# INSTRUMENTATION AND CONTROL

## TUTORIAL 8 – STABILITY AND THE ‘s’ PLANE

This tutorial is of interest to any student studying Control System Engineering and is set at NVQ level 5 and 6

On completion of this tutorial, you should be able to do the following.

- Define Poles and Zero’s
- Explain the Characteristic Equation of a Transfer Function.
- Explain and interpret Root Locus Diagrams.
- Use The Rules for Graphical Construction of Root Loci.
- Construct Root Loci.
- Explain and apply the Routh-Hurwitz criteria of stability.

If you are not familiar with instrumentation used in control engineering, you should complete the tutorials on Instrumentation Systems.
In order to complete the theoretical part of this tutorial, you must be familiar with basic mechanical and electrical science.
You must also be familiar with the use of transfer functions and the Laplace Transform (see maths tutorials).

***Contents***

## 1. *Poles and Zeros*

Most higher order transfer functions may be expressed as a polynomial on the top and bottom line such that

$$G(s) = \frac{\theta_o}{\theta_i} = \frac{(s + z_1)(s + z_2) \ldots \ldots}{(s + p_1)(s + p_2) \ldots \ldots}$$

The values of s that make the denominator zero are called poles and the values of s that make the numerator zero are called zeros. The zeros are hence $-z_1$, $-z_2$ … and the poles are $-p_1$, $-p_2$ …

If s = p in any of the factors, then the denominator is zero and G(s) = ∞. If this happens, or if any value of z makes the numerator infinity, the system is unstable. The analysis however, is conducted not on the closed loop but on the characteristic equation defined next.

## 2. *Characteristic Equation*

The characteristic equation is the bottom line (denominator) of the closed loop transfer function when equated to zero.

The diagram shows a system with an open loop transfer function G(s) and feedback transfer function H(s).

When H(s) = 1 we have ***Unity Feed Back***.

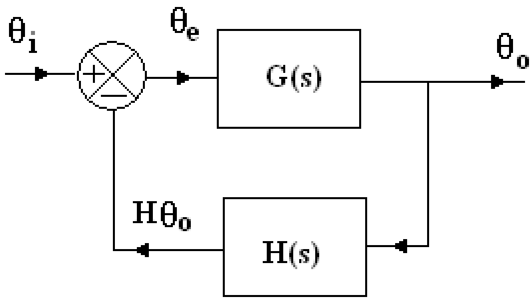



Figure 1

$$G_{cl} = \frac{G}{GH + 1} \text{ or } = \frac{1}{H + 1/G}$$

The characteristic equation is GH + 1 = 0 or H + 1/G = 0

If GH has poles and zeros such that

$$GH = \frac{(s + z_1)(s + z_2) \ldots \ldots}{(s + p_1)(s + p_2) \ldots \ldots}$$

Instability occurs when

$$GH + 1 = 0 = \frac{(s + z_1)(s + z_2) \ldots \ldots}{(s + p_1)(s + p_2) \ldots \ldots} + 1$$

It is important not to confuse the poles and zeros of the characteristic equation with those of the closed loop transfer function.

We learn a lot about the stability and response of a system by examining the poles and the roots of the characteristic equation.

## *3. The Affect of Poles and Zeros on Dynamic Responses*

***Real Poles***

Consider the simple open loop transfer function

$$G(s) = \frac{\theta_o}{\theta_i}(s) = \frac{1}{(s+\sigma)}$$

There is only one pole and this is s = -σ

It is normal to use an impulse input to study how a system responds, especially regarding stability.

If the input to the system is an impulse, the output in the time domain is simply the reverse Laplace transform giving $\theta_o(t) = e^{-\sigma t}$

Plotting the output for different values of p show:-

(i) The larger the value of σ, the quicker it decays.
(ii) The smaller the value of σ, the longer it takes to decay.
(iii) When the values of σ become negative the output grows with time.
(iv) The more negative the value of σ, the quicker it grows with time.

This is demonstrated on the following plot of output against time for various values of σ.

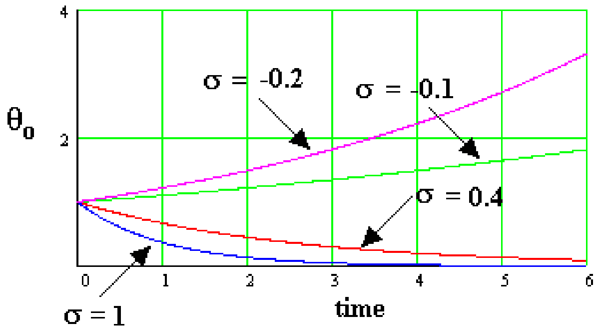



Figure 2

-σ is the pole of the transfer function so we learn that the position of the pole on the s plane greatly influences the system. Negative poles (positive σ) are stable and positive poles are unstable.

The same evidence is produced for a step input later.

**Real Zeros**

Let's examine the closed loop transfer function

$$G(s) = \frac{\theta_o}{\theta_i}(s) = \frac{s+A}{\left(s+1(s+5)\right)}$$

There is a zero at s = -A and two poles at s = -1 and s = -5

Again examine the time response for an impulse input. This is simply the case of inverse Laplace transform.

$$\theta_o(t) = A\frac{e^{-t}}{4} - \frac{e^{-t}}{4} + 5\frac{e^{-5t}}{4} - A\frac{e^{-5t}}{4}$$

This is plotted in the next diagram.

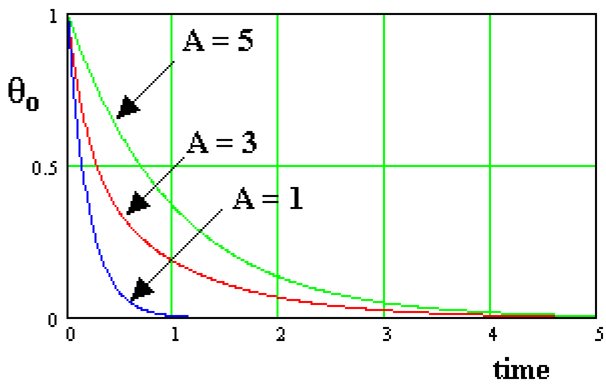



Figure 3

Clearly if A = 1 then the zero will cancel one pole. If A = 5 it will cancel the other. If A is between 1 and 5 the result is mid way between the other two. We conclude that the zero moves the response toward one pole or another.
The affect of zeros on the root locus plot is more difficult to understand and is covered later.

### *Complex Poles*

Complex poles in the characteristic equation produce oscillatory responses to impulse and step inputs. Consider the following open loop transfer function.

$$G(s) = \frac{k}{(s+1)(s+2)(s+3)}$$

With unity feedback we get the following.

$$G(s)_{cl} = \frac{G(s)}{G(s)+1}$$

The characteristic equation of the closed loop transfer function is

$$G(s) = \frac{k}{(s+1)(s+2)(s+3)} + 1 = 0$$

Hence

$$(s+1)(s+2)(s+3) + k = 0$$

If any of the brackets are zero then the system is unstable. If k = 0, this will clearly happen when s =-1 or -2 or -3. For all other values of k we must solve the equation

$$s^3 + 6s^2 + 11s + 6 + k = 0$$

If this is plotted for various values of k we get the result below. It can be seen that when k = 0 there are three real roots at -1, -2 and -3. As the value of k increases the roots at -1 and -2 converge to a single value (around k = 0.4) and after that they become complex roots. The other root is always real.

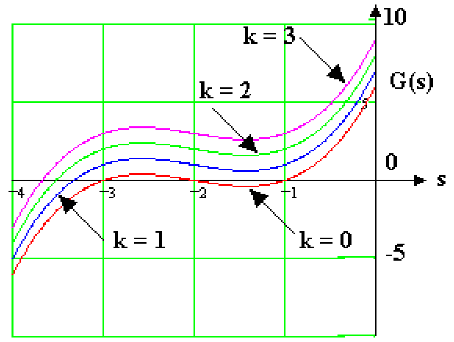



Figure 4

If we expressed G(s) as

$$G(s) = \frac{k}{(s-p_1)(s-p_2)(s-p_3)}$$

$p_1$, $p_2$ and $p_3$ are the ***Poles*** and these are the roots of the characteristic equation when k = 0. It follows that if

$$G(s) = \frac{k}{(s+1)(s+2)(s+3)} +$$

the poles are at -1, -2 and -3.

Consider the case k = 1. The characteristic equation has a zero values at one real point (root) around s = - 3.4. By trial and error with a calculator it is possible to arrive at the solution.

We are finding the roots of $s^3 + 6s^2 + 11s + 7 = 0$ Guessing s = -3.5 we get

| | | |
|---|---|---|
| | s = - 3.5 | - 42.875 + 73.500 - 38.50 + 7 = - 0.875 |
| First correction | s = - 3.4 | - 39.300 + 69.360 - 37.40 + 7 = - 0.34 |
| Next correction | s = - 3.3 | - 35.937 + 65.340 - 36.30 + 7 = 0.103 |
| | s = -3.35 | - 37.593 + 67.335 - 36.85 + 7 = - 0.108 |
| | s = -3.33 | - 36.926 + 66.533 - 36.63 + 7 = - 0.023 ≈ 0 |

So the real root is -3.33. Now we need to determine the remaining complex roots.

We now know that (s + 3.33)(remaining quadratic) = $s^3 + 6s^2 + 11s + 7$

The quadratic will have the form $(as^2 + bs + c)$ so

$$(s + 3.33)(as^2 + bs + c) = s^3 + 6s^2 + 11s + 7$$

Multiply out and $as^3 + s^2 (b + 3.33a) + s (c + 3.33b) + 3.33c = s^3 + 6s^2 + 11s + 7$

Comparing coefficients we see that $a\,s^3 = s^3$ so a = 1

(b + 3.33a) = 6 hence b = 2.67
(c + 3.33b) = 11 hence c = 2.1
Checking 3.33c = 7 is correct.

The equation is $(s + 3.33)(s^2 + 2.67s + 2.1) = 0$

One root is at -3.33 the other roots are found by solving $(s^2 + 2.67s + 2.1) = 0$

Using the quadratic equation we find

$$s = \frac{-2.67 \pm \sqrt{2.67^2 - 4 \times 1 \times 2.1}}{2 \times 1} = \frac{-2.67 \pm \sqrt{-1.27}}{2} = -1.335 \pm 0.564j$$

Note that the two complex roots are conjugate numbers, in other words they are mirror images about the real axis. This is always true for complex poles; they only exist in conjugate pairs. To summarise, the characteristic equation is

$$\frac{k}{(s+1)(s+2)(s+3)} + 1 = 0$$

In which case $s^3 + 6s^2 + 11s + 6 + k = 0$

For the case k = 1 we have found that the values of s that produces zero values are:

$s_1 = -3.33$   $s_2 = -1.335 + 0.564\,j$   $s_3 = -1.335 - 0.564\,j$ and these are the roots.

Note that it is quite possible to have complex poles (when k = 0) as well as complex roots.

A root locus plot would require you to evaluate the roots for many values of k and without a suitable calculation aid this would be very difficult. A graphical method is outlined later. The next section shows the complete plot for the example just examined.

### *4. Root Locus Diagrams – The 's' Plane*

Real and complex roots can be plotted on an Argand diagram for different values of k in the previous section. Note that a locus can be plotted for other parameters such as damping ratio (see tutorial 5). If the points are joined we obtain a locus known as the ***Root Locus Diagram.*** This is laborious unless a computer package is used but as this is not available under exam conditions it may be as well to learn how to sketch them. The result for the case in hand is shown below and because there are three roots, we have 3 loci. The plane is usually referred to as the s plane.

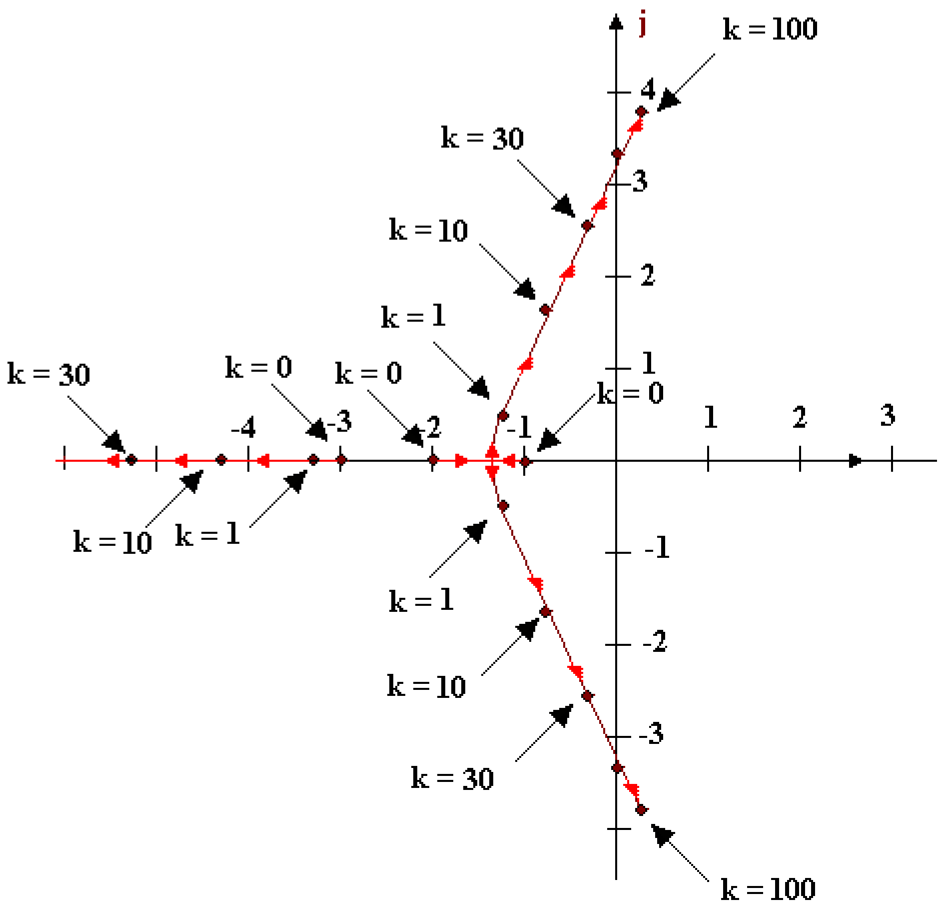



Figure 5

The three loci are plotted for k values between zero and -∞. One locus runs from -3 to -∞ along the real axis. The other two loci approach each other from -2 and -1 and depart from the real axis at a point between them to form mirror images.

We will look at rules for the construction of root loci later in the tutorial.

## 5 *Interpretation of Root Locus Diagrams*

### 5.1 *Intercept with the Imaginary Axis*

Referring to the root locus plot in section 4, the value of k where the locus cuts the imaginary axis is the critical value where the system changes from stable to unstable. It is the point corresponding to -1 on the Nyquist diagram. In the example, this occurs at k = 60 and the ***Nyquist*** plot of the system confirms it. Below is the Nyquist plot for

$$G = \frac{k}{(s+1)(s+2)(s+3)}$$

k = 1 and k = 60. When k = 60 the plot passes through the -1 point.

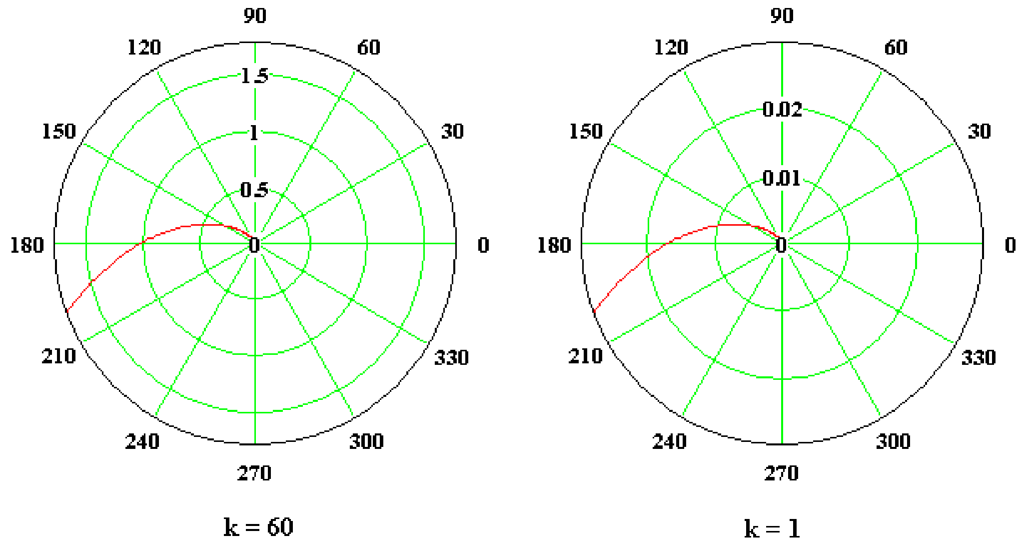



Figure 6

### 5.2 *Position of the Poles*

The reason for sketching these loci is to give an insight into the stability of a system. If a pole appears to the right of the imaginary axis, the output will grow indefinitely and the system is unstable. If all the poles are to the left of the imaginary axis, the system is stable. The poles closest to the origin on the left side of the axis are the *Most Significant* as they indicate the output will take a long time to decay (die away) and dominate the system. The poles furthest from the origin on the positive (right) side indicate greatest instability. The reasoning behind this was covered earlier in this tutorial but here is a more detailed look at it.

Remember that the pole p is the value of s that makes the bracket zero. Hence if we had a bracket (s + 2) the pole is at s = -2 so we might write the bracket as (s – p).

Consider that a closed loop transfer function is of the form:

$$\frac{\theta_o}{\theta_i} = G(s) = \frac{K}{(s-p_1)(s-p_2)(s-p_3)\ \ldots\ldots(s-p_n)}$$

For a unit step input $\theta_i(s) = 1/s$ the output $\theta_o(s)$ is

$$\theta_o(s) = \frac{K}{s(s-p_1)(s-p_2)(s-p_3)}$$

Change into partial fractions

$$\theta_o(s) = \frac{A_0}{s} + \frac{A_1}{s-p_1} + \frac{A_2}{s-p_2} + \frac{A_3}{s-p_3}$$

Inverse Laplace Transform

$$\theta_o(t) = A_0 + A_1e^{p_1t} + A_2e^{p_2t} + A_3e^{p_3t}$$

Repeating the process for a unit ramp we get

$$\theta_o(t) = B_0 + tB_1 + B_2e^{p_1t} + B_3e^{p_2t} + B_4e^{p_3t}$$

The initial terms $A_0$ $B_0$ and $tB_1$ or others resulting from the solution, are the steady state components (constant or proportional to time).

With real poles the exponential terms such as $Ae^{pt}$ are transient terms. At t = 0, $Ae^{pt}$ = A. If p is negative, it will decay with time. If p is positive it will grow with time. The rate at which it decays or grows depends on the value of p. This is illustrated on the diagram below.

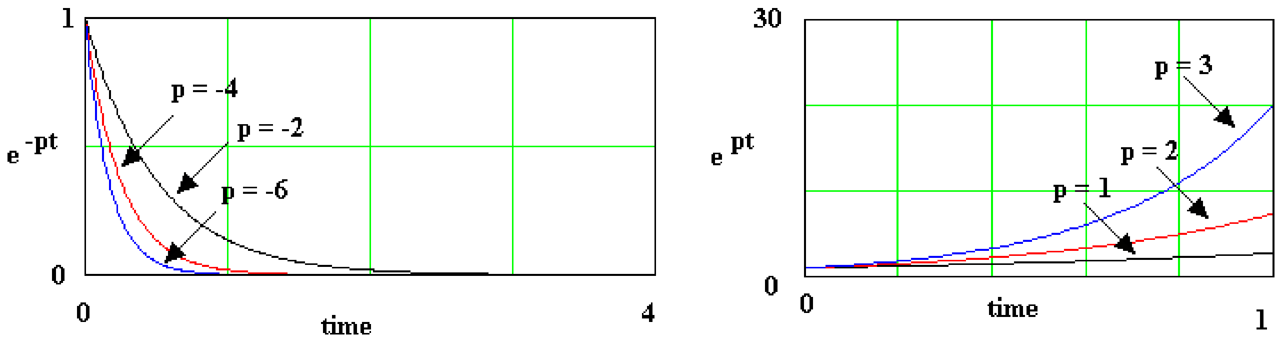



Figure 7

It follows that if p lays on the negative real axis of the s plane, the transient will die away and the further it is from the origin, the quicker it will take. If p lays on the positive real axis, the transient will grow and the further it is from the origin, the faster it will grow.

When p is a complex number, the forgoing still applies but to the real part of the number. Without proof, it can be shown that the smaller the imaginary part of the number, the larger the damping and vice versa.

The affect of the position of the poles on the s plane may be summarised as shown on the diagram.

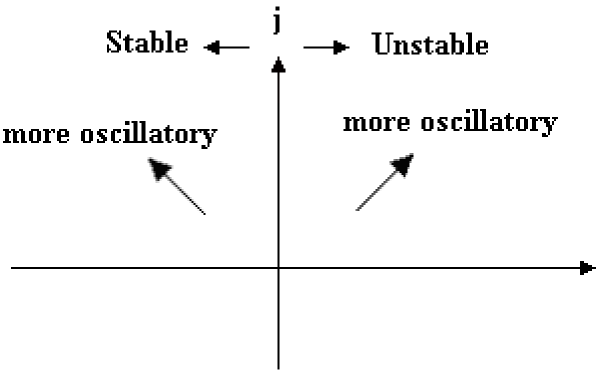



Figure 8

## *5.3 Affect of the Zero on the Root Locus*

We have studied the root locus of the closed loop system with an open loop transfer function of

$$G(s) = \frac{k}{(s+1)(s+2)(s+3)}$$

Now consider the open loop transfer function

$$G(s) = \frac{k(s+1.5)}{(s+1)(s+2)(s+3)}$$

We now have a zero at s = - 1.5. What affect does this have on the root locus?

The closed loop transfer function is

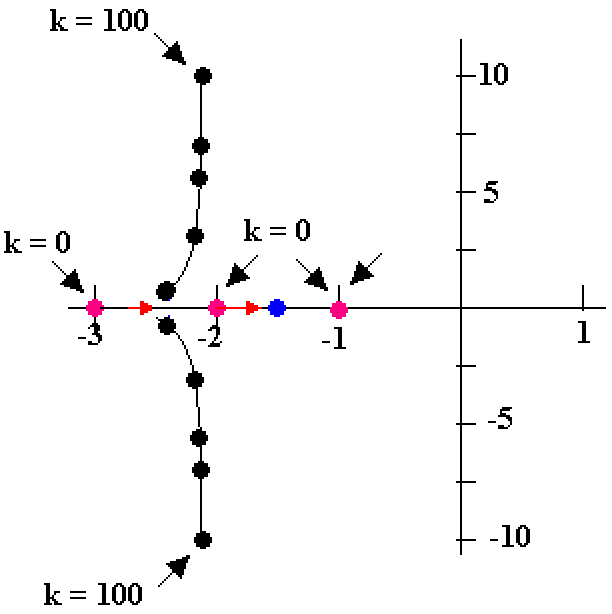
$$G_{cl} = \frac{G}{1+G} \text{ or } \frac{1}{1+1/G} = \frac{1}{1+\dfrac{(s+1)(s+2)(s+3)}{k(s+1.5)}} = \frac{1}{k(s+1.5)+(s+1)(s+2)(s+3)}$$

The characteristic equation of the closed loop transfer function is now:

$$(s+1)(s+2)(s+3) + k(s+1.5) = 0$$

Any value of k that satisfies this equation will produce instability. We are solving the roots

The result is shown below. The first thing we notice is that no value of k will produce instability as the root loci never cross into the positive real regions. The first root starts at -3 and ends at -1.5 (the zero). One complex root starts at -2 and moves to -3 before becoming complex. The other starts at -1 and moves to 2.5 before becoming complex.




| | root 1 | root 2 | root 3 |
|---|---|---|---|
| k = 0 | -3 | -2 | -1 |
| k = 0.1 | -1.02 | -2.9 | -2.0 |
| k = 0.2 | 1.05 | -2.8 | -2.1 |
| k = 0.5 | 1.11 | -2.4 ± 0.29 j | |
| k = 1 | -1.21 | -2.4 ± 0.69 j | |
| k = 10 | -1.46 | -2.27 ± 3j | |
| k = 30 | -1.49 | -2.26 ± 5.4j | |
| k = 50 | -1.49 | -2.25 ± 7j | |
| k = 100 | -1.5 | -2.25 ± 10 j | |

Figure 9

The real locus runs from -3 to 1.5. The complex loci have asymptotes at $90^o$ and $270^o$ and intercept the real axis at -2.25. The affect of the zero is stabilizing the system in this case by reducing the affect of the complex roots. Increased gain makes the system more oscillatory.

## 6 *General Rules for Constructing the Root Locus Diagrams*

In an examination situation, it is unlikely that a student will have access to a calculator capable of solving complex roots to give the answers listed above.

1 Arrange the closed-loop characteristic equation into the form 1 + KF(s) = 0 where

$$F(s) = \frac{M(s)}{N(s)}$$

$$M(s) = (s - z_1)\ (s - z_2)\ \ldots\ (s - z_m)$$

and

$$N(s) = (s - p_1)(\ s - p_2)\ \ldots\ (s - p_n),$$

so that $z_1, z_2, \ldots, z_m$, are the m zeros of F(s) and $p_1$' $p_2, \ldots p_n$ are the n poles of F(s).

2 For positive K every point on the root loci satisfies the magnitude and angle criteria:

$|F(s)| = 1/K$
$\angle F(s) = (1 + 2h) \times 180°$, where h = 0, ± 1, ±2.....

3 (a) The number of separate loci is equal to n. There is one locus for each root.

(b) For m = n the n loci start at poles when K = 0 and terminate at zeros when $K \to \infty$

(c) For m < n, and r = n - m, m of the loci start at poles when K = 0 and terminate at zeros when $K \to \infty$, and r of the loci start at poles when K = 0 and approach asymptotes as $K \to \infty$

(d) Pairs of loci are mirrored in the real axis of the s-plane.

4 The asymptotes intersect the real axis at a single point whose coordinate is

$$s_0 = \frac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i}{r}$$

When they are complex use the real part in this calculation

5 The angles of the asymptotes to the positive real axis are

$$\theta_q = \frac{180(1 + 2q)}{r}$$

where q = 0, 1, 2; ..., (r - 1).

For 1 pole the angle is $180^o$.
For 2 poles the angles are $90^o$ and $270^o$.
For 3 poles the angles are $60^o$ and $180^o$ and $300^o$
For 4 poles the angles are $45^o$, $135^o$, $225^o$ and $315^o$ and so on.

6 The loci, or segments of loci, coincide with the real axis at points for which the total number of poles and zeros on the axis to the right is odd.

7 The loci branch to and from the real axis at points given by the real solutions to

$$N_s \frac{dM(s)}{ds} - M(s)\frac{dN(s)}{ds} = 0$$

This must satisfy rule 6.

8 The locus departs from a complex pole, $p_i$, at an angle $\theta_{Di} = 180° + \angle F'(p_i)$, where

$$F'(s) = (s - p_i)F(s)$$

9 The locus arrives at a complex zero, $z_i$, at an angle $\theta_{Ai} = 180° - \angle F''(z_i)$ where

$$F''(s) = \frac{F(s)}{s - z_i}$$

Explanation of rule 8

CASE 1
Consider the case of 4 poles and no zeros. Two poles are complex.

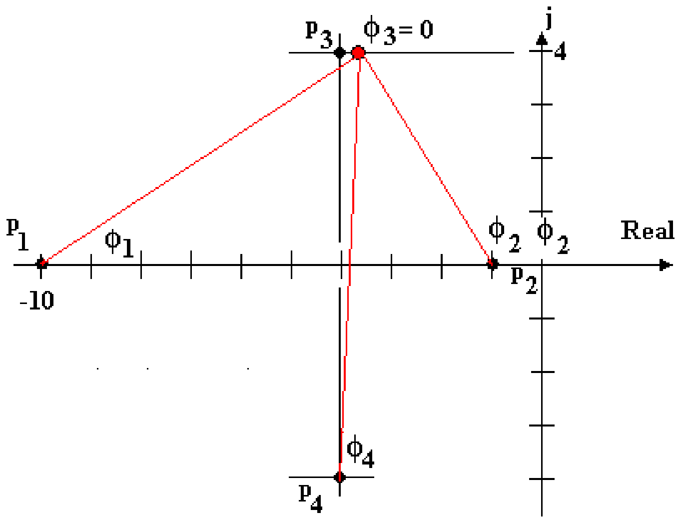



Figure 10

Choose a point s just to the right of the complex pole as shown. Draw a line from each pole to this point. If it is close to a pole as shown, the angle $\phi_3$ is zero. Measure or calculate the other angles. Add the angles together and then calculate the next largest angle θ as a multiple of $180^o$.
The breakaway angle of the locus from the pole is $\theta - \sum\phi$

$p_1 = -10 \quad p_2 = -1$
$p_3 = -4 + 4j$ and $p_4 = p_3 = -4 - 4j$
$\phi_1 = \tan^{-1} 4/6 = 33.7^o$
$\phi_2 = 180^o - \tan^{-1} 4/3 = 126.9^o$
$\phi_3 = 0^o$
$\phi_4 = 90^o$
$\sum\phi = 250.6^o$

The next largest multiple of $180^o$ is $360^o$ so the break away angle is $360 - 250.6 = 109.4^o$

CASE 2 WITH A ZERO

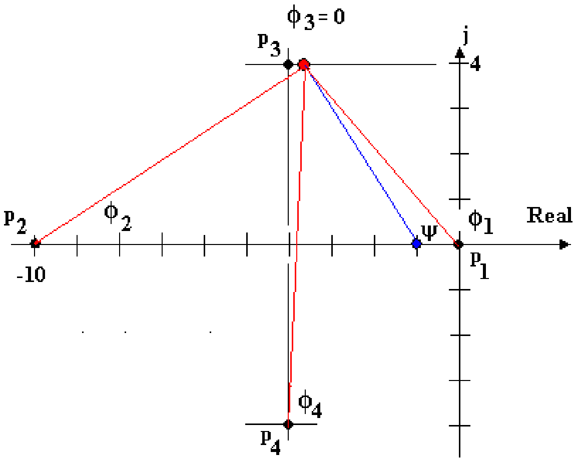



Figure 11

$p_1 = 0 \quad p_2 = -10$
$p_3 = -4 + 4j$ and $p_4 = p_3 = -4 - 4j$
$z_1 = -1$
$\phi_1 = 180^o - \tan^{-1} 4/4 = 45^o$
$\phi_2 = \tan^{-1} 4/6 = 33.7^o$
$\phi_3 = 0^o$
$\phi_4 = 90^o$
$\Sigma\phi = 250.6^o$
$\psi = 180 - \tan^{-1}4/3 = 126.9^o$
$\Sigma\psi$
$\Sigma\psi - \Sigma\phi = 126.9 - 250.6 = -123.7^o$
The next multiple of $180^o$ is $0^o$ so the break away angle is $0 - (-123.7) = -123.7^o$

CASE 3 with 3 POLES

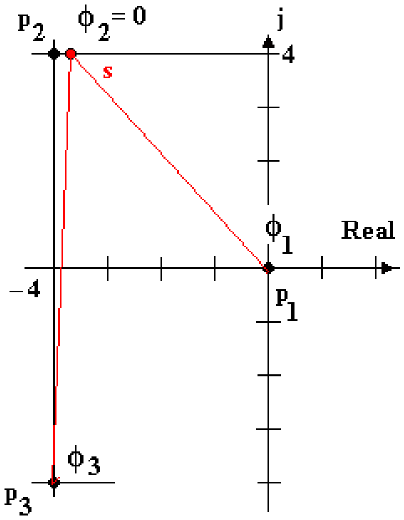



Figure 12

Now consider the case of a 3 pole system given $p_1 = 0$,
$p_2 = -4 + 4j$ and $p_3 = -4 - 4j$
$\phi_1 = 180^o - \tan^{-1}4/4 = 180^o - 45^o = 135^o$
$\phi_2 = 0^o$
$\phi_3 = 90^o$
$\Sigma\phi = 225^o$
The next largest multiple of $180^o$ is $360^o$ so the break away angle is $360 - 225 = 135^o$

**WORKED EXAMPLE No. 1**

A system has an open loop transfer function of

$$G(s) = \frac{k}{(s+1)(s+2)(s+3)}$$

Sketch the root locus of the characteristic equation.

**SOLUTION**

This is the same problem covered earlier with a full plot.

$$G_{cl}(s) = \frac{G}{G+1}$$

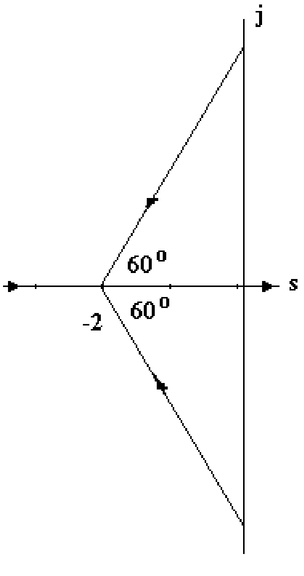
The characteristic equation is G + 1 = 0 so the correct form is

$$\frac{k}{(s+1)(s+2)(s+3)} + 1 = 0$$

One locus is along the real axis from $-\infty$ to -3. The others are loci are asymptotes to lines at $60^o$ and $300^o$ respectively that intersect the real axis at

$$\frac{(p_1 + p_2 + p_3)}{3} = \frac{(-3-2-1)}{3} = -2$$

The diagrams show the simplified and actual plots.



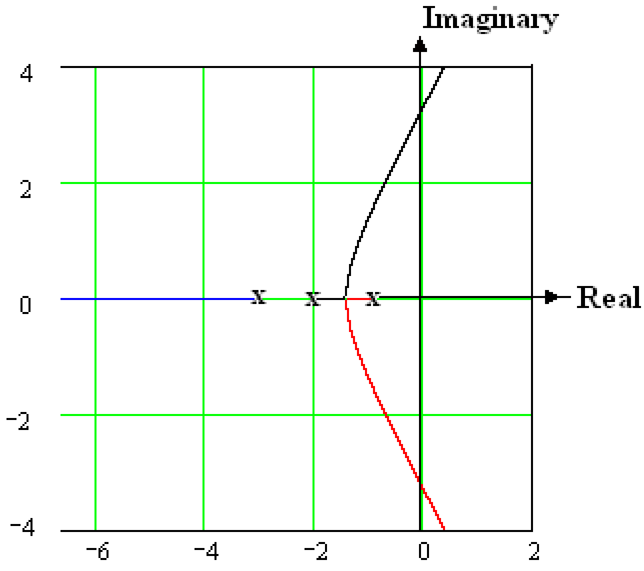




Figure 13

**WORKED EXAMPLE No. 2**

A system has an open loop transfer function of

$$G(s) = \frac{k(s+2)}{(s+1)(s+3)(s+5)}$$

Sketch the root locus of the characteristic equation.

**SOLUTION**

$$G_{cl}(s) = \frac{G}{G+1}$$

The characteristic equation is

$$\frac{k(s+2)}{(s+1)(s+3)(s+5)} + 1 = 0$$

m = number of zeros = 1   n = number of poles = 3.   r = n - m = 2

Intercept

$$\frac{-1-3-5-(-2)}{r} = \frac{-7}{2} = -3.5$$

Angles

$$\frac{180 \times (1+2q)}{r} = \frac{180 \times (1+2q)}{2} = 90(1+2q)$$

For q = 0  $\theta = 90^o$   For q = 1  $\theta = 270^o$   For q = 2  $\theta = 450^o$

The poles are at -1, -3 and -5   the zero is at -2. This is enough information to sketch the root locus

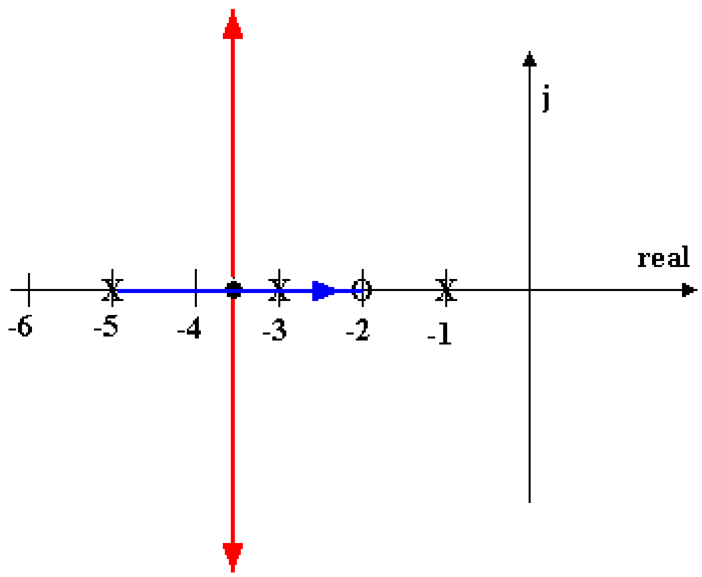



Figure 14

**WORKED EXAMPLE No. 3**

A system has a forward transfer function of

$$G(s) = \frac{k}{(0.5s + 5)(s^2 + 4s + 16)}$$

It also has a feed-back transfer function

$$H(s) = \frac{1}{(s + 1)}$$

Sketch the root locus of the characteristic equation and discuss the importance of the pole positions.

**SOLUTION**

$$G_{cl}(s) = \frac{G}{GH + 1}$$

The characteristic equation is GH + 1=0

$$\frac{k}{(s + 1)(0.5s + 5)(s^2 + 4s + 16)} + 1 = 0$$

Multiply the top and bottom by 2 to obtain unity s value

$$\frac{2k}{(s + 1)(s + 10)(s^2 + 4s + 16)} + 1 = 0$$

Factorise the quadratic

$$\frac{5k}{(s + 1)(s + 10)(s - 2 + 3.46j)(s - 2 - 3.46j)} + 1 = 0$$

This is now in the correct form

There are 4 poles and no zeros m = 0 n = 4 r = 4

The intercept is

$$\frac{-1 - 10 - 2 - 2}{4} = \frac{-15}{4} = -3.75$$

Two loci start on the real axis at -1 and -10 and break away to two asymptotes. The other two loci start at the complex poles -2 ± 3.46j

The asymptotes for 4 loci are 45°, 135°, 225° and 315°

The complex poles break away at an angle θ found as follows.

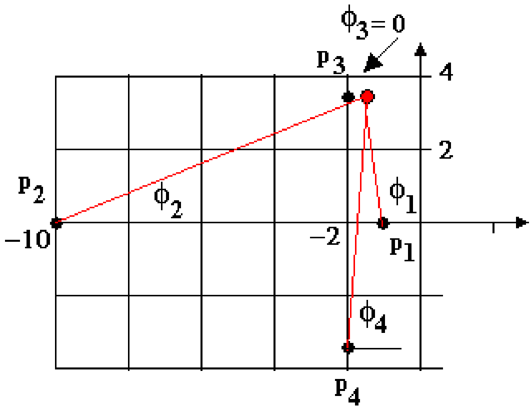



Figure 15

$\phi_1 = 180 - \tan^{-1}(3.46/1) = 106^o$     $\phi_2 = \tan^{-1}(3.46/8) = 23.4^o$     $\phi_3 = 0$     $\phi_4 = 90^o$

$\Sigma\phi = 219.4^o$     Break away angle = 360 - 219.4 = 140.6$^o$

Now we can attempt to sketch the loci. Without further work we cannot determine the break away point or the precise point where the loci meld with the asymptotes and the student should be aware that creating accurate plots requires a lot of practise and experience.

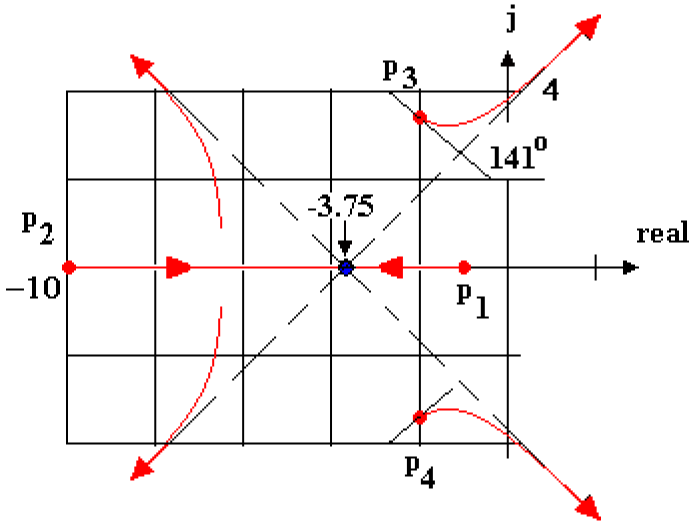



Figure 16

The dominant pole is $p_1$ at -1 when the values of K are small. As the value of K increases the real roots have less dominance but the complex pair of poles $p_3$ and $p_4$ becomes more dominant and at some critical point where they cross the imaginary axis, the system becomes unstable. More work is needed to find this point.

**SELF ASSESSMENT EXERCISE No. 1**

1. Sketch the approximate root locus diagram for a closed loop system with unity feed back when the forward loop transfer function is:

$$G = \frac{k}{s(s+2)(s+3)}$$

Determine the poles and the approximate point where the locus cuts the imaginary axis. Comment on the stability.

(0, -2, -3 and $\pm$ j 3 )

2. Plot the root locus with respect to k for a system with a forward transfer function

$$G = \frac{k}{3s^2+5s+1}$$

It has unity feedback. (Answer on next page)

3. Plot the root locus with respect to k for a system with a forward transfer function

$$G = \frac{k}{s(3s^2+5s+1)}$$

It has unity feedback. Comment on the difference to the result for Q2
(Answer on next page)

4. A system with unity feed back has a forward transfer function of

$$G = \frac{k}{(s+4)(s^2+2s+4)}$$

Determine the largest value of k for which the system is stable.

5. A system with unity feed back has an open loop transfer function

$$G_{ol} = \frac{80(1+s\tau_d)}{(s^2+8s+80)}$$

Plot the root locus of the characteristic equation with τ being the variable.

Answer Q2

The two loci run from poles at -0.232 ans -1.434 and converge at -0.833 running off at 90°. The system is always stable.

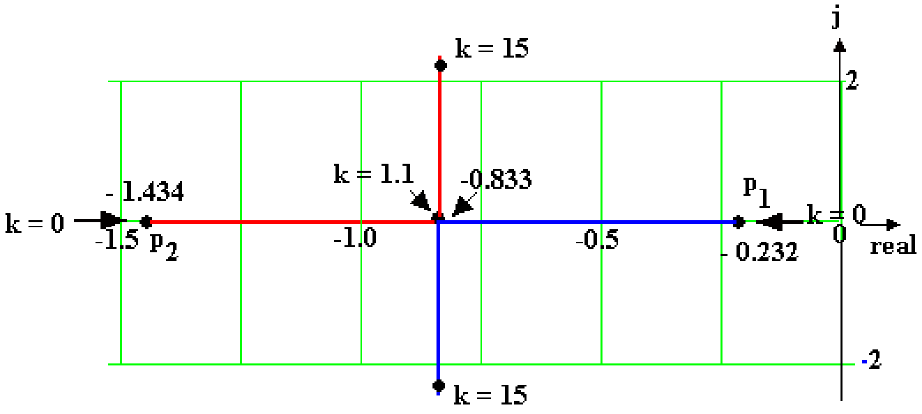



Figure 17

Answer Q 3

There is one extra pole at 0,0 giving an intercept at 0.55 and angles of 60° and 300°. The extra s term produces the possibility of instability at high gains.

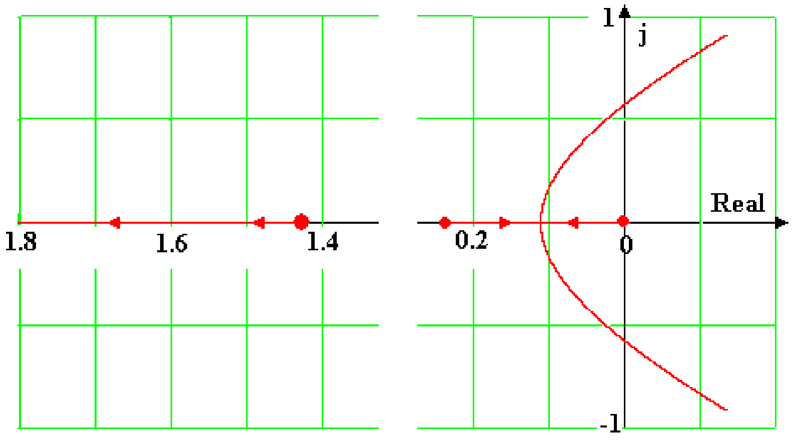



Figure 18 The Complete Plot

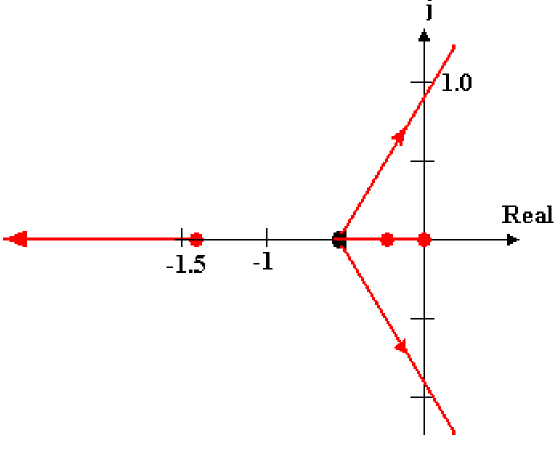



Figure 19 The Graphical Construction

Solution Q4

Poles are at -4, (-1 + √3) and (-1 +√3)

Intercept is (-4 -1 -1)/3 = -2

Angles for 3 poles are 60° to the real axis

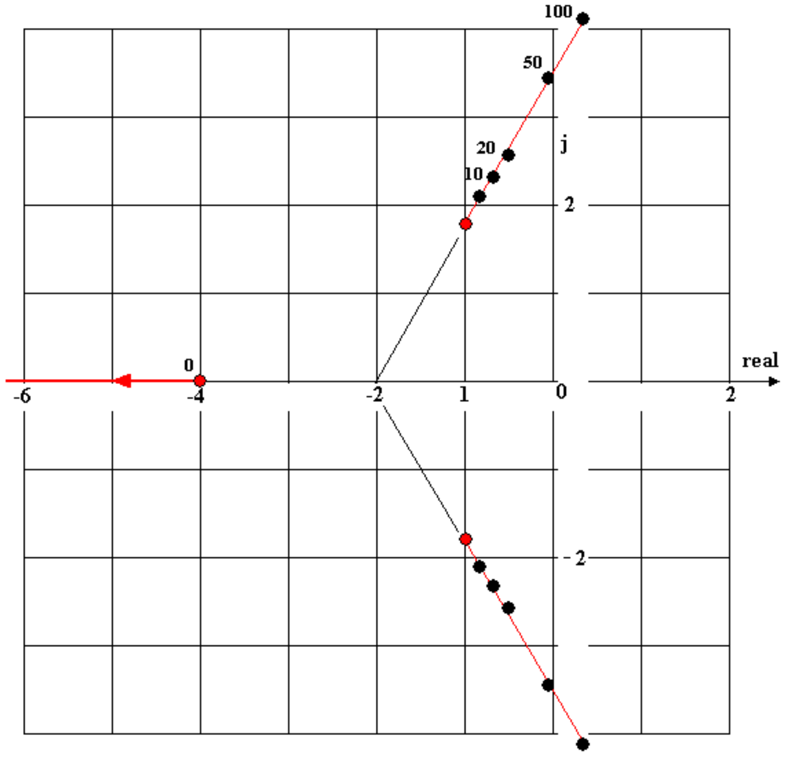
The roots are given by

$$R1(k) := (-8 - k)^{\left(\frac{1}{3}\right)} - 2$$

$$R2(k) := \frac{-1}{2}\cdot(-8 - k)^{\left(\frac{1}{3}\right)} - 2 + \frac{1}{2}\cdot i\cdot\sqrt{3}\cdot(-8 - k)^{\left(\frac{1}{3}\right)}$$

$$R3(k) := \frac{-1}{2}\cdot(-8 - k)^{\left(\frac{1}{3}\right)} - 2 - \frac{1}{2}\cdot i\cdot\sqrt{3}\cdot(-8 - k)^{\left(\frac{1}{3}\right)}$$

k = 50 is the limit for stability.



## 7. *Routh-Hurwitz Criterion for Stability*

This is a mathematical way of determining if the system is unstable. We know that stability depends on where the roots lay on the s plane. The Routh-Hurwitz criterion is based on this and offered here with out proof. The characteristic equation should be arranged as a polynomial with descending orders of s. It should have a form of:

$$a_n s^n + a_{n-1}s^{n-1} + a_{n-2} s^{n-2} + \ldots\ldots\ a_3 s^2 + a_2 s^2 + a_1 s^1 + s^o$$

The coefficients should then be set up as a matrix as follows.

| | | | | |
|---|---|---|---|---|
| $a_n$ | $a_{n-2}$ | $a_{n-4}$ | $a_{n-6}$........ | (the first, third, fifth and so on) |
| $a_{n-1}$ | $a_{n-3}$ | $a_{n-5}$ | $a_{n-7}$ ......... | (the second, fourth, sixth and so on) |
| $b_1$ | $b_2$ | $b_3$ | ...................... | (a new row created from the others) |
| $c_1$ | $c_2$ | $c_3$ | ...................... | (a new row created from the others) |
| $d_1$ | $d_2$ | ........................ | ............ | (a new row created from the others) |

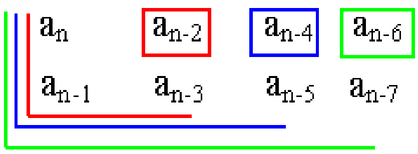
We calculate

**$b_1 = a_{n-2} - (a_n/a_{n-1})\ a_{n-3}$**

**$b_2 = a_{n-4} - (a_n/a_{n-1})\ a_{n-5}$**

**$b_3 = a_{n-6} - (a_n/a_{n-1})\ a_{n-7}$**

This is a way of visualising it.

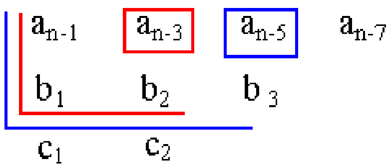



Next we calculate

**$c_1 = a_{n-3} - (a_{n-1}/\ b_1)\ b_2$**

**$c_2 = a_{n-5} - (a_{n-1}/\ b_1)\ b_3$**

This is a way of visualising it.




This process is repeated until all the coefficients are found.

$d_1 = b_2 - (b_1/c_1)c_2$

When the matrix is set up, the criterion is that the system is stable if the coefficients in the first column are all positive. One or more negative values in the first column indicate instability.

**SIMPLEST CASE**

If the highest power of the characteristic equation is 3 the criteria may be simplified as follows.

Characteristic equation = $(as^3 + bs^2 + cs + d)$

If the next coefficient is negative the system is unstable and this is given by $R = c - ad/b$

**WORKED EXAMPLE No. 4**

A system has a characteristic equation $2s^6 + 4s^5 + 2s^4 - s^3 + 2s - 2 = 0$
Determine if it is a stable system.

**SOLUTION**
Note that the coefficient for $s^2$ is zero since it doesn't exist. The equation could be written in full as $2s^6 + 4s^5 + 2s^4 - s^3 + 0s^2 + 2s - 2 = 0$

The highest power is 6

$a_6 = 2$ $a_4 = 2$ $a_2 = 0$ $a_0 = -2$
$a_5 = 4$ $a_3 = -1$ $a_1 = 2$

$b_1 = a_4 - (a_6/a_5)\,a_3 = 2 - (2/4)(-1) = 2.5$
$b_2 = a_2 - (a_6/a_5)\,a_1 = 0 - (2/4)(2) = -1$
$b_3 = a_0 - (a_6/a_5)\,a_{-1} = -2 - 0 = -2$
$c_1 = a_3 - (a_5/b_1)\,b_2 = -1 - (4/2.5)(-1) = 0.6$
$c_2 = a_1 - (a_5/b_1)\,b_3 = 2 - (4/2.5)(-2) = 5.2$
$c_3 = a_3 - (a_5/b_1)\,b_2 = -1 - (4/2.5)(-1) = 0.6$
$d_1 = b_2 - (b_1/c_1)c_2 = -1 - (2.5/0.6)(5.2) = -22.67$

The complete matrix is like this.

| | | | |
|---|---|---|---|
| 2 | 2 | 0 | -2 |
| 4 | -1 | 2 | |
| 2.5 | -1 | -2 | |
| 0.6 | 5.2 | 0.6 | |
| -22.67 | -2 | | |

The first column has a negative value so the system is unstable.

**WORKED EXAMPLE No. 5**

The characteristic equation of a closed loop system is $3s^3 + s^2 + 0.2s + 1$. Determine if the system is stable.

**SOLUTION**

The highest power is 3 so use the simplified test.

$a = 3$ $b = 1$ $c = 0.2$ $d = 1$

$R = c - ad/b = 0.2 - 3 \times 1/1 = -2.8$

Since R is negative the system is unstable.

**WORKED EXAMPLE No. 6**

A closed loop system with velocity feed back has a transfer function of

$$G(s) = \frac{200K}{s^3 + 12s^2 + 50(1 + 4K\alpha)s + 200K}$$

Determine the value of 'α' which makes the system stable when K = 8.

**SOLUTION**

The characteristic equation is

$$s^3 + 12s^2 + 50(1 + 4K\alpha)s + 200K$$

$a = 1$  $b = 12$  $c = 50(1 + 4K\alpha)$  $d = 200K$

$R = c - ad/b = 50(1 + 4K\alpha) - 1 \times 200K/12$   Put K = 8   $R = 50(1 + 32\alpha) - 1\,600/12$

At the limit of stability R = 0 so $12 \times 50(1 + 32\alpha) = 1\,600$

$1 + 32\alpha = 1\,600/600 = 2.6667$

$32\alpha = 1.6667$   $\alpha = 1.667/32 = 0.05208$

**WORKED EXAMPLE No.7**

Find the value of k where the root locus cuts the imaginary axis for worked example No.3 where

$$G(s) = \frac{k}{(0.5s + 5)(s^2 + 4s + 16)} \quad \text{and} \quad H(s) = \frac{1}{(s + 1)}$$

**SOLUTION**

The characteristic equation is

$$\frac{5k}{(s + 1)(s + 10)(s^2 + 4s + 16)} + 1 = 0$$

$$5k(s + 1)(s + 10)(s^2 + 4s + 16) = 0$$

$$s^4 + 15s^3 + 30s^2 + 216s + (160 + 5k) = 0$$

The highest power is 4

$a_4 = 1$   $a_2 = 30$   $a_0 = (160 + 5k)$
$a_3 = 15$   $a_1 = 216$
$b_1 = 30 - (1/15)(216) = 15.6$   $b_2 = (160 + 5k)$
$c_1 = 216 - (15/15.6)(160+5k)$

The complete matrix is like this.

| | | |
|---|---|---|
| 1 | 30 | 160 + 5k |
| 15 | 216 | |
| 15.6 | 160 + 5k | |
| 216 – (15/15.6) (160 + 5k) | | |

The only term in the first column that can be negative is the last so limit of stability is reached when
216 – (15/15.6) (160 + 5k) = 0
224.64 = (160+5k)
5k = 64.64
k = 13

The root locus crosses the imaginary axis when k = 13

**SELF ASSESSMENT EXERCISE No. 2**

1. Using the Routh – Hurwitz criterion, verify that k = 60 where the root locus cuts the imaginary axis for the characteristic equation (the example used earlier)

$$s^3 + 6s^2 + 11s + 6 + k = 0$$

2. Determine if the closed loop systems described by the following transfer functions are stable.

a.

$$G(s) = \frac{1}{2s^3 + 3s^2 + 0.8s + 1}$$

(R = 0.113)

b.

$$G(s) = \frac{1}{s^3 + 12s^2 + 2s + 1\,000}$$

(R = -81.3)

3. A system with velocity feed back has a transfer function of

$$G(s) = \frac{30K}{s^3 + 10s^2 + 200(1 + 2K\alpha)s + 5K}$$

Determine the value of ‘α’ which makes the system stable when K = 40. (-0.011)

4. A system with velocity feed back has a transfer function of

$$G(s) = \frac{10K}{2s^3 + 10s^2 + 500(1 + K\alpha)s + 20K}$$

Determine the value of ‘α’ which makes the system stable when K= 4. (-0.242)