ENGINEERING COUNCIL
DIPLOMA LEVEL
MECHANICS OF SOLIDS D209

TUTORIAL 1 - ADVANCED SECOND MOMENTS OF AREA

This tutorial provides a more advanced study of cross sectional shapes used in beams and compression members. The material covers elements of the following syllabi.

You should judge your progress by completing the self assessment exercises. These may be sent for marking at a cost (see home page).

On completion of this tutorial you should be able to do the following.

- Define the Product Second Moment of Area (Moment of inertia).
- Calculate the Product Second Moment of Area (Moment of inertia).
- Define Principal Axes.
- Calculate the angle of the Principal Axes.
- Calculate the maximum and minimum second moments of area.
- Define the polar second moment of area for various sections.

It is assumed that students doing this tutorial already understand the basic principles of moments, shear force, stress and moments of area.

INTRODUCTION

When dealing with beams or columns with sections such as ‘L’ shapes, we need to understand some more advanced concepts about moments of inertia and the axis about which bending occurs. We need to look first at moments of inertia.

SECOND MOMENTS OF AREA

Consider a simple rectangular section with two axes of symmetry passing through the centroid. The second moment of area about the x and y axes are respectively:

$$I_x = \int_{-D/2}^{D/2} y^2 dA = \frac{BD^3}{12}$$

$$I_y = \int_{-B/2}^{B/2} x^2 dA = \frac{DB^3}{12}$$

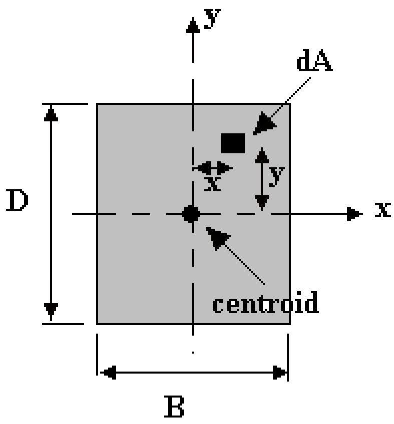

Figure 1

PRODUCT SECOND MOMENT OF AREA

In advanced work we need another concept called the product second moment of inertia. This is defined as:-

$$I_{xy} = \int xy\,dA$$

For our rectangle dA = dx dy

$$I_{xy} = \int xy\,dx\,dy = \left[\frac{x^2}{2}\right]_{-D/2}^{D/2} + \left[\frac{y^2}{2}\right]_{-B/2}^{B/2} = 0$$

And this is zero for all shapes that are symmetrical in two axes.

PARALLEL AXIS THEOREM

The second moment of area about an axis x x parallel to an axis x' x' is $I_{xx} = I_{x'x'} + A\bar{y}^2$

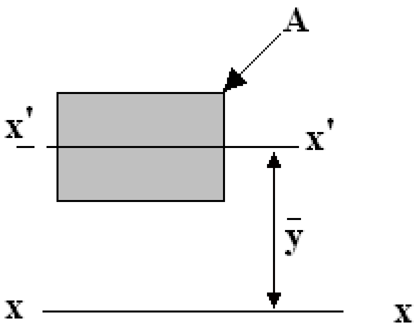

Figure 2

There is a corresponding theory for the product second moment.

Note that the axes $x'x'$ and $y'y'$ are the axis through the centroid.

$I_{xy} = I_{x'y'} + A\overline{xy}$

$I_{x'y'}$ is the product value about the centroid of the area.

If the area is symmetrical about two axis then $I_{x'y'} = 0$ and $I_{xy} = A\overline{xy}$

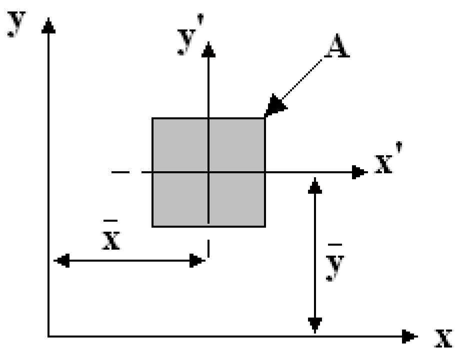

Figure 3

WORKED EXAMPLE No.1

Calculate the product second moment of area for the ‘L’ shape shown about axes parallel to its sides passing through the centroid.

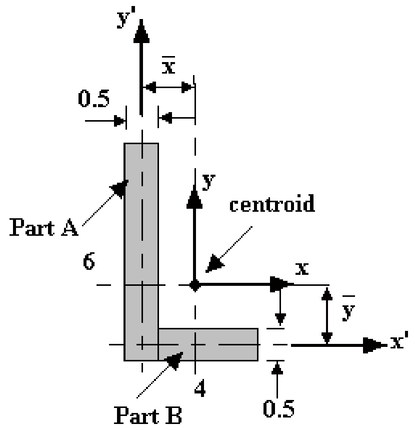

Figure 4

SOLUTION

Divide into two parts A and B. Each part is symmetrical about two axes through their own centroid so $I_{x'y'}$= 0 for both parts.
$I_{xy} = A\,\bar{y}\,\bar{x}$

Calculate the values of $\bar{y}$ and $\bar{x}$ in the usual way. First find it relative to the x′ axis.

	Area	$\bar{y}$	$A\bar{y}$	
A	3	2.75	8.25	
B	1.75	0	0	
Totals	4.75		8.25	$\bar{y} = 8.25/4.75 = 1.737$

Next find it relative to the y′ axis.

	Area	$\bar{x}$	$A\bar{x}$	
A	3	0	0	
B	1.75	2	3.5	
Totals	4.75		3.5	$\bar{x} = 3.5/4.75 = 0.737$

$I_{xy} = A\,\bar{y}\,\bar{x} = 4.75 \times 1.737 \times 0.737 = 6.08$

MAXIMIMUM AND MINIMUM SECOND MOMENTS OF AREA

The axes that give the maximum and minimum second moment of areas are called the principal axes. If a section is symmetrical in two axes, the principle axis will be the axis of symmetry.

If the section is not symmetrical then the required axis will be inclined to the x and y directions but will remain normal to each other. The product of inertia is always zero when computed with respect to the principal axes. Consider the L section.

General Equations (for any rotated pair of axes):

$$I_{x'} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta$$

$$I_{y'} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta + I_{xy}\sin 2\theta$$

$$I_{x'y'} = \frac{I_x - I_y}{2}\sin 2\theta + I_{xy}\cos 2\theta$$

You may recognise these equations are identical to those used in Mohr's circle of stress and strain and the same graphical construction techniques may be used.

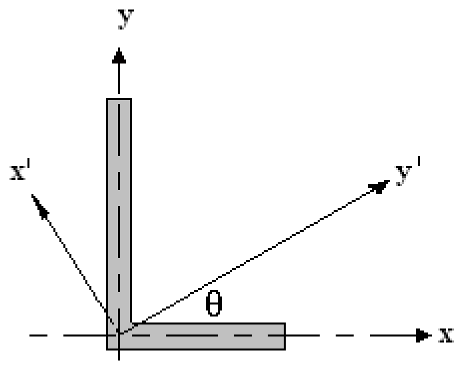

Figure 5

The angle θ is found from the equation $\tan 2\theta = \frac{-2I_{xy}}{I_x - I_y}$

The maximum and minimum values of I are given by the equation

$$I_{max/min} = \frac{I_x + I_y}{2} \pm \sqrt{\frac{(I_x - I_y)^2}{2} + I_{xy}^2}$$

WORKED EXAMPLE No.2

Calculate the maximum and minimum second moment of area for the L section shown and determine the angle of the principal axes.

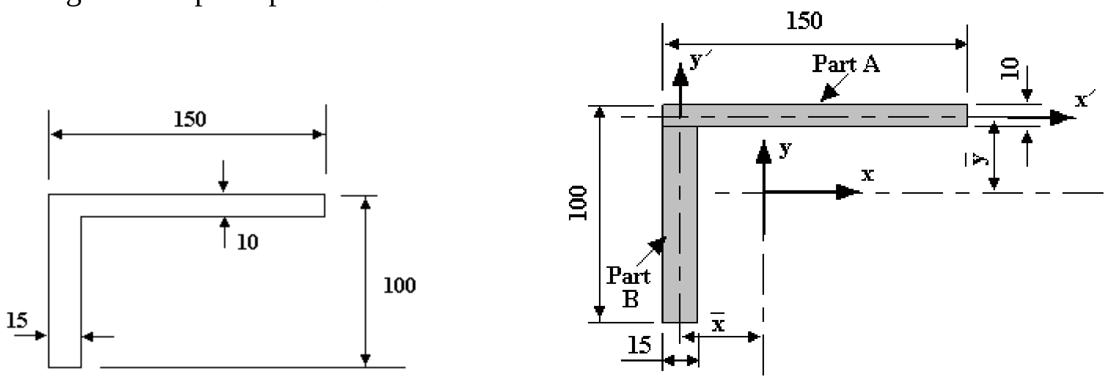

Figure 6

SOLUTION

Divide into two parts A and B.

First solve the first moments about the axis x′. y′ is the distance to the centroid of each part.

	Area	$\bar{y}$	$A\bar{y}$
Section A	1500	0	0
Section B	1350	50	67500
Totals	2850		67500

The overall $\bar{y}$ is 67500/2850 = 23.68 mm

Now repeat about the y′ axis. x′ is the distance to the centroid of each part.

	Area	$\bar{x}$	$A\bar{x}$
Section A	1350	75	101250
Section B	1500	0	0
Totals	2850		101250

The overall $\bar{x}$ is 101250/2850 = 35.53 mm

The centroid is 28.68 mm from the top edge and 43.03 mm from the left edge.
We must calculate the minimum I. The formula is:

$$I_{max/min} = \frac{I_x + I_y}{2} \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2}$$

About the centroid of the whole shape $I_{xy} = A\,\bar{y}\,\bar{x}$ (from the parallel axes theorem)

$\mathbf{I_{xy} = A\,\bar{y}\,\bar{x} = 2850 \times 23.68 \times 35.53 = 2.398 \times 10^6\ mm^4}$

This may be positive or negative but it doesn’t matter in the following work.

Next calculate I_x and I_y.

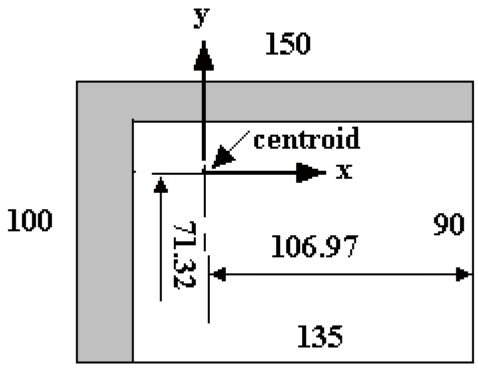

Figure 7

Next take second moments about the bottom edge.

$I_{base} = 150 \times 100^3/3 - 135 \times 90^3/3 = 17.195 \times 10^6\ mm^4$

$I_x = 17.195 \times 10^6 - 2850 \times 71.32^2 = 2.698 \times 10^6 mm^4$

Now find I_y by repeating about the right edge.

$I_{edge} = 100 \times 150^3/3 - 90 \times 135^3/3 = 38.689 \times 10^6\ mm^4$

$I_y = 38.689 \times 10^6 - 2850 \times 106.97^2 = 6.078 \times 10^6\ mm^4$

$$I_{max/min} = \frac{I_x + I_y}{2} \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2}$$

$$I_{max/min} = \frac{6.078 \times 10^6 + 2.698 \times 10^6}{2} \pm \sqrt{\left(\frac{6.078 \times 10^6 - 2.698 \times 10^6}{2}\right)^2 + \left(2.398 \times 10^6\right)^2}$$

$$I_{max/min} = 4.388 \times 10^6 \pm \sqrt{2.856 \times 10^{12} + \left(5.75 \times 10^{12}\right)}$$

$$I_{max/min} = 4.388 \times 10^6 \pm 2.933 \times 10^6$$

$I_{min} = 1.454 \times 10^6\ mm^4$ $\qquad$ $I_{max} = 7.321 \times 10^6\ mm^4$

Finally the angle of the principal axes is found.

$$\tan 2\theta = \frac{-2I_{xy}}{I_x - I_y} = \frac{-2 \text{ x } 2.398}{2.698 - 6.078} = 1.419$$

$$2\theta = 54.82^o \quad \theta = 27.4^o$$

SELF ASSESSMENT EXERCISE No.1

Calculate the minimum second moment of area for the section shown and the angle of the principal axis.

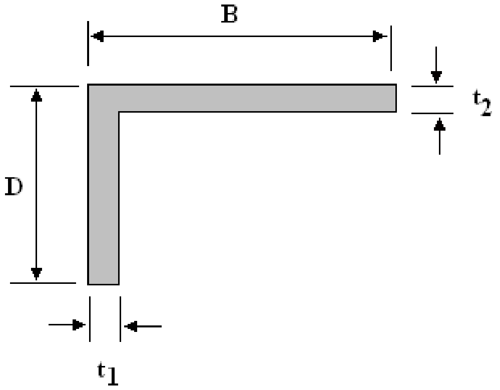

Figure 8

1. B = 50 mm D = 50 mm t_1=10 mm t_2 = 10 mm (Answer 85.288 x 10^3 mm4 and 45^o)

2. B = 60 mm D = 40 mm t_1=10 mm t_2 = 20 mm (Answer 95.55 x 10^3 mm4 and 13^o)

POLAR SECOND MOMENTS OF AREA FOR NON-CIRCULAR SECTION

The polar second moment of area J is taken about the centroid and is found from $J = \int r^2\, dA$ and for a circular section diameter D this is easily shown to be $\pi D^4/32$

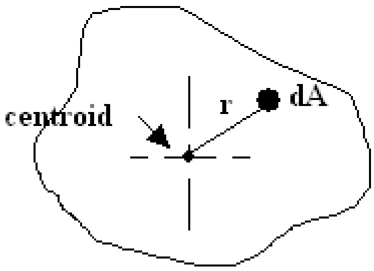

Figure 9

For non-circular sections this is much more difficult. For a rectangle B by D the second moment of area about the axis through the centroid parallel to the edges are $BD^3/12$ and $DB^3/12$ The perpendicular axis theorem gives the polar second moment of area simply by adding them.

$J = BD^3/12 + DB^3/12 = BD(D^2+B^2)/12$ and when it's a square of side D, $J = D^4/6$

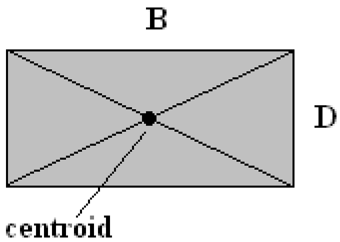

Figure 10

For a triangle the result is $J = BH^3/36$ and the centroid is 1/3 of the perpendicular height from any edge.

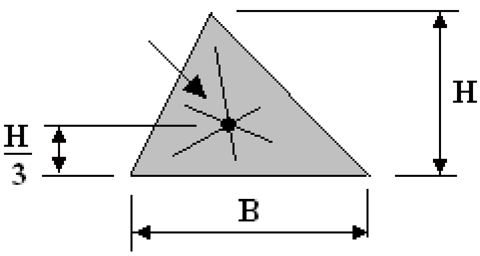

Figure 11