ENGINEERING COUNCIL
DIPLOMA LEVEL
MECHANICS OF SOLIDS D209

TUTORIAL 9 - COMPLEX STRESS AND STRAIN

You should judge your progress by completing the self assessment exercises.

- On completion of this tutorial you should be able to do the following.
- Explain a complex stress situation.
- Derive formulae for complex stress situations.
- Analyse and solve stresses in a complex stress situation.
- Solve problems using a graphical method (Mohr's Circle)
- Explain the use of strain gauge rosettes to determine principal strains and stresses.
- Apply the theory to combined bending and torsion problems.

1. COMPLEX STRESS

Materials in a stressed component often have direct and shear stresses acting in two or more directions at the same time. This is a complex stress situation. The engineer must then find the maximum stress in the material. We will only consider stresses in two dimensions, x and y. The analysis leads on to a useful tool for solving complex stress problems called Mohr's' Circle of Stress.

1.1 DERIVATION OF EQUATIONS

Consider a rectangular part of the material. Stress σ_x acts on the x plane and σ_y acts on the y plane. The shear stress acting on the plane on which σ_x acts is τ_x and τ_y act on the plane on which σ_y acts. The shear stresses are complementary and so must have opposite rotation. We will take clockwise shear to be positive and anti-clockwise as negative. Suppose we cut the material in half diagonally at angle θ as shown and replace the internal stresses in the material with applied stresses σ_θ and τ_θ. In this case we will do it for the bottom half. The dimensions are x, y and t as shown.

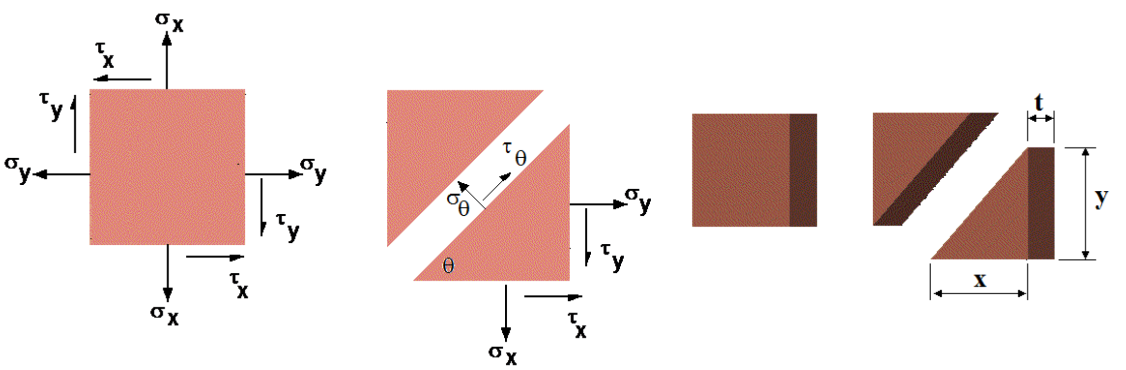


Figure 1

Now turn the stresses into force. If the material is t m thick normal to the paper then the areas are t x and t y on the edges and t y/sinθ or t x/cosθ on the sloping plane.

The forces due to the direct stresses and shear stresses are stress x area and as shown.

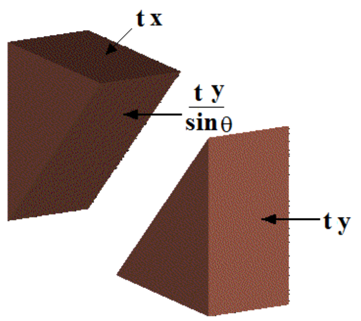


Figure 2

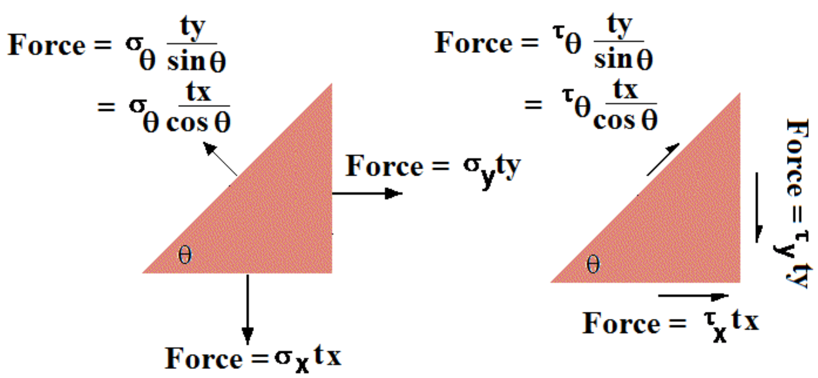


Figure 3

The material is in equilibrium so all the forces and moments on the plane must add up to zero. We now resolve these forces perpendicular and parallel to the plane. To make it easier the forces are labelled a, b, c, d, e, f, g, and h.

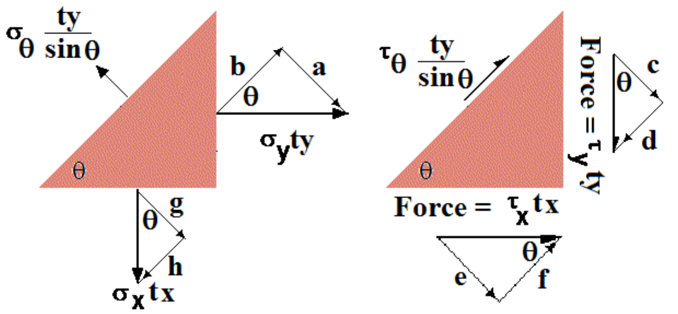


Figure 4

$a = t\,y\,\sigma_y \sin\theta$.	$b = t\,y\,\sigma_y \cos\theta$
$c = t\,y\,\tau_y \cos\theta$.	$d = t\,y\,\tau_y \sin\theta$
$e = t\,x\,\tau_x \sin\theta$	$f = t\,x\,\tau_x \cos\theta$
$g = t\,x\,\sigma_x \cos\theta$	$h = t\,x\,\sigma_x \sin\theta$

All the forces normal to the plane must add up to $(t\,y/\sin\theta)\,\sigma_\theta$.

Balancing we have $\quad a + c + e + g = (t\,y/\sin\theta)\,\sigma_\theta$

All the forces parallel to the plane must add up to $(t\,y/\sin\theta)\,\tau_\theta$

$$- f + h + b - d = (t\,y/\sin\theta)\,\tau_\theta$$

Making the substitutions and conducting algebraic process will yield the following results.

$$\frac{ty}{\sin\theta}\sigma_\theta = ty\sigma_y \sin\theta + ty\tau_y \cos\theta + tx\tau_x \sin\theta + tx\sigma_x \cos\theta$$

$$\sigma_\theta = \sigma_y \sin^2\theta + \tau_y \sin\theta\cos\theta + \frac{x}{y}\tau_x \sin^2\theta + \frac{x}{y}\sin\theta\cos\theta$$

$$\sigma_\theta = \sigma_y \frac{1-\cos 2\theta}{2} + \tau_y \frac{\sin 2\theta}{2} + \tau_x \frac{\sin^2\theta}{\tan\theta} + \sigma_x \frac{\sin\theta\cos\theta}{\tan\theta}$$

$$\sigma_\theta = \sigma_y \frac{1-\cos 2\theta}{2} + \tau_y \frac{\sin 2\theta}{2} + \tau_x \frac{\sin 2\theta}{2} + \sigma_x \frac{1+\cos 2\theta}{2}$$

$$\sigma_\theta = \frac{\sigma_y}{2} - \sigma_y \frac{\cos 2\theta}{2} + \tau_y \frac{\sin 2\theta}{2} + \tau_x \frac{\sin 2\theta}{2} + \frac{\sigma_x}{2} + \sigma_x \frac{\cos 2\theta}{2}$$

Perhaps the point should have been made earlier that the shear stress on both planes are equal (read up complementary stress) so we denote them both $\tau_{xy} = \tau_x = \tau_y$

$$\sigma_\theta = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)\cos 2\theta}{2} + \tau_{xy}\sin 2\theta \qquad \text{............(1.1)}$$

Repeating the process for the shear stress we get

$$\tau_\theta = \frac{(\sigma_x - \sigma_y)\sin 2\theta}{2} - \tau_{xy}\cos 2\theta \text{ ....(1.2)}$$

Consider the case where σ_x=125 MPa, σ_y=25 MPa and τ_{xy}=100 MPa

$$\sigma_\theta = 75 + 50\cos 2\theta + 100\sin 2\theta \qquad \tau_\theta = 50\sin 2\theta - 100\cos 2\theta$$

If the value of σ_θ and τ_θ are plotted against θ the resulting graphs are as shown below.

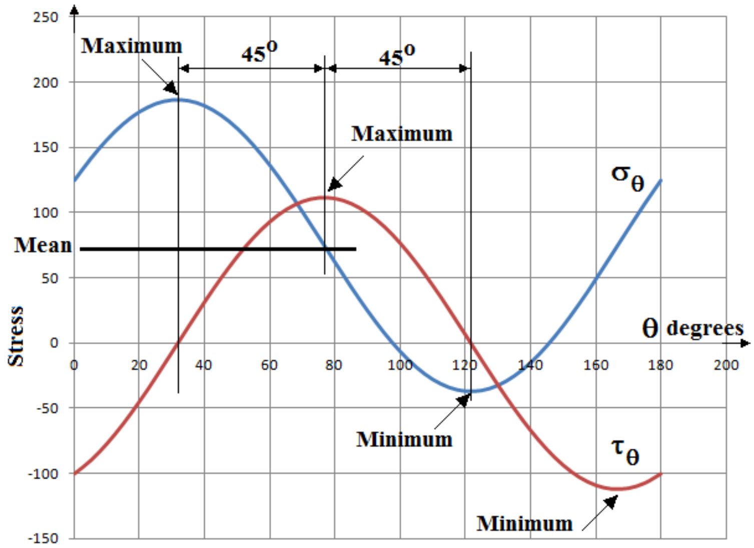


Figure 5

In this example the maximum value of σ_θ is about 190 MPa. The plane with this stress is at an angle of about 32°. The maximum shear stress is about 112 MPa on a plane at angle 77°.

These general results are the same what ever the values of the applied stresses. The graphs show that σ_θ has a maximum and minimum value and a mean value not usually zero. These are called the **PRINCIPAL STRESSES**. The principal stresses occur on planes 90° apart. These planes are called the **PRINCIPAL PLANES**.

The shear stress τ_θ has an equal maximum and minimum value with a mean of zero. The max and min values are on planes 90° apart and 45° from the principal planes. This is of interest because brittle materials fail on these planes. For example, if a brittle material is broken in a tensile test, the fracture occurs on a plane at 45° to the direction of pull indicating that they fail in shear. Further it can be seen that the principal planes have no shear stress so this is a definition of a principle plane.

τ_θ = 0 when σ_θ = on the principal planes where it is maximum or minimum.

There are several theories about why a material fails usually. The principle stresses and maximum shear stress are used in those theories.

1.2 DETERMINING THE PRINCIPAL STRESSES AND PLANES

You could plot the graph and determine the values of interest as shown in the previous section but is convenient to calculate them directly. The stresses are a maximum or minimum on the principal planes so we may find these using max and min theory (the gradient of the function is zero at a max or min point). First differentiate the function and equate to zero.

$$\frac{d\sigma_\theta}{d\theta} = -(\sigma_x - \sigma_y)\sin 2\theta + 2\tau\cos 2\theta = 0$$

$$2\tau\cos 2\theta = (\sigma_x - \sigma_y)\sin 2\theta$$

$$\tan 2\theta = \frac{2\tau}{\sigma_x - \sigma_y} \quad \text{.............(1.3)}$$

There are two solutions to this equation giving answers less than 360° and they differ by 90°. From this the angle of the principal plane may be found. If this angle is substituted into equation (1.1) and algebraic manipulation conducted the stress values are then the principal stresses and are found to be given as

$$\sigma_{max} = \sigma_1 = \frac{(\sigma_x + \sigma_y)}{2} + \frac{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2{}_{xy}}}{2} \quad \text{.............(1.4)}$$

$$\sigma_{min} = \sigma_2 = \frac{(\sigma_x + \sigma_y)}{2} - \frac{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2{}_{xy}}}{2} \quad \text{............(1.5)}$$

Repeating the process for equation (1.2), we get

$$\tau_{max} = \frac{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2{}_{xy}}}{2} = \frac{\sigma_1 - \sigma_2}{2} \quad \text{.................(1.6)}$$

$$\tau_{min} = \frac{-\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2{}_{xy}}}{2} = -\left[\frac{\sigma_1 - \sigma_2}{2}\right] \quad \text{...........(1.7)}$$

WORKED EXAMPLE No. 1

Using the formulae, find the principal stresses for the case shown below and the position of the principal plane.

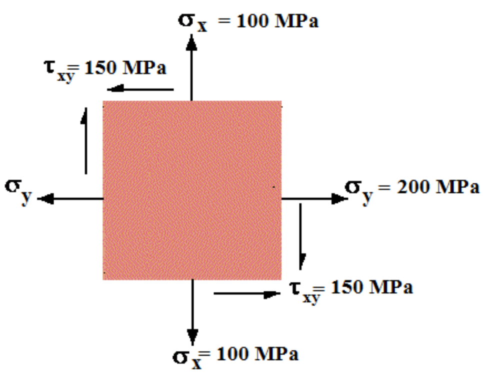


Figure 6

SOLUTION

$$\frac{2\tau}{\sigma_x - \sigma_y} = -\tan 2\theta = \frac{2 \times 150}{200 - 100} = -3$$

2θ = -71.6° or 108.4°   θ = -35.8° or 54.2°

Putting this into equations 1.1 and 1.2 we have

$$\sigma_\theta = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)\cos 2\theta}{2} + \tau_{xy}\sin 2\theta$$

$$\sigma_\theta = \frac{(200+100)}{2} + \frac{(200-100)\cos(-71.6^o)}{2} + 150\sin(-71.6^o)$$

$\sigma_\theta = 308.1\,\text{MPa}$ $\sigma_1 = 308.1\,\text{MPa}$

Using the other angle.

$$\sigma_\theta = \frac{(200+100)}{2} + \frac{(200-100)\cos(108.4^o)}{2} + 150\sin(108.4^o)$$

$\sigma_\theta = -8.1\,\text{MPa}$ $\sigma_2 = -8.1\,\text{MPa}$

$$\tau_\theta = \frac{(\sigma_x - \sigma_y)\sin 2\theta}{2} - \tau_{xy}\cos 2\theta$$

$$\tau_\theta = \frac{(100-200)\sin 71.6^o}{2} + 150\cos 71.6^o = 0 \text{ (This is as expected)}$$

From equations 1.6 and 1.7 we have

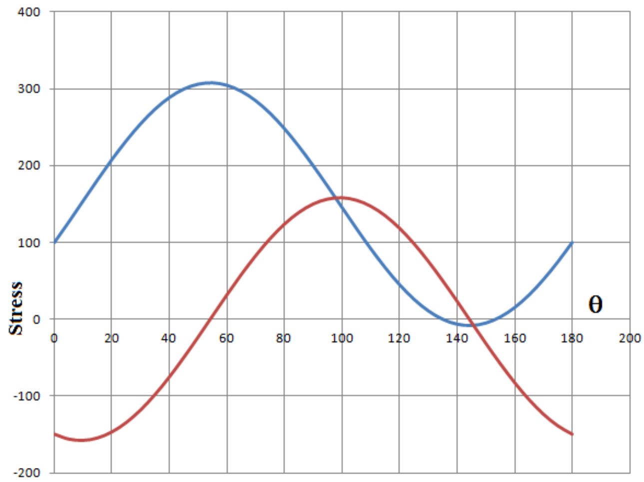
$\tau_{max} = (1/2)(\sigma_1 - \sigma_2) = 158.1$ MPa $\tau_{min} = -(1/2)(\sigma_1 - \sigma_2) = -158.1$ MPa

It is easier to use equations 1.4 and 1.5 as follows.

$$\sigma_{max} = \sigma_1 = \frac{(\sigma_x + \sigma_y)}{2} + \frac{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}{2}$$

$$\sigma_1 = \frac{(200+100)}{2} + \frac{\sqrt{(200-100)^2 + 4 \times 150^2}}{2} = 308.1\,\text{MPa}$$

$$\sigma_{min} = \sigma_2 = \frac{(200+100)}{2} - \frac{\sqrt{(200-100)^2 + 4 \times 150^2}}{2} = -8.1\,\text{MPa}$$



The plot confirms these results.

1.3. MOHR'S CIRCLE OF STRESS

Although we can solve these problems easily these days with computer programmes or calculators, it is still interesting to study this method of solution. Mohr found a way to represent equations 1.1 and 1.2 graphically. The following rules should be used.

1. Draw point 'O' at a suitable position (which is possible to see with experience)

2. Measure σ_x and σ_y along the horizontal axis using a suitable scale and mark A and B.

3. Find the centre (M) half way between the marked points.

4. Draw τ_{xy} up at A and down at B if it is positive or opposite if negative (as in this example).

5. Draw the circle centre M and radius MC. It should also pass through D.

6. Draw the diagonal CD which should pass through M.

6. Measure OE and this is σ_2. Measure OF and this is σ_1. The angle 2θ is shown. τ_{max} is the radius of the circle.

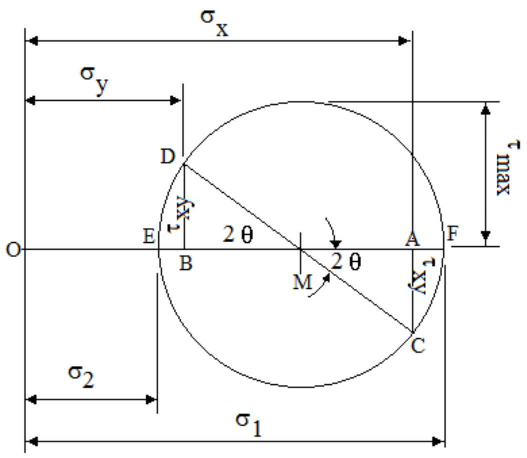


Figure 7

Consider the triangle. The sides are as shown.

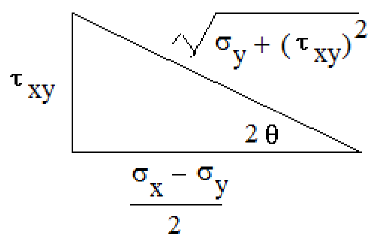


Figure 8

Applying trigonometry we find the same results as before so proving the geometry represents the equations.

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \text{ ....(1.3)}$$

$$\sigma_{max} = \sigma_1 = \frac{(\sigma_x + \sigma_y)}{2} + \frac{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}{2} \text{ ....(1.4)}$$

$$\sigma_{min} = \sigma_2 = \frac{(\sigma_x + \sigma_y)}{2} - \frac{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}{2} \text{ ....(1.5)}$$

WORKED EXAMPLE No.2

Show that the solution in worked example No.1 may be obtained by constructing the circle of stress.

SOLUTION

Following the rules for construction figure 8 is obtained.

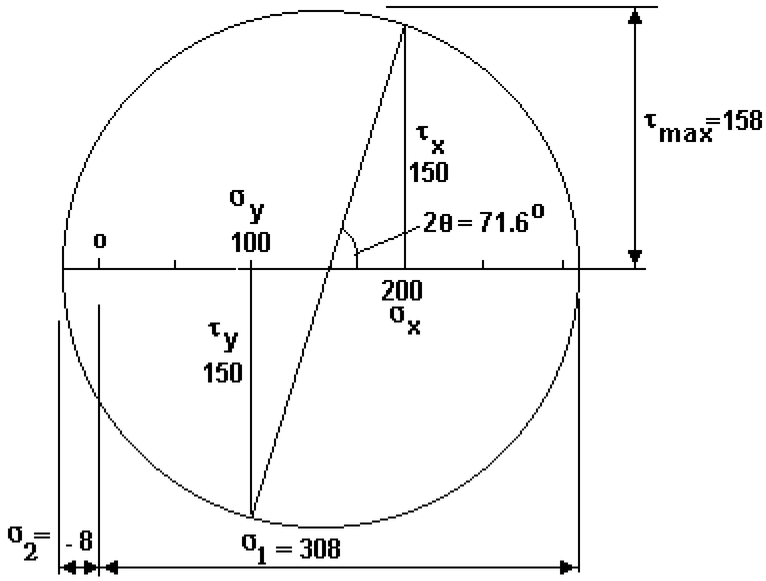


Figure 9

WORKED EXAMPLE No.3

A material has direct stresses of 120 MPa tensile and 80 MPa compressive acting on mutually perpendicular planes. There is no shear stress on these planes. Draw Mohrs' circle of stress and determine the stresses on a plane 20^o to the plane of the larger stress.

SOLUTION

Since there is no shear stress, σ_x and σ_y are the principal stresses and are at the edge of the circle.
Select the origin '0' and plot +120 to the right and -80 to the left.

Draw the circle to encompass these points and mark the centre of the circle.

Using a protractor draw the required plane at 40^o to the horizontal (assumed anti-clockwise) through the middle of the circle.

Draw the verticals. Scale the direct stress σ_θ and shear stress τ_θ . These may be checked with the formulae if confirmation is required.

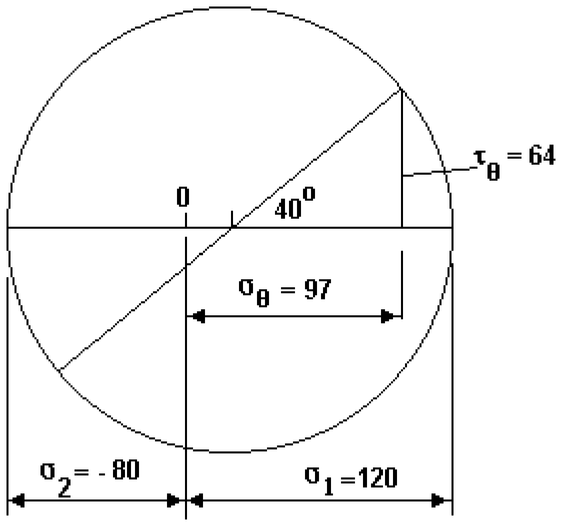


Figure 10

SELF ASSESSMENT EXERCISE No.1

a) Figure 10 shows an element of material with direct stresses on the x and y planes with no shear stress on those planes. Show by balancing the forces on the triangular element that the direct and shear stress on the plane at angle θ anti-clockwise of the x plane is given by

$$\sigma_\theta = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)\cos 2\theta}{2}$$

$$\tau_\theta = \frac{(\sigma_x - \sigma_y)\sin 2\theta}{2}$$

Show how Mohr's circle of stress represents this equation.

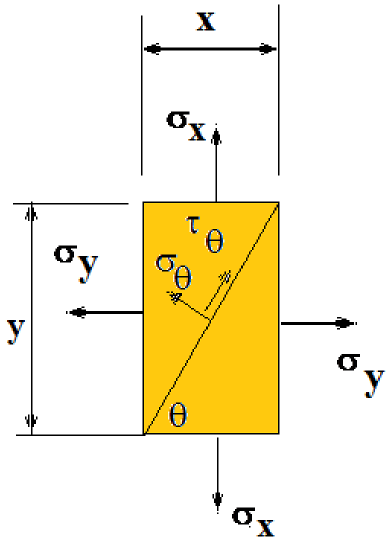


Figure 11

b) An elastic material is subjected to two mutually perpendicular stresses 80MPa tensile and 40 MPa compressive. Determine the direct and shear stresses acting on a plane 30^o to the plane on which the 80 MPa stress acts.

(Hint for solution) The derivation is the same as in the notes but with no shear stress, i.e. the stresses shown on fig.9 are the principal stresses.

(50 MPa and 52 MPa)

2. Define the terms principal stress and principal plane.

A piece of elastic material has direct stresses of 80 MPa tensile and 40 MPa compressive on two mutually perpendicular planes. A clockwise shear stress acts on the plane with the 80 MPa stress and an equal and opposite complementary shear stress acts on the other plane.
The maximum principal stress in the material is 100 MPa tensile. Construct Mohrs' circle of stress and determine the following.

i. The shear stress on the planes. (52 MPa)

ii. The maximum shear stress. (80 MPa)

iii. The minimum principal stress. (-60 MPa)

iv. The position of the principal planes. (20^o C.W. of 80 MPa direction)

(Hint for solution. Make the larger stress act on the x plane and the other on y plane. Mark off σ_x, σ_y and σ_1. This is enough data to draw the circle.)

2. COMPLEX STRAIN

2.1 PRINCIPAL STRAINS

In any stress system there are 3 mutually perpendicular planes on which only direct stress acts and there is no shear stress. These are the principal planes and the stresses are the principal stresses. These are designates σ_1, σ_2 and σ_3 . The corresponding strains are the principal strains ε_1, ε_2 and ε_3 .

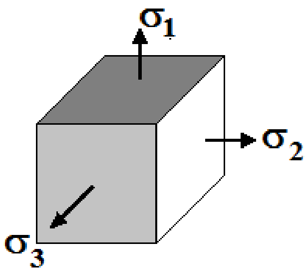


Figure 12

The strain in each direction is given by

$$\varepsilon_1 = \frac{1}{E}(\sigma_1 - \nu\sigma_2 - \nu\sigma_3) = \frac{1}{E}[\sigma_1 - \nu(\sigma_2 + \sigma_3)]$$

$$\varepsilon_2 = \frac{1}{E}(\sigma_2 - \nu\sigma_1 - \nu\sigma_3) = \frac{1}{E}[\sigma_2 - \nu(\sigma_1 + \sigma_3)]$$

$$\varepsilon_3 = \frac{1}{E}(\sigma_3 - \nu\sigma_1 - \nu\sigma_3) = \frac{1}{E}[\sigma_3 - \nu(\sigma_1 + \sigma_2)]$$

Most often we only study 2 dimensional systems in which case $\sigma_3 = 0$ so

$$\varepsilon_1 = \frac{1}{E}[\sigma_1 - \nu(\sigma_2)] \text{.............(2.1)}$$

$$\varepsilon_2 = \frac{1}{E}[\sigma_2 - \nu(\sigma_1)] \text{.............(2.2)}$$

It is more practical to measure strains and convert them into stresses as follows. From equation 2.2 $E\varepsilon_2 + \nu\sigma_1 = \sigma_2$

Substituting for σ_2 in equation 1 we have

$$\varepsilon_1 = \frac{1}{E}[\sigma_1 - \nu(E\varepsilon_2 + \nu\sigma_1)]$$

$$E\varepsilon_1 = [\sigma_1 - \nu E\varepsilon_2 + \nu^2\sigma_1]$$

$$E\varepsilon_1 = \sigma_1(1 - \nu^2) - \nu E\varepsilon_2$$

$$E\varepsilon_1 + \nu E\varepsilon_2 = \sigma_1(1 - \nu^2)$$

$$\sigma_1 = \frac{E}{1 - \nu^2}(\varepsilon_1 + \nu\varepsilon_2)$$

Similarly we can show

$$\sigma_2 = \frac{E}{1 - \nu^2}(\varepsilon_2 + \nu\varepsilon_1)$$

These formulas should be used for converting principal strains into principal stresses.

SELF ASSESSMENT EXERCISE No.2

1. The principal strains acting on a steel component are 12 με and 6 με. Determine the principal stresses.

   E = 205 GPa υ = 0.32 (Ans. 2.247 MPa and 3.179 MPa)

2. The principal strains acting on a steel component are -100 με and 160 με. Determine the principal stresses.

   E = 205 GPa υ = 0.32 (Ans. 29.2 MPa and -11.14 MPa)

2.2 MOHR'S CIRCLE OF STRAIN

Two dimensional strains may be analysed in much the same way as two dimensional stresses and the circle of strain is a graphic construction very similar to the circle of stress. Strain is measured with electrical strain gauges (not covered here) and a typical single gauge is shown in the picture.

First consider how the equations for the strain on any plane are derived. Figure 13

Consider a rectangle A,B,C,D which is stretched to A',B',C',D under the action of two principal stresses. The diagonal rotates an angle β from the original direction. The plane under study is this diagonal at angle θ to the horizontal.

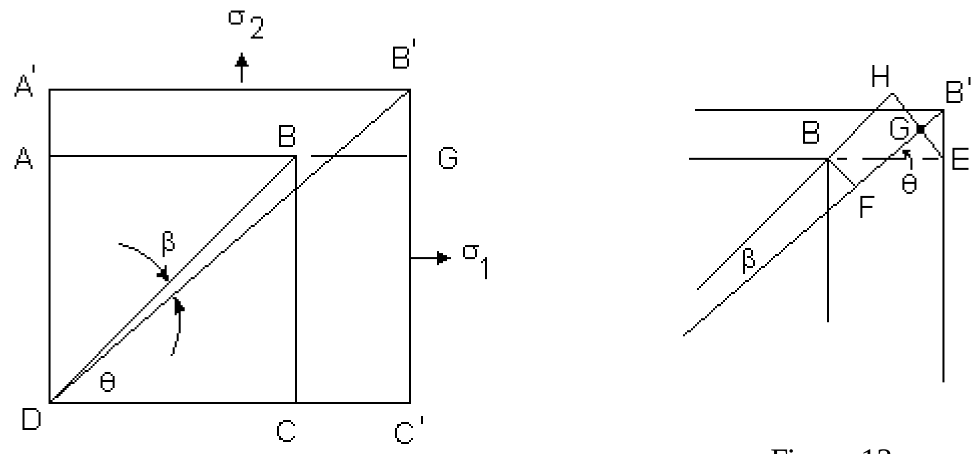


Figure 13

The principal strains are ε_1 = BE/AB so BE = ε_1 AB and ε_2 = EB'/CB so EB' = ε_2 CB

BH is nearly the same length as FG.
FB' = FG + GB' = BE cosθ + EB' sin θ = ε_1AB cosθ + ε_2CB sinθ
The strain acting on the plane at angle θ is ε_θ = FB'/DB

$$\varepsilon\theta = \varepsilon_1\cos^2\theta + \varepsilon_2\sin^2\theta$$

$$\varepsilon_\theta = ½(\varepsilon_1+\varepsilon_2) + ½(\varepsilon_1-\varepsilon_2)\cos 2\theta \text{ ....(2.3)}$$

BF is nearly the same length as HG so BF = HE - GE = BE $\sin\theta$ - EB' $\cos\theta$
BF = ε_1AB $\sin\theta + \varepsilon_2$CB $\cos\theta$

β = BF/DB = $\varepsilon_1\cos\theta\sin\theta - \varepsilon_2\cos\theta\sin\theta$

$$\beta = ½(\varepsilon_1-\varepsilon_2)\sin2\theta \text{ ....(2.4)}$$

Equations 2.3 and 2.4 may be compared to the equations for 2 dimensional stress which were

$$\sigma_\theta= ½(\sigma_1 + \sigma_2) + ½\{(\sigma_1 - \sigma_2)\}\cos 2\theta \text{ ....(2.5)}$$

$$\tau_\theta= ½\{(\sigma_1 - \sigma_2)\}\sin 2\theta \quad \text{...........(2.6)}$$

It follows that a graphical construction may be made in the same way for strain as for stress. There is one complication. Comparing equations 4 and 6, τ_θ is the shear stress but β is not the shear strain. It can be shown that the rotation of the diagonal (in radians) is in fact half the shear strain on it (γ) and negative so equation 4 becomes

$$-½ \gamma= ½(\varepsilon_1-\varepsilon_2)\sin2\theta \text{ ....(2.7)}$$

These equations may be used to determine the strains on two mutually perpendicular planes x and y by using the appropriate angle. Alternatively, they may be solved by constructing the circle of strain.

2.3 CONSTRUCTION

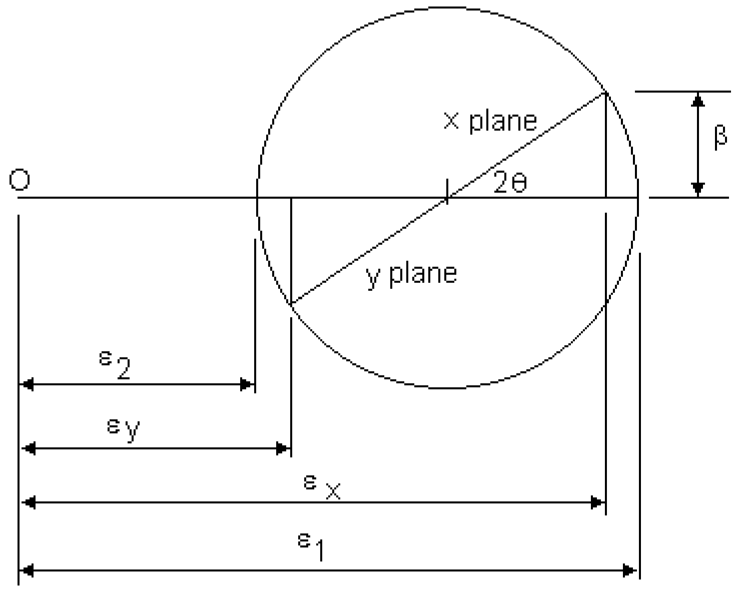


Figure 15

1. Draw point 'O' at a suitable position (which is possible to see with experience)

2. Measure ε_1 and ε_2 along the horizontal axis using a suitable scale.

3. Find the centre half way between the marked points.

4. Draw the circle.

5. Draw in the required plane at the double angle.

6. Measure ε_x and ε_y and β.

SELF ASSESSMENT EXERCISE No.3

1. The principal strains in a material are 500 $\mu\varepsilon$ and 300 $\mu\varepsilon$. Calculate the direct strain and shear strain on a plane 30^o anti clockwise of the first principal strain.
(Answers 450 $\mu\varepsilon$ and -173.2 $\mu\varepsilon$.)

2. Repeat the problem by constructing the circle of strain.

3. The principal strains in a material are 600 $\mu\varepsilon$ and -200 $\mu\varepsilon$. Determine the direct strain and shear strain on a plane 22.5^o clockwise of the first principal strain.
(Answers 480 $\mu\varepsilon$ and -566 $\mu\varepsilon$.)

3. STRAIN GAUGE ROSETTES

It is possible to measure strain but not stress. Strain gauges are small surface mounting devices which, when connected to suitable electronic equipment, enable strain to be measured directly.

In order to construct a circle of strain without knowing the principal strains, we might expect to use the strains on two mutually perpendicular plains (x and y) and the accompanying shear strain. Unfortunately, it is not possible to measure shear strain so we need three measurements of direct strain in order to construct a circle. The three strain gauges are conveniently manufactured on one surface mounting strip and this is called a strain gauge rosette. There are two common forms. One has the gauges at 45^o to each other and the other has them at 60^o to each other. The method for drawing the circle of strain is different for each.

We shall label the gauges A, B and C in an anti clockwise direction as shown.

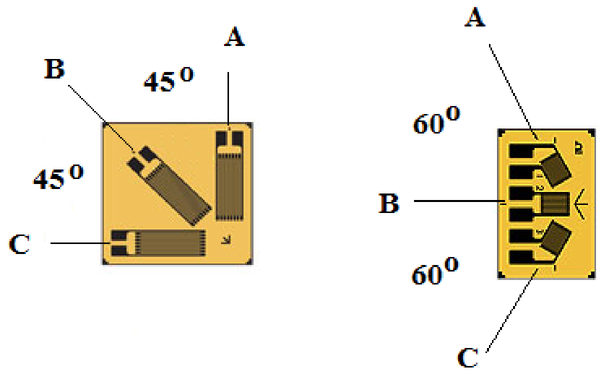


Figure 16

3.1 CONSTRUCTION OF THE 45^o CIRCLE

We shall use a numerical example to explain the construction of the circle. Suppose the three strains are $\varepsilon_A = 700\ \mu\varepsilon$ $\varepsilon_B = 300\ \mu\varepsilon$ $\varepsilon_C = 200\mu\varepsilon$

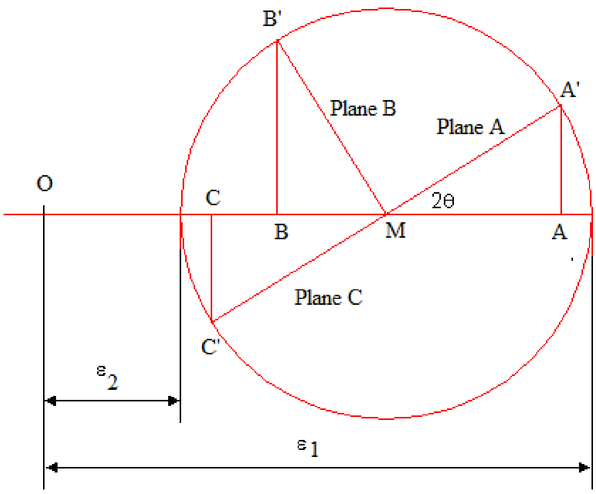


Figure 17

1. Choose a suitable origin O
2. Scale off horizontal distances from O for ε_A, ε_B and ε_C and mark them as A, B and C.
3. Mark the centre of the circle M half way between A and C.
4. Construct vertical lines through A, B and C
5. Measure distance BM
6. Draw lines A A' and C C' equal in length to BM
7. Draw circle centre M and radius M A' = M C'
8. Draw B B'

Scaling off the values we find $\varepsilon_1 = 742\ \mu\varepsilon$, $\varepsilon_2 = 158\ \mu\varepsilon$ and the angle $2\theta = 30^o$
The first principal plane is hence 15^o clockwise of plane A.

SELF ASSESSMENT EXERCISE No.4

1. The results from a 45^o strain gauge rosette are
   $\varepsilon_A = 200\ \mu\varepsilon$
   $\varepsilon_B = -100\ \mu\varepsilon$
   $\varepsilon_C = 300\ \mu\varepsilon$

Draw the strain circle and deduce the principal strains. Determine the position of the first principal plane. Go on to convert these into principal stresses given E = 205 GPa and $\nu = 0.29$.

(Answers $\varepsilon_1 = 603\ \mu\varepsilon$ $\varepsilon_2 = -103\ \mu\varepsilon$ $\sigma_1 = 128.3$ MPa $\sigma_2 = 16$ MPa
The first principal plane is 49^o clockwise of plane A)

3.2 CONSTRUCTION OF A 60^o ROSETTE

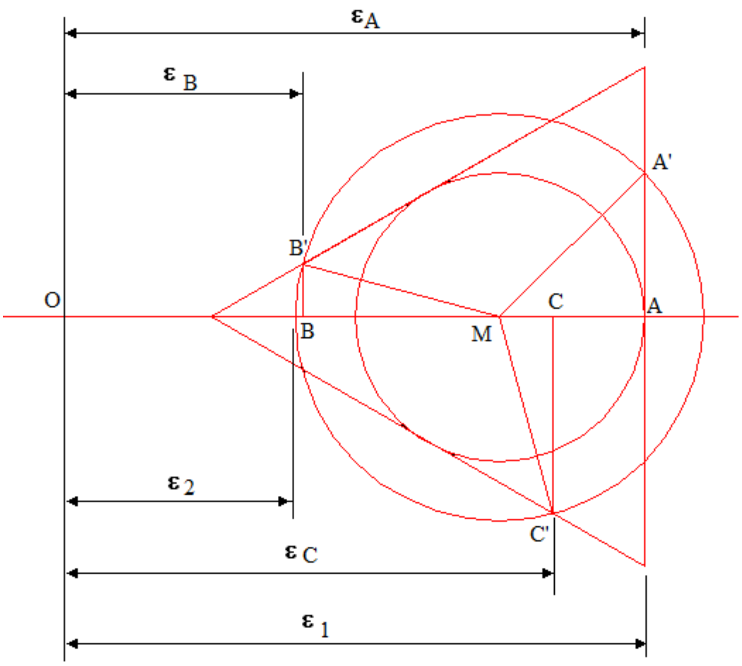


Figure 18

1. Select a suitable origin O
2. Scale of A, B and C to represent the three strains ε_A, ε_B and ε_C
3. Calculate OM = (A + B + C)/3
4. Draw the inner circle radius MA
5. Draw the triangle (60^o each corner)
6. Draw outer circle passing through B' and C'.
7. Make sure that the planes A, B and C are in an anti clockwise direction because you can obtain an upside down version for clockwise directions.
8. Scale off principal strains.

SELF ASSESSMENT EXERCISE No.5

1. The results from a 60^o strain gauge rosette are
$\varepsilon_A = 700\ \mu\varepsilon$
$\varepsilon_B = 200\ \mu\varepsilon$
$\varepsilon_C = -100\ \mu\varepsilon$

Draw the strain circle and deduce the principal strains. Determine the position of the first principal plane. Go on to convert these into principal stresses given E = 205 GPa and $\nu = 0.25$.

(Answers $\varepsilon_1 = 733\ \mu\varepsilon$ $\varepsilon_2 = -198\ \mu\varepsilon$ $\sigma_1 = 149.4$ MPa $\sigma_2 = -3.6$ MPa
The first principal plane is 11^o anti clockwise of plane A)

4. COMBINED BENDING, TORSION AND AXIAL LOADING.

When a material is subjected to a combination of direct stress, bending and torsion at the same time, we have a complex stress situation. A good example of this is a propeller shaft in which torsion is produced. If in addition there is some misalignment of the bearings, the shaft will bend as it rotates. If a snap shot is taken, one side of the shaft will be in tension and one in compression. The shear stress direction depends upon the direction of the torque being transmitted.

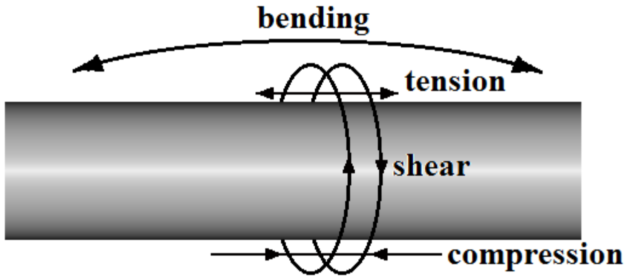


Figure 19

The bending and shear stresses on their own are a maximum on the surface but they will combine to produce even larger stresses. The maximum stress in the material is the principal stress and this may be found with the formulae or by constructing Mohr's circle. In earlier work it was shown that the maximum direct and shear stress was given by the following formula.

$$\sigma_{max} = \sigma_1 = \frac{(\sigma_x + \sigma_y)}{2} + \frac{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2{}_{xy}}}{2} \qquad \tau_{max} = \frac{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2{}_{xy}}}{2}$$

If there is only one direct stress in the axial direction σ_x and an accompanying shear stress τ (assumed positive), then putting $\sigma_y = 0$ we have the following.

$$\sigma_{max} = \frac{(\sigma)}{2} + \frac{\sqrt{(\sigma)^2 + 4\tau^2}}{2} \qquad \tau_{max} = \frac{\sqrt{(\sigma)^2 + 4\tau^2}}{2}$$

If the axial stress is only due to bending, then $\sigma = \sigma_B$. From the bending and torsion equations we have formula for σ_B and τ as follows.

$$\sigma_B = \frac{My}{I} = \frac{MD}{2I} = \frac{32M}{\pi D^3} \text{ and } \tau = \frac{TR}{J} = \frac{TD}{2J} = \frac{16T}{\pi D^3}$$

Substituting for σ and τ we get the following result.

$$\sigma_{max} = \frac{16}{\pi D^3}\left[M + \sqrt{T^2 + M^2}\right] \text{ and } \tau_{max} = \frac{16}{\pi D^3}\sqrt{T^2 + M^2}$$

It was shown earlier that the angle of the principal plane could be found from the following formula. $\tan 2\theta = \dfrac{2\tau}{\sigma_x - \sigma_y}$. Putting $\sigma_x = \sigma$ and $\sigma_y = 0$ this becomes $\tan 2\theta = \dfrac{2\tau}{\sigma}$. If there is only bending stress and torsion we may substitute for σ_B and τ as before, we get the following formula.

$$\tan 2\theta = \frac{T}{M}$$

If the direct stress is due to bending and an additional axial load, (e.g. due to the propeller pushing or pulling), the direct stresses should be added together first to find σ as they are in the same direction. You could draw Mohr's circle to solve these problems or use the appropriate formulae.

WORKED EXAMPLE No.4

A propeller shaft has a bending stress of 7 MPa on the surface. Torsion produces a shear stress of 5 MPa on same point of the surface. The propeller pushes and puts a compressive stress of 2 MPa in the shaft.

Determine the following.

- The principal stresses on the surface.
- The position of the principal plane.

SOLUTION

Since we have stress values, the problem is best solved by drawing Mohr's circle. At the point considered we have two a direct stresses and a shear stress. The total direct stress is 7 - 2 = 5 MPa. Let this be σ_x and let the shear stress be positive on this plane. σ_y will be zero and the shear stress will be negative on the y plane.

Construction of the circle yields principle stresses of 8.09 and -3.09 MPa. The principle plane is 31.7^o clockwise of the x plane.

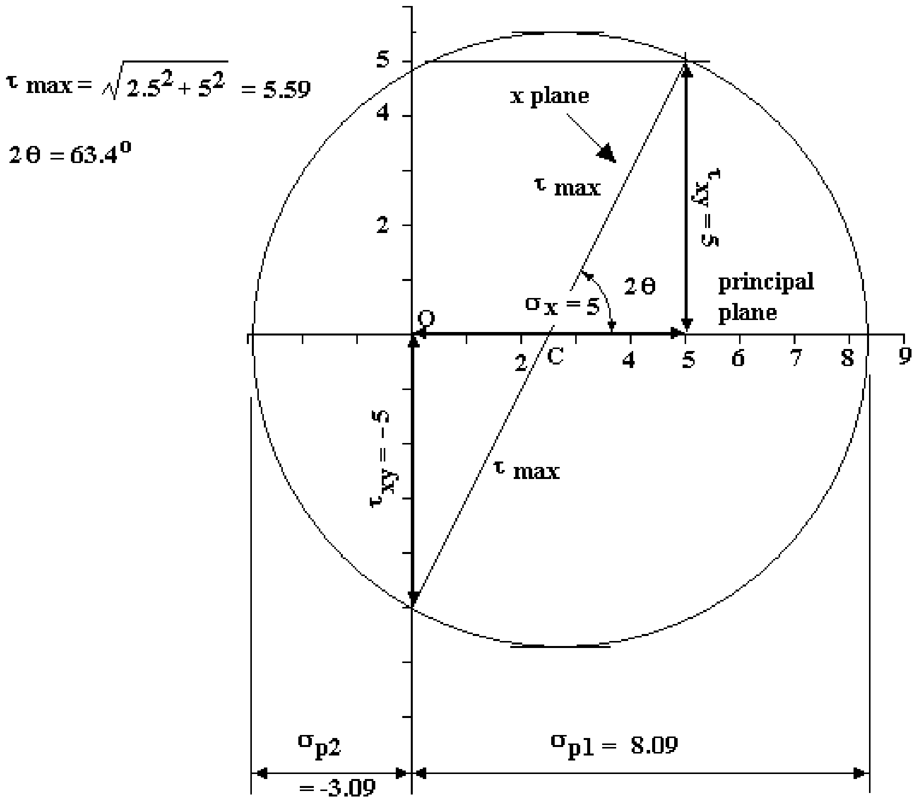


Figure 20

Check the answers by using the formulae.

$$\sigma_{max} = \frac{(\sigma)}{2} + \frac{\sqrt{(\sigma)^2 + 4\tau^2}}{2} = \frac{5}{2} + \frac{\sqrt{5^2 + 4 \text{ x } 5^2}}{2} = 8.09 \text{ MPa}$$

$$\tau_{max} = \frac{\sqrt{(\sigma)^2 + 4\tau^2}}{2} = \frac{\sqrt{5^2 + 4 \text{ x } 5^2}}{2} = 5.59 \text{ MPa}$$

$$\tan 2\theta = \frac{2\tau}{\sigma} = \frac{2 \text{ x } 5}{5} = 2 \qquad 2\theta = 63.43 \quad \theta = 31.7^o$$

WORKED EXAMPLE No.5

A solid circular shaft 100 mm diameter is subjected to a bending moment of 300 Nm and a Torque of 400 Nm. Calculate the maximum direct stress and shear stress in the shaft.

SOLUTION

As there is no other axial load we can use the simpler formulae.

$$\sigma_{max} = \frac{16}{\pi D^3}\left[M + \sqrt{T^2 + M^2}\right] = \frac{16}{\pi(0.1)^3}\left[300 + \sqrt{400^2 + 300^2}\right]$$

$$\sigma_{max} = 4.07 \text{ MPa}$$

$$\tau_{max} = \frac{16}{\pi D^3}\sqrt{T^2 + M^2} = \tau_{max} = \frac{16}{\pi(0.1)^3}\sqrt{400^2 + 300^2}$$

$$\tau_{max} = 2.55 \text{ MPa}$$

$$\tan 2\theta = \frac{T}{M} = \frac{400}{300} = 1.333 \quad 2\theta = 53.13^o \quad \theta = 26.6^o$$

SELF ASSESSMENT EXERCISE No.6

1. A shaft is subjected to bending and torsion such that the bending stress on the surface is 80 MPa and the shear stress is 120 MPa. Determine the maximum stress and the direction of the plane on which it occurs relative to the axis of the shaft.

   (166.5 MPa 35.8^o clockwise of the axis)

2. A solid circular shaft 120 mm diameter is subjected to a torque of 2000 Nm and a bending moment of 8000 Nm. Calculate the maximum direct and shear stress and the angle of the principal plane.

   (47.9 MPa, 24.3 MPa and 7^o)

3. A solid circular shaft 60 mm diameter transmits a torque of 50 Nm and has a bending moment of 9 Nm. Calculate the maximum direct and shear stress and the angle of the principal plane.

   (1.41 MPa, 1.2 MPa and 39.9^o)

4. A solid circular shaft is 90 mm diameter. It transmits 8 kNm of torque and in addition there is a tensile axial force of 8 kN and a bending moment of 3 kNm. Calculate the following.

i. The maximum stress due to bending alone. (41.9 MPa)
ii. The shear stress due to torsion alone. (55.9 MPa)
iii. The stress due to the axial force alone. (1.26 MPa)
iv. The maximum direct stress. (81.5 MPa)
v. The maximum shear stress. (59.9 MPa)
vi. The position of the principal plane. (34.4^o relative to the axis)