INSTRUMENTATION AND CONTROL

TUTORIAL 6 – SINUSOIDAL RESPONSES

This tutorial is of interest to any student studying control systems and in particular the EC module D227 – Control System Engineering.

On completion of this tutorial, you should be able to do the following.

- Use the Fourier transform to analyse harmonic responses.
- Produce polar plots for basic transfer functions.
- Produce polar plots for the standard first order system.
- Produce polar plots for the standard second order system.

If you are not familiar with instrumentation used in control engineering, you should complete the tutorials on Instrumentation Systems.

In order to complete the theoretical part of this tutorial, you must be familiar with basic mechanical and electrical science.

You must also be familiar with the use of transfer functions and the Laplace Transform (see maths tutorials).

1. INTRODUCTION

The standard first and second order system transfer functions are as follows.

First order form $\frac{\theta_o}{\theta_i}(s) = \frac{k}{Ts + 1}$

Second Order form $\frac{\theta_o}{\theta i}(s) = \frac{k}{T^2s^2 + 2T\delta s + 1}$

When we have a sinusoidal input the steady state condition will produce an output that is also sinusoidal but with a different amplitude and a phase angle between them. Both signals may be represented by phasors as shown below. If the input has unit amplitude and held at zero degrees, the output phasor will form a phase angle ϕ to it and the length of the phasor will be the ratio of the amplitudes. Plotting vectors for all frequencies from $\omega = 0$ to $\omega = \infty$ produces a polar plot from which the amplitude ratio and phase angle can be seen for any frequency. This is one form of a frequency response diagram.

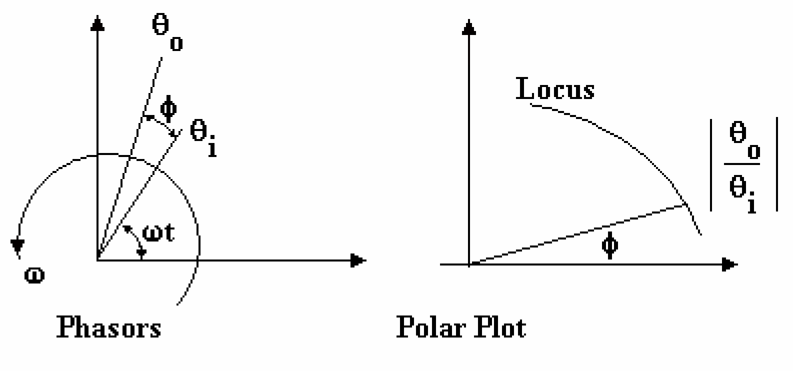

Figure 1

In order to produce this plot we change the transfer function into a function of jω by replacing s with jω. The standard transfer functions become:

$$\frac{\theta_o}{\theta_i}(j\omega) = \frac{k}{j\omega T + 1}$$

$$\frac{\theta_o}{\theta i}(j\omega) = \frac{k}{T^2(j\omega)^2 + 2T\delta j\omega + 1}$$

This is the Fourier transform. When this is done, the transfer function may be made into a complex number in the form G = A + jB. We may then plot the vector on the complex plane.

NOTE that in the steady state when $\omega = 0$ $G(j\omega) = k$ and this is the D.C. gain. T is a time constant and δ is the damping ratio.

The polar plot can be made for a closed loop or open loop system. The open loop is usually the easiest and there is an important reason for doing this which you will see later on when we look at stability.

First let's look at the polar plot of some basic blocks.

2. BASIC BLOCKS

2.1 INTEGRATOR

Consider a control system element that integrates the input.

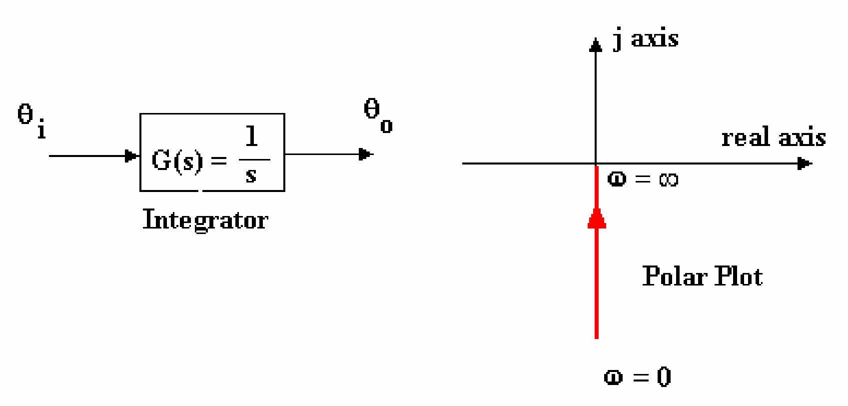

Figure 2

$\theta_o(t) = \int \theta_i\, dt$ $\theta_o(s) = s^{-1}\theta_i$ $\theta_o/\theta_i = 1/s$ Substitute $s = j\omega$

$$\frac{\theta_o}{\theta_i}(j\omega) = \frac{1}{j\omega}$$

Multiply by the conjugate. $\frac{\theta_o}{\theta_i}(j\omega) = \frac{1}{j\omega} \times \frac{-j\omega}{-j\omega} = \frac{-j}{\omega}$

The polar plot would be a line running from $-\infty$ to 0 on the $-j$ axis as shown. The radius is $1/\omega$ and the angle is -90^o for all ω.

2. DIFFERENTIATOR

Consider a control system element that differentiates the input.

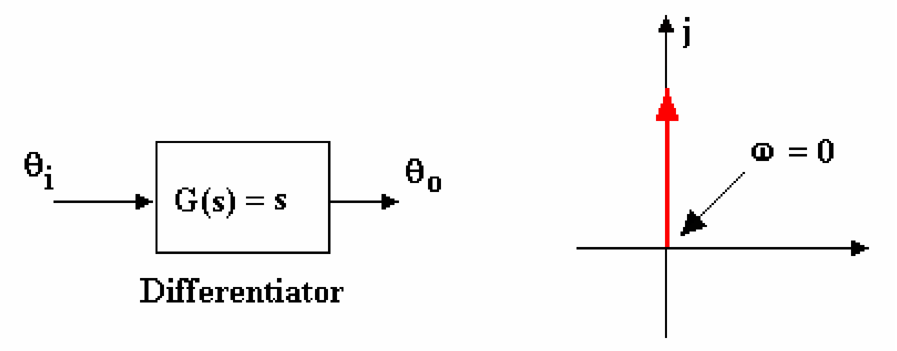

Figure 3

$\theta_o(t) = d\theta_i/dt$ $\theta_o(s) = s\theta_i$ $\theta_o/\theta_i = s$

Substitute $s = j\omega$ $\frac{\theta_o}{\theta_i}(j\omega) = j\omega$

The polar plot would be a line running from 0 to ∞ on the $+j$ axis as shown. The radius is ω and the angle is $+90^o$ for all ω.

2.3 EXPONENTIAL DELAY

$\theta_o\,(t) = e^{-t}\,\theta_i\,dt \quad \frac{\theta_o}{\theta_i}(s) = \frac{1}{1+s}$

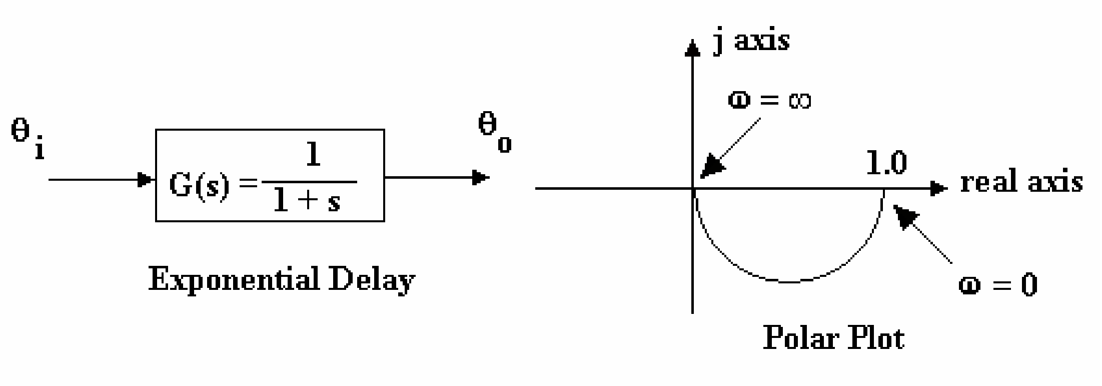

Figure 4

In the jω form the transfer function becomes $\frac{\theta_o}{\theta_i}(j\omega) = \frac{1}{1 + j\omega}$

This is converted into a complex number by multiplying the top and bottom by the conjugate number.

$$\frac{\theta_o}{\theta_i}(j\omega) = \frac{(1 - j\omega)}{(1 + j\omega)(1 - j\omega)} = \frac{(1 - j\omega)}{(1 + \omega^2)} = \frac{1}{1 + \omega^2} - j\frac{\omega}{1 + \omega^2}$$

$$\frac{\theta_o}{\theta_i}(j\omega) = A - jB \qquad A = \frac{1}{(1 + \omega^2)} \qquad B = \frac{\omega}{(1 + \omega^2)}$$

The polar plot is shown and formed by plotting the coordinates A and B for all values of ω.

The radius is $\frac{1}{\sqrt{1 + \omega^2}}$ and the angle is $-\tan^{-1}(\omega)$

3. THE STANDARD FIRST ORDER TRANSFER FUNCTION

In the jω form the transfer function becomes $\frac{\theta_o}{\theta_i}(j\omega) = \frac{1}{1 + j\omega T}$

This is converted into a complex number by multiplying the top and bottom by the conjugate number.

$$\frac{\theta_o}{\theta_i}(j\omega) = \frac{(1 - j\omega T)}{(1 + j\omega T)(1 - j\omega T)} = \frac{(1 - j\omega T)}{(1 - \omega^2 T^2)} = \frac{1}{1 - \omega^2 T^2} - j\frac{\omega T}{1 - \omega^2 T^2}$$

$$\frac{\theta_o}{\theta_i}(j\omega) = A - jB \qquad A = \frac{1}{(1 + \omega^2 T^2)} \qquad B = \frac{\omega T}{(1 + \omega^2 T^2)}$$

To obtain a polar plot we plot A horizontally and B vertically. If this is done for all frequencies over the range 0 to infinity, the resulting locus is called a frequency response diagram. This takes the form of a semi circle and reveals that at $\omega = 0$ the output and input have the same amplitude and the phase angle is zero. As the frequency increases, the amplitude of the output reduces to zero and the phase angle tends to 90^o.

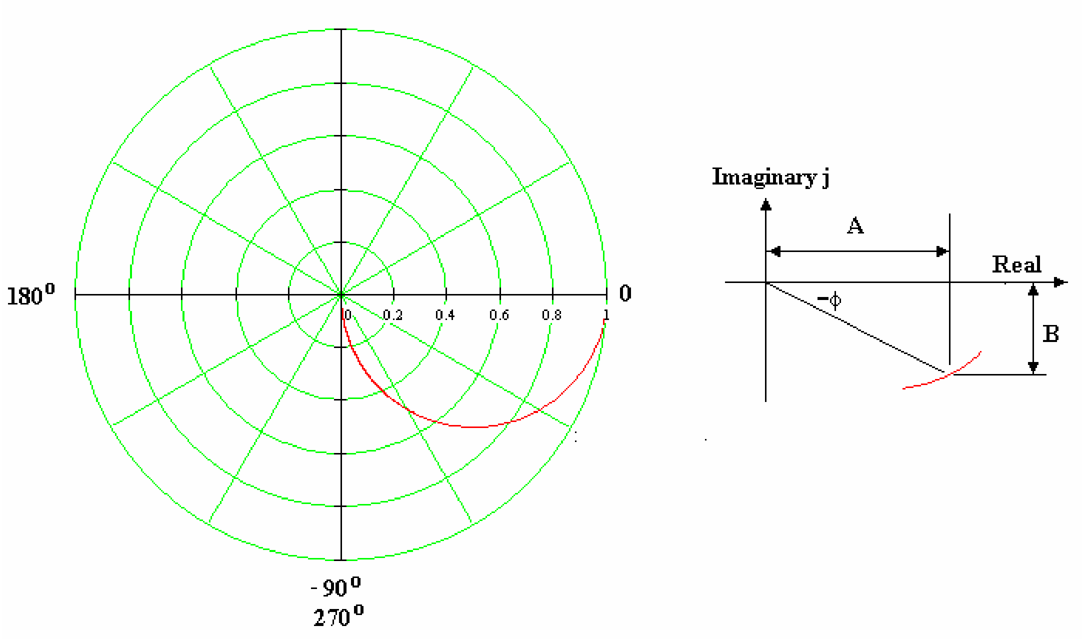

Figure 5

From the geometry we find $\tan(-\phi) = B/A = \omega T$ so $-\phi = \tan^{-1}\omega T$

The ratio of the output amplitude to the input amplitude is $\frac{a_2}{a_1} = \sqrt{B^2 + A^2} = \frac{1}{\sqrt{(1 + \omega^2 T^2)}}$

This is the length of the vector. For any given frequency, the steady state plot θ_o and θ_i against time is typically as shown.

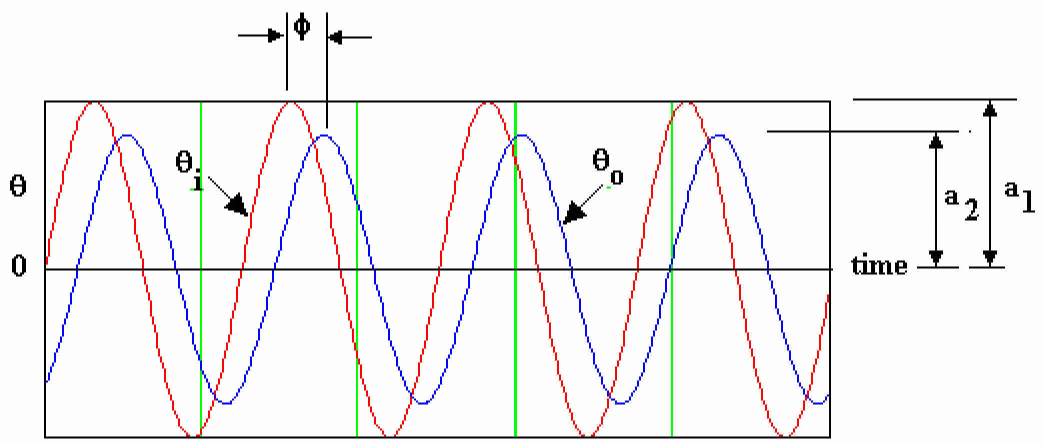

Figure 6

If the transfer function is presented in the form $\frac{\theta_o}{\theta_i}(s) = \frac{K}{n + T's}$ and we change it into the standard form $\frac{\theta_o}{\theta_i}(s) = \frac{k}{1 + Ts}$ the DC gain is k = K/n and the primary time constant is T = T'/n. Going through the above process again would show that the resulting radius and angle are

$$\text{Radius} = \frac{k}{\sqrt{1 + T^2\omega^2}} = \frac{K/n}{\sqrt{1 + \frac{T'^2\omega^2}{n^2}}}$$

$$\phi = -\tan^{-1}(\omega T) = -\tan^{-1}\left(\frac{\omega T'}{n}\right)$$

WORKED EXAMPLE No.1

A simple first order system has the transfer function $\theta_o/\theta_i(s) = 1/(Ts + 1)$

The input is a sinusoidal signal. The system time constant is 0.2 seconds. Produce a polar plot of the locus of $\theta_o/\theta_i(j\omega)$ for frequencies from 0 to 100 rad/s.

SOLUTION

ω (Rad/s)	0	1	5	10	20	50	70	100
Mod(θ_o/θ_i)	1	0.98	0.707	0.447	0.243	0.1	0.07	0.05
ϕ(degrees)	0	-11.3	-45	-63.4	-75.96	-84.29	-85.91	-87.1

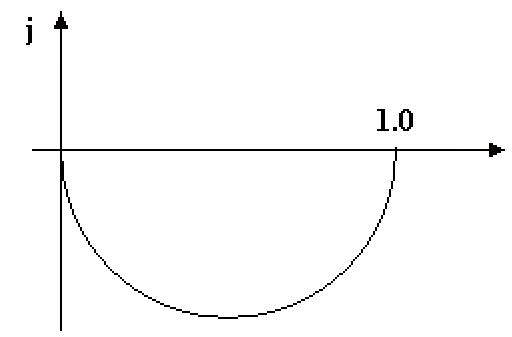

Figure 7

4. THE STANDARD 2nd ORDER TRANSFER FUNCTION

The basic transfer function is

$$G(s) = \frac{1}{T^2 s^2 + 2T\delta s + 1}$$

$$G(j\omega) = \frac{1}{T^2 (j\omega)^2 + 2T\delta j\omega + 1}$$

$$G(j\omega(= \frac{1}{\left(1 - T^2\omega^2\right) + 2T\delta j\omega}$$

$$G(j\omega) = \frac{1}{A + jB}$$

where $A = \left(1 - T^2\omega^2\right)$ and $B = 2T\delta\omega$

Multiply the top and bottom line by the conjugate number A - jB

$$G(j\omega) = \frac{A - jB}{(A + jB)(A - jB)}$$

$$G(j\omega) = \frac{A - jB}{\left\{A^2 + B^2\right\}} = C - jD$$

$$C = \frac{A}{A^2 + B^2} = \frac{\left(1 - \omega^2 T^2\right)}{\left\{\left(1 - \omega^2 T^2\right) + (2\delta\omega T)^2\right\}}$$

$$D = \frac{B}{A^2 + B^2} = \frac{2\delta T\omega}{\left\{\left(1 - T^2\omega^2\right)^2 + (2\delta\omega T)^2\right\}}$$

The polar plot is formed by plotting C horizontally and D vertically. If we plot for all frequencies from ω = 0 to ω = infinity we get the following diagram.

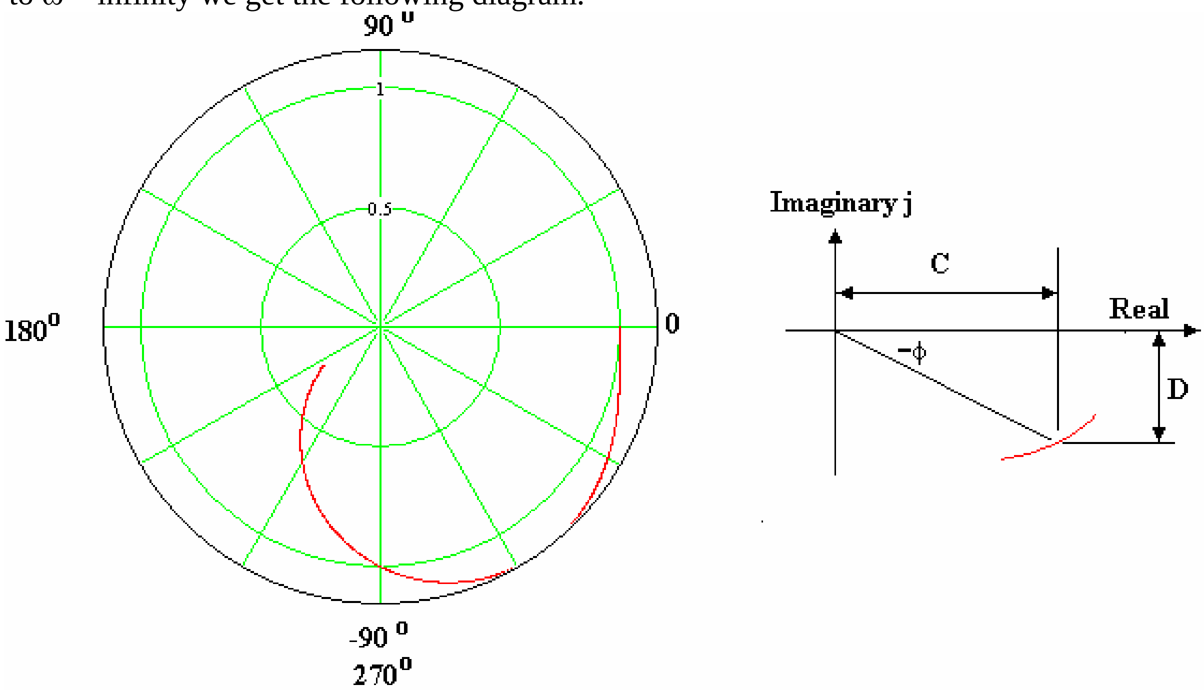

Figure 8

The phase angle is found from $\phi = \tan^{-1} D/C$. The length of the vector is found from $G(s) = \sqrt{\{C^2 + D^2\}}$. This is the ratio of the amplitude of the output to the input. The overall result shows that at ω = 0 the amplitudes are the same and in phase but as ω increases the amplitude of the output grows (depending on the value of δ) and then shrinks to zero as the phase angle increases to 180^o.

WORKED EXAMPLE No.2

A hydraulic cylinder is controlled by the transfer function $\theta_o/\theta_i = 1/(T^2s^2 + 2\delta Ts + 1)$
The time constant T is 0.02 Seconds and the damping ratio $\delta = 0.5$. The input is varied harmonically as $x_i = 20\sin(\omega t)$ at 15 rad/s, calculate the phase shift and amplitude of the output.

SOLUTION

$\omega = 15 \quad \delta = 0.5 \quad t = 0.02$

$$A = \left(1 - T^2\omega^2\right) = 1 - 0.02^2 \times 15^2 = 0.91 \qquad B = 2T\delta\omega = j\,2 \times 0.02 \times 0.5 \times 15 = 0.3$$

$$C = \frac{A}{A^2 + B^2} = \frac{0.91}{0.91^2 + 0.3^2} = 0.991 \qquad D = \frac{B}{A^2 + B^2} = \frac{0.3}{0.91^2 + 0.3^2} = 0.327$$

$\phi = \tan^{-1} D/C = \tan^{-1} 0.327/0.991 = -18.25^o$

Mod $\theta_o/\theta_i = \sqrt{\{C^2 + D^2\}} = \sqrt{\{0.991^2 + 0.327^2\}} = 1.043$
It follows that the output amplitude is 1.043 x 20 = 20.87 and the phase angle is -18.25^o
$\theta_o(t) = 20.87 \sin(\omega t - 18.25^o)$

SELF ASSESSMENT EXERCISE No.1

1. An electrical circuit has a resistor and capacitor has shown. Show that the transfer function is;

   $(V_o/V_i)(s) = 1/(Ts+1)$ where T = RC

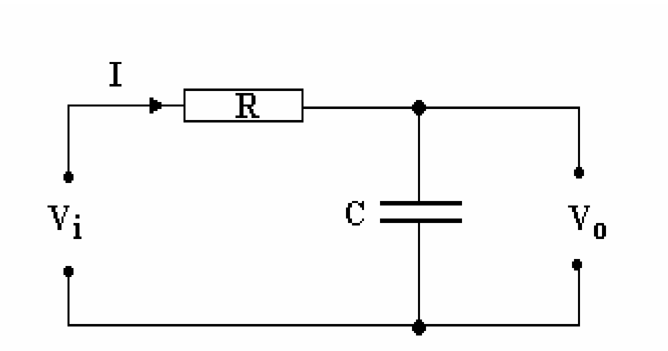

Figure 9

Given that R = 47 Ω and C = 20μF determine the output voltage when the input is sinusoidal such that $v_i = 5 \sin(2000 t)$.

Answer $2.35\sin(2000t - 62^o)$

2. A standard second order system has the transfer function $x_o/x_i = 1/(T^2s^2 + 2\delta Ts + 1)$

   The time constant T is 0.4 Seconds and the damping ratio $\delta = 0.2$. The input is varied harmonically as $\theta_i = 6\sin(\omega t)$ at 2.5 rad/s. Calculate the phase shift and amplitude of the output.

   (Answer 90^o and 15)