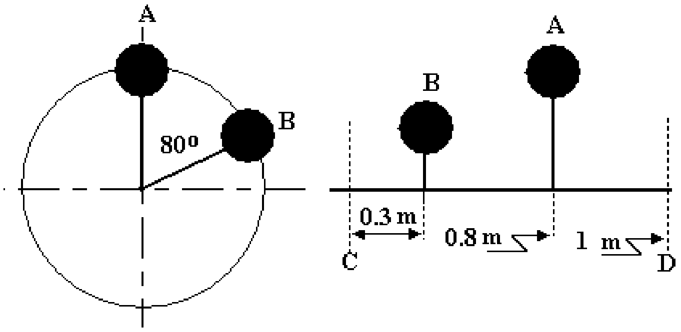
There is complete force balance but there are unbalanced moments of

$$F_p = 0.866c\ M\omega^2R\sin\theta - 1.5c\ M\omega^2R\cos\theta = c\ M\omega^2R\{0.866\sin\theta - 1.5\cos\theta)$$

$$F_s = -1.5c\ M\omega^2(R/n)\cos2\theta - 0.866c\ M\omega^2(R/n)\sin2\theta = -c\ M\omega^2(R/n)\{1.5\cos2\theta + 0.866\sin2\theta$$

WORKED EXAMPLE No. 12

Two lines of reciprocating parts at A and B are to be balanced for primary forces and couples by two lines of reciprocating parts C and D. Given M_A = 500 g M_B = 750 g and $\alpha = 80^o$, find the masses and angles for C and D. Determine the unbalanced secondary components.



SOLUTION

We must modify equations equation 1, 2, 5 and 6 to take account of the different masses and distances. They become:

(1) $M_A + M_B \cos\alpha + M_C \cos\beta + M_D \cos\gamma = 0$ primary force
(2) $M_B \sin\alpha + M_C \sin\beta + M_D \sin\gamma = 0$ primary force
(5) $M_A x_A + M_B x_B \cos\alpha + M_C x_C \cos\beta + M_D x_D \cos\gamma = 0$ primary moment
(6) $M_B x_B \sin\alpha + M_C x_C \sin\beta + M_D x_D \sin\gamma = 0$ primary moment

β is the angle of M_C and γ is the angle of M_D
(1) $0.5 + 0.75 \cos 80^o + M_C \cos\beta + M_D \cos\gamma = 0$
(2) $0.75 \sin 80^o + M_C \sin\beta + M_D \sin\gamma = 0$
(5) $(0.5 \text{ x } 0.8) + (0.75 \text{ x } 0.3)\cos 80^o + 0 + M_D \cos\gamma = 0$
(6) $(0.75 \text{ x } 0.3)\sin 80^o + 0 + M_D \sin\gamma = 0$

Rearrange

(1) $M_C \cos\beta + M_D \cos\gamma = -0.63$
(2) $M_C \sin\beta + M_D \sin\gamma = -0.74$
(5) $M_D \cos\gamma + 0 = -0.439$
(6) $M_D \sin\gamma = -0.222$

From (5) and (6) $\tan\gamma = \sin\gamma/\cos\gamma = 0.22/0.439 = 0.506$ $\gamma = 26.8^o$ or 206.8^o
Since $\sin\gamma$ and $\cos\gamma$ are both negative, γ must lie between 180^o and 270^o
From (6) $M_D = -0.439/\cos\gamma = 0.492$ kg

From (1) $M_C \cos\beta + 0.492 \cos 206.8^o = -0.63$ $M_C \cos\beta = -0.191$
From (2) $M_C \sin\beta + 0.492 \sin 206.8^o = -0.74$ $M_C \sin\beta = -0.518$
$\tan\beta = \sin\beta/\cos\beta = 0.518/0.191 = 2.71$ $\beta = 69.8$ or 249.8
Since $\sin\beta$ and $\cos\beta$ are negative it must be the angle between 180^o and 270^o
It follows that $M_C = -0.518/\sin 249.8^o = 0.552$ kg

Secondary Components

We must modify equations (3), (4), (7) and (8)

(3)	$M_A + M_B\cos(2\alpha) + M_C\cos(2\beta) + M_D\cos(2\gamma) = 0$	secondary force balanced
(4)	$M_B\sin(2\alpha) + M_C\sin(2\beta) + M_D\sin(2\gamma) = 0$	secondary force balanced
(7)	$M_A x_A + M_B x_B \cos 2\alpha + M_C x_C \cos 2\beta + M_D x_D \cos 2\gamma = 0$	secondary moment not balanced
(8)	$M_B x_B \sin 2\alpha + M_C x_C \sin 2\beta + M_D x_D \sin 2\gamma = 0$	secondary moment not balanced

From (3) $M_A + M_B\cos(2\alpha) + 0.492\cos(2\beta) + 0.552\cos(2\gamma)$

$0.5 + 0.75\cos(160^o) + 0.552\cos(413.6^o) + 0.552\cos(499.6^o) = -0.333$ kg

From (4) $M_B\sin(2\alpha) + M_C\sin(2\beta) + M_D\sin(2\gamma)$

$0.75\sin(160^o) + 0.492\sin(499.6^o) + 0.492\sin(413.6^o) = 1.01$ kg

From (7) $M_A x_A + M_B x_B \cos 2\alpha + M_C x_C \cos 2\beta + M_D x_D \cos 2\gamma$

$(0.5 \times 0.8) + (0.75 \times 0.3)\cos(160) + (0.552 \times 0)\cos(499.6^o) + (0.492 \times 1)\cos(413.6^o)$

$= 0.4 - 0.211 + 0 + 0.29 \quad = 0.481$ kg m

From (8) $M_B x_B \sin 2\alpha + M_C x_C \sin 2\beta + M_D x_D \sin 2\gamma = 0$

$(0.75 \times 0.3) \sin(160) + (0.552 \times 0)\sin(499.6^o) + (0.492 \times 1)\sin(413.6^o)$

$= 0.0769 + 0 + 0.396 = 0.473$ kg m

The unbalanced moment is $M\omega^2(R/n)(0.481 \cos 2\theta - 0.473 \sin 2\theta)$

Further studies in this area would include cylinders not in one line such as the Vee configuration.