OUTCOME 3 – MASS – SPRING SYSTEMS

TUTORIAL 3 FORCED VIBRATIONS

3 Be able to determine the behavioural characteristics of translational and rotational mass-spring systems

Natural vibrations: mass-spring systems; transverse vibrations of beams and cantilevers; torsional vibrations of single and two-rotor systems; determination of natural frequency of vibration; whirling of shafts

Damped vibrations: representative second-order differential equation for mass-spring system with damping; transient response of a mass-spring system to an impulsive disturbance; degrees of damping; frequency of damped vibrations; logarithmic decrement of amplitude

Forced vibrations: representative second-order differential equation for a damped mass-spring system subjected to a sinusoidal input excitation; transient and steady state solutions; amplitude and phase angle of the steady state output; effect of damping ratio; conditions for resonance

On completion of this tutorial you should be able to do the following.

- Define a forced vibration in general terms.
- Solve problems involving mass – spring – damper systems.
- Analyse the case of a harmonic disturbing force.
- Analyse the case of a harmonic disturbance of the support.
- Analyse the frequency response for the same.
- Define and use phasors.

This tutorial covers the theory of forced vibrations. You should study the tutorial on free vibrations before commencing.
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1. **INTRODUCTION**

In the tutorial on damped oscillations, it was shown that a free vibration dies away with time because the energy trapped in the vibrating system is dissipated by the damping. The equation for the displacement in a damped oscillation was derived and given as

\[ x = Ce^{-\delta \omega_n t} \cos(\omega t) \]

\( \delta \) is the damping ratio and \( \omega_n \) the natural angular frequency. The following cases were described.

When \( \delta > 1 \) we have an over damped system.
When \( \delta = 1 \) we have a critically damped oscillation.
When \( \delta < 1 \) we have a damped oscillation that dies away with time.
When \( \delta = 0 \) we have a system with no damping and a steady oscillation occurred.

It might be inferred from this pattern that if \( \delta < 0 \) we get an oscillation that grows with time. The diagram illustrates this pattern.

![Figure 1](image_url)

In order for the damping ratio \( \delta \) to be less than zero, that is, to be negative, we would have to have the opposite of damping, something that puts energy into the system instead of taking it out. As the energy is added to the system the amplitude grows and grows. The energy is added by an outside source and such oscillations are called forced, (the object of this tutorial). A good example of such an oscillation is a child on a swing. If nothing is done, friction will make the swing come to a halt. If someone gives the swing a small push at the start of each swing, energy is added to the system and the swing goes higher and higher. This phenomenon is also known as excitation.

In engineering, many structures are prone to vibrate when excited at or near the natural frequency. A good example is what happens to a car when the wheels are out of balance or when you drive along a corrugated surface. If the disturbance is close to the natural frequency of the suspension system the vehicle might bounce out of control. Vehicles are fitted with dampers to prevent this. The wind blowing around chimney stacks, cooling towers and suspended cables can excite them into catastrophic oscillations.
2. **FORCED VIBRATIONS**

We must examine two common types of forced vibrations, first when a mass has a disturbing force acting on it and second when the spring support is disturbed harmonically.

2.1. **HARMONIC DISTURBING FORCE**

Consider an ideal system as shown. A mass M is suspended on a spring and a damper is placed between the spring and the support. The support does not move. Located on the mass is a small rotating machine that is out of balance. It has the equivalent of a small mass m rotating at radius r that produces an out of balance force due to the centripetal/centrifugal affect. The magnitude of this force is \( F_0 = mr^2\omega^2 \). The main mass is constrained in guides so that it will only move up and down (one degree of freedom). At the position shown with rotation through angle \( \theta \), the component of \( F_0 \) acting vertically is \( F = F_0 \sin \theta \).

Any force applied to the mass to make it move must overcome the inertia, damping and spring force. The applied force is hence

\[
F = F_i + F_d + F_s
\]

\[
F = M \frac{d^2x}{dt^2} + c \frac{dx}{dt} + kx
\]

In this case the mass can only move vertically so the only force applied to it in this direction is the vertical component of the centrifugal force.

\[
F_0 \sin(\omega t) = M \frac{d^2x}{dt^2} + c \frac{dx}{dt} + kx
\]
2.1.1 PHASOR REPRESENTATION

We may assume (and it is known from observations) that the mass is going to oscillate up and down with a sinusoidal oscillation of amplitude $A$. Let's assume that time starts when the oscillation passes through the rest position. The displacement is given by

$$x = A \sin \omega t$$

where $A$ is the amplitude.

The velocity is then

$$v = \frac{dx}{dt} = A\omega \cos \omega t$$

where $A\omega$ is the amplitude.

The acceleration is

$$a = \frac{dv}{dt} = -A\omega^2 \sin \omega t$$

where $A\omega^2$ is the amplitude.

The displacement $x$, velocity $v$ and acceleration are plotted against time in the diagram below. Each graph may be generated by a vector rotating at $\omega$ rad/s and with a length equal to the amplitude. Such vectors are called phasors. At a given moment in time, the tip of each vector is projected across to the appropriate point as shown.

![Diagram showing phasor representation of oscillation](image)

**Figure 3**

We can see that in order to produce the result, the velocity vector is $90^\circ$ in front of the displacement and the acceleration is $90^\circ$ in front of the velocity.

The spring force is directly proportional to displacement $x$ so it must be in phase with $x$. The damping force is directly proportional to the velocity $v$ so it must be in phase with $v$. The inertia force is directly proportional to the acceleration $a$ so it must be in phase with $a$. It follows that the three forces can also be represented by phasors all rotating at angular velocity $\omega$ rad/s. We can choose a moment in time when the displacement is horizontal as shown.

The spring force is in phase with the movement so we draw the vector horizontally. The other vectors are $90^\circ$ and $180^\circ$ ahead respectively.

![Diagram showing forces in phasor form](image)

**Figure 4**

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The sum of these three vectors is $F_0$ so adding them we get a typical vector diagram as shown.

![Figure 5](image_url)

The diagram shows that the applied force $F_0$ is at an angle $\phi$ to the horizontal so it must be displaced by a phase angle $\phi$ relative to $x$. Applying trigonometry we have

$F_0^2 = \left( kA - MA\omega^2 \right)^2 + (cA\omega)^2$ \hspace{1cm} Pythagoras

$F_0^2 = A^2\left( k - M\omega^2 \right)^2 + A^2(c\omega)^2$

$F_0^2 = A^2\left( k - M\omega^2 \right)^2 + A^2(c\omega)^2$ \hspace{1cm} Divide every term by $M^2$

$\frac{F_0^2}{M^2} = A^2\left( \frac{k}{M} - \omega^2 \right)^2 + \left( \frac{c\omega}{M} \right)^2$

It has been shown in the tutorial on damped vibrations that $\omega_n^2 = \frac{k}{M}$ and $\frac{c}{M} = 2\delta \omega_n$

$A^2 = \left( \frac{F_0}{M} \right)^2 \left\{ \frac{1}{\left( \omega_n^2 - \omega^2 \right)^2 + \left( 2\delta \omega \omega_n \right)^2} \right\}$

From the triangle we also get the phase angle.

$\tan \phi = \frac{2\delta \omega \omega_n}{\omega_n^2 - \omega^2}$

Plotting $x$ and $F_0$ for a given applied frequency against time produces a graph similar to below.

![Figure 6](image_url)
3.1.2 **FREQUENCY RESPONSE DIAGRAMS**

Suppose we start the out of balance machine and gradually increase the speed $\omega$ from zero. Taking a typical value of $\omega_n = 10$ and plotting $\phi$ against $\omega$ for various values of $\delta$ produces the graph below.

![Frequency Response Diagram](image)

Figure 7

The plot shows that the phase angle $\phi$ starts at zero and reaches $90^\circ$ when $\omega = \omega_n$. As the speed increases to large values, the phase angle approaches $180^\circ$.

Now consider what happens to the amplitude $A$. Plotting $A$ against $\omega$ for various values of $\delta$ gives the graph below.

![Amplitude vs Frequency](image)

Figure 8

The following analyses the equation for $A$ at the three obvious points

$$A^2 = \left(\frac{F_0}{M}\right)^2 \frac{1}{\left(\omega_n^2 - \omega^2\right)^2 + \left(2\delta \omega \omega_n\right)^2}$$

when $\omega = 0$ this reduces to

$$A = \left(\frac{F_0}{M}\right) \frac{1}{\omega_n^2}$$

This has a finite value (2 on the diagram) and this is the same for all values of $\delta$. 

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When \( \omega = \omega_a \) the amplitude becomes

\[ A = \left( \frac{F_0}{M} \right) \left( \frac{1}{2\delta \omega^2} \right) \]

and the value depends upon the value of \( \delta \).

The smaller the value of the damping ratio, the greater the peak value of \( A \) becomes. If \( \delta = 0 \) then in theory \( A = \infty \).

When the frequency becomes very large, the amplitude tends to die away to zero for all values of \( \delta \).

We may conclude from this, that when the out of balance machine rotates very fast, there is very little disturbance to the system but when it approaches the natural frequency of the system the amplitude might become very large depending on the damping. It should also be noted that the frequency at which the amplitude peaks is called the resonant frequency and this is not quite the same as the natural frequency.

**WORKED EXAMPLE No.1**

The diagram shows a mass-spring-dashpot system. The mass has a harmonic disturbing force applied to it given by the equation \( F' = 400 \sin(30 t) \) Newton.

Determine the amplitude of the mass and the phase angle.

![Diagram](image-url)
SOLUTION

From the question we know that \( k = 10000 \), \( M = 5 \) and \( c = 150 \)

\[
\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{10000}{5}} = 44.72 \text{ rad/s}
\]

\[
c_c = \sqrt{4Mk} = \sqrt{4 \times 5 \times 10000} = 44.72
\]

\[
\delta = \frac{c}{c_c} = \frac{150}{447.21} = 0.335
\]

From the equation of motion \( F_0 = 400 \text{ N} \) and \( \omega = 30 \text{ rad/s} \)

\[
A^2 = \left( \frac{F_0}{M} \right)^2 \left( \frac{1}{(\omega_n^2 - \omega^2)^2 + (2\delta \omega \omega_n)^2} \right)
\]

\[
A^2 = \left( \frac{400}{5} \right)^2 \left( \frac{1}{(44.72^2 - 30^2)^2 + (2 \times 0.335 \times 30 \times 44.72)^2} \right)
\]

\[
A^2 = 0.00317 \quad A = 0.056 \text{ m or 56 mm}
\]

\[
\tan \phi = \frac{2\delta \omega \omega_n}{\omega_n^2 - \omega^2} = \frac{2 \times 0.335 \times 30 \times 44.72}{44.72^2 - 30^2} = 0.818
\]

\[
\phi = 39.3^\circ
\]

SELF ASSESSMENT EXERCISE No.1

1. A mass of 12 kg rests on a springy base of stiffness 8 kN/m. There is a damper between the mass and the support with a damping coefficient of 400 N s/m. The support is subjected to a harmonic disturbing force given by \( F = 200\sin(30t) \). Calculate the amplitude of the mass and the phase angle.

(16 mm and 103.1 degrees).

2. A mass of 150 kg rests on a springy base of stiffness 60 kN/m. There is a damper between the mass and the support with a damping coefficient of 5000 N s/m. The support is subjected to a harmonic disturbing force given by \( F = 800\sin(25t) \). Calculate the amplitude of the mass and the phase angle.

(6.2 mm and 105.1 degrees).
2.2 HARMONIC MOVEMENT OF THE SUPPORT

The diagram shows a mass spring damper system. The mass can only move vertically. The support is made to move up and down by a cam that rotates at $\omega$ rad/s with amplitude $a$. If time starts when the support passes through the mean position, the motion of the support is described by the equation $y = a \sin(\omega t)$.

We may assume that the mass is going to move up and down harmonically with an amplitude $A$ but we cannot assume that the motion is in phase with the support so the equation of motion will be given by the equation $x = A \sin(\omega t + \phi)$ where $\phi$ is the phase angle.

![Diagram showing mass spring damper system](image)

At any given moment in time the spring is stretched or shortened by an amount $x - y$ at any time. The spring force is hence $F = k(x-y)$. The three forces acting on the mass are:

- Spring force $k(x-y)$
- Damping force $c \frac{dx}{dt}$
- Inertia force $M \frac{d^2x}{dt^2}$

In this case there is no applied so the force balance gives

$$0 = M \frac{d^2x}{dt^2} + c \frac{dx}{dt} + k(x - y)$$

$$0 = M \frac{d^2x}{dt^2} + c \frac{dx}{dt} + kx - ky$$

$$ky = M \frac{d^2x}{dt^2} + c \frac{dx}{dt} + kx$$

$$k(a \sin \omega t) = M \frac{d^2x}{dt^2} + c \frac{dx}{dt} + kx$$

Compare this with the previous case. $F_0 \sin(\omega t) = M \frac{d^2x}{dt^2} + c \frac{dx}{dt} + kx$

This is the same except the term $ka$ replaces the term $F_0$. It follows that the solutions are the same with this substitution.

$$A^2 = \left( \frac{ka}{M} \right)^2 \left[ \frac{1}{(\omega^2 - \omega_n^2)^2 + (2\delta \omega \omega_n)^2} \right]$$

$$\tan \phi = \frac{2\delta \omega \omega_n}{\omega_n^2 - \omega^2}$$
2.2.1 MAGNIFICATION FACTOR

The magnification factor is the ratio \( \frac{A}{a} \) when the support is excited. The last equation may be
arranged into the following form.

\[
MF = \frac{A}{a} = \left( \frac{k}{M} \right) \sqrt{\frac{1}{\left( \frac{\omega_n^2}{\omega^2} \right)^2 + \left( 2\delta \omega_n \right)^2}}
\]

Since \( k/M = \omega_n^2 \) then

\[
MF = \sqrt{\frac{\left( \frac{\omega_n^2}{\omega^2} \right)^2}{\left( \frac{\omega_n^2}{\omega^2} \right)^2 + \left( 2\delta \omega_n \right)^2}} = \frac{1}{\sqrt{\left( 1 - \frac{\omega^2}{\omega_n^2} \right)^2 + \left( 2\delta \omega_n \right)^2}}
\]

This formula also applies to the case when a harmonic disturbing force is applied since \( k_A = F_0 \) and
it follows that \( MF = \) Maximum force in spring/F_0

The response graph is shown below and could apply to either case. At low speeds the support and
mass move up and down together. As \( \omega \) approaches \( \omega_n \) the amplitude of the mass grows and the
phase angle approaches 90°. As the speed passes resonance, the amplitude of the mass reduces and
eventually becomes almost static. The phase angle tends to 180° at high speeds. The magnification
is greatest at resonance and as before, the resonant frequency is not quite the same as the natural
frequency.

![Graph showing the magnification factor](image)

Figure 11

The PEAK MF occurs when \( \left( 1 - \frac{\omega^2}{\omega_n^2} \right)^2 + \left( 2\delta \frac{\omega}{\omega_n} \right)^2 \) is a maximum. This can be found by max and
min theory. Simplify this to \( \left( 1 - \frac{\omega^2}{\omega_n^2} \right)^2 \) is a maximum. This can be found by max and

\[
d\left[ \left( 1 - \frac{\omega^2}{\omega_n^2} \right)^2 + \left( 2\delta \frac{\omega}{\omega_n} \right)^2 \right] = 2(1 - \frac{\omega^2}{\omega_n^2})(-2\delta \omega_n) + 8\delta^2 \frac{\omega}{\omega_n} \frac{dr}{dr}
\]

Equate to zero and

\[
2(1 - \frac{\omega^2}{\omega_n^2})(-2\delta \omega_n) + 8\delta^2 \omega_n = 0
\]

\[
r = \sqrt{1 - 2\delta^2}
\]

Peak MF occurs when

\[
\omega = \omega_n \sqrt{1 - 2\delta^2}
\]
WORKED EXAMPLE No.2

The diagram shows a mass-spring-dashpot system. The support is moved with a motion of \( y = 6 \sin(40t) \) mm. Determine the amplitude of the mass and the phase angle.

\[ k = 10000 \text{ N/m} \]
\[ M = 5 \text{ kg} \]
\[ c = 150 \text{ Ns/m} \]

Figure 12

**SOLUTION**

From the question we know that \( k = 10000 \), \( M = 5 \) and \( c = 150 \)

\[ \omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{10000}{5}} = 44.72 \text{ rad/s} \]

\[ c_c = \sqrt{4Mk} = \sqrt{4 \times 5 \times 10000} = 447.21 \]

\[ \delta = \frac{c}{c_c} = \frac{150}{447.21} = 0.335 \]

From the equation of motion \( a = 6 \) mm and \( \omega = 40 \text{ rad/s} \)

\[ A = \left( \frac{k}{M} \right)^{\frac{1}{2}} \frac{1}{\left( \omega_n^2 - \omega^2 \right)^2 + (2\delta \omega \omega_n)^2} \]

\[ A = \left( \frac{10000}{5} \right)^{\frac{1}{2}} \frac{1}{\left( 44.72^2 - 40^2 \right)^2 + (2 \times 0.335 \times 40 \times 44.72)^2} \]

\[ A = 1.581 \quad A = 1.581 \times 6 = 9.487 \text{ mm} \]

\[ \tan \varphi = \frac{2\delta \omega \omega_n}{\omega_n^2 - \omega^2} = \frac{2 \times 0.335 \times 40 \times 44.72}{44.72^2 - 40^2} = 3 \]

\[ \varphi = 71.56^\circ \]
2.2.2 TRANSMISSIBILITY

When a mass vibrates on an elastic support, a force is transmitted through the spring and damper to the frame or ground. This is the sum of the spring and damping force. This may be illustrated with the vector diagram.

From the vector diagram we deduce that the transmitted force is \( F_T = \sqrt{(F_s^2 + F_d^2)} \)

\( F_s = kA \) and \( F_d = cA\omega \) \( F_T = \sqrt{(kA)^2 + (cA\omega)} \)

The ratio \( F_T/F_o \) is called the transmissibility ratio.

The phase angle between the transmitted force and the applied force is \( \phi_T = \phi - \tan^{-1}(F_d/F_s) \)

The above work applies to both harmonic disturbing forces and harmonic motion of the support if the substitution \( F_o = ka \)

WORKED EXAMPLE No.3

Calculate the transmitted force and the phase angle of the transmitted force for example No.2.

SOLUTION

In example 2 we calculated \( A = 56 \text{ mm} \) \( k = 10 \text{ kN/m} \) and \( \omega = 30 \text{ rad/s} \) \( \phi = 39.3^\circ \)

\( F_s = kA = 10000 \times 0.056 = 560 \text{ N} \)

\( F_d = cA\omega = 150 \times 0.056 \times 30 = 252 \text{ N} \)

\( F_T = \sqrt{(560^2 + 252^2)} = 614.1 \text{ N} \)

\( \phi_T = 39.3 - \tan^{-1}(252/560) = 15.1^\circ \)
SELF ASSESSMENT EXERCISE No.2

1. A mass of 500 kg rests on a springy base of stiffness 40 kN/m. The damping ratio is 0.25. The support is moved harmonically with an amplitude of 0.2 mm at 6 Hz. Calculate the amplitude of the mass and the phase angle.

\[(0.0118 \text{ mm and } -7.165 \text{ degrees})\]

Calculate the transmitted force and the phase angle to the motion of the support.

\[(1.104 \text{ N and } -71.8^\circ)\]

2. A mass of 60 kg hangs from a spring of stiffness 100 kN/m. The damping ratio is 0.2. The support is moved harmonically with an amplitude of 3 mm at 6 Hz. Calculate the amplitude of the mass and the phase angle.

\[(7.544 \text{ mm and } 68.3^\circ)\]

Calculate the transmitted force and the phase angle to the motion of the support.

\[(804 \text{ N and } 48^\circ)\]