INSTRUMENTATION AND CONTROL

TUTORIAL 5 – SYSTEM RESPONSE

This tutorial is a continuation of tutorial 4 and is of interest to any student studying control systems and in particular the EC module D227 – Control System Engineering.

On completion of this tutorial, you should be able to do the following.

- Explain the standard second order transfer function in detail
- Examine the various ways of writing the standard second order transfer function.
- Define the roots and poles of the standard second order transfer function.
- Explain and calculate the response of a standard 2^{nd} order system to a step change in the input.
- Explain and define design parameters of the second order step response.
- Explain and calculate the response of a standard 2^{nd} order system to a ramp (velocity) change in the input.
- Explain the response of a standard 2^{nd} order system to a sinusoidal change in the input.
- Explain how to find the overall gain of a system.
- Explain how to find the steady state error of systems.
- Explain the relationship between pole positions and dynamic response.
- Explain the relationship of the complex pole position with the damping ration and natural frequency.
- Explain the root locus diagram for the complex poles with respect to damping ratio.

If you are not familiar with instrumentation used in control engineering, you should complete the tutorials on Instrumentation Systems.

In order to complete the theoretical part of this tutorial, you must be familiar with basic mechanical and electrical science.

You must also be familiar with the use of transfer functions and the Laplace Transform (see maths tutorials).

1. STANDARD 2ND ORDER SYSTEM

1.1 FORMS OF THE STANDARD TRANSFER FUNCTION

The standard second order transfer function can be expressed in many ways. Here are the most useful.

In terms of the time constant T $\quad \frac{\theta_o}{\theta i}(s) = \frac{k}{T^2 s^2 + 2T\delta s + 1}$

In terms of the natural frequency ω_n $\quad \frac{\theta_o}{\theta i}(s) = \frac{\omega_n^2 k}{s^2 + 2\,\delta\,\omega_n s + \omega_n^2}$

In terms of the poles (polynomial) $\quad \frac{\theta_o}{\theta i}(s) = \frac{\omega_n^2 k}{(s - p_1)(s - p_2)}$

In terms of two parameters σ and ω_r $\quad \frac{\theta_o}{\theta i}(s) = \frac{\omega_n^2 k}{(s+\sigma)^2 + \omega_r^2} = \frac{\omega_r^2 + \sigma\omega_r^2}{(s+\sigma)^2 + \omega_r^2}$

Remember that δ is the damping ratio (ξ is often used in the exam and other texts)

If you are happy with this you should skip the following proofs.

TIME CONSTANT FORM

T is the second order time constant of the system resulting from various parameters of a given system.

$\frac{\theta_o}{\theta i}(s) = \frac{k}{T^2 s^2 + 2T\delta s + 1}$ is the form that usually results from the analysis of a real system and is our starting point.

NATURAL FREQUENCY FORM

The natural frequency of the system is by definition $\omega_n = 1/T$. We simply substitute $T = 1/\omega_n$ and rearrange to get $\frac{\theta_o}{\theta i}(s) = \frac{\omega_n^2 k}{s^2 + 2\,\delta\,\omega_n s + \omega_n^2}$

POLYNOMIAL FORM

The denominator of the transfer function is a quadratic equation so it should factorise into two brackets to give $\frac{\theta_o}{\theta i}(s) = \frac{\omega_n^2 k}{(s+x)(s+y)} = \frac{\omega_n^2 k}{s^2 + s(x+y) + xy}$

We may compare $s^2 + s(x + y) + xy$ with $(s^2 + 2\delta\omega_n s + \omega_n^2)$ and it is apparent that:-

$x + y = 2\delta\,\omega_n$ and $xy = \omega_n^2$

$x = \omega_n^2/y \qquad \omega_n^2/y + y = 2\delta\,\omega_n \qquad \omega_n^2 + y^2 = y\,2\delta\,\omega_n \qquad y^2 - y\,2\delta\,\omega_n + \omega_n^2 = 0$

or

$y = \omega_n^2/x \qquad x + \omega_n^2/x = 2\delta\,\omega_n \qquad x^2 + \omega_n^2 + x^2 = x\,2\delta\,\omega_n \qquad x^2 - x\,2\delta\,\omega_n + \omega_n^2 = 0$

Since the solution of the quadratic is the same for both, it follows that x and y are the same and must be the roots of the denominator.

Solve the quadratic equation $\quad y = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a} \quad a = 1 \quad b = -2\delta\omega_n \quad c = \omega_n^2$ hence:

$$y = x = \frac{2\,\delta\,\omega_n \mp \sqrt{(2\,\delta\,\omega_n)^2 - 4\omega_n^2}}{2} = \delta\,\omega_n \mp \sqrt{(\,\delta\,\omega_n)^2 - \omega_n^2}$$

$$y = x = \delta\omega_n \mp \omega_n\sqrt{\delta^2 - 1} = \delta\omega_n \mp j\omega_n\sqrt{1 - \delta^2}$$

$$\frac{\theta_o}{\theta i}(s) = \frac{\omega_n^2 k}{\left\{s + \delta\omega_n + j\omega_n\sqrt{1-\delta^2}\right\}\left\{s + \delta\omega_n - j\omega_n\sqrt{1-\delta^2}\right\}}$$

σ AND ω_r FORM

This is simply the substitution $\sigma = \delta\omega_n$ and $\omega_r = \omega_n\sqrt{1-\delta^2}$ in the last equation.

ω_r is the damped resonant frequency at which the output value peaks. σ (sigma) has no name.

The transfer function is $\frac{\theta_o}{\theta i}(s) = \frac{\omega_n^2 k}{\{s+\sigma+j\omega_r\}\{s+\sigma-j\omega_r\}}$ and this further simplifies to

$\frac{\theta_o}{\theta i}(s) = \frac{\omega_n^2 k}{(s+\sigma)^2+\omega_r^2}$ (Try multiplying out to see it is the same)

This form is often manipulated further to make it a recognisable form for Inverse Laplace transforms. Using the definitions $\sigma = \delta\omega_n$ and $\omega_r = \omega_n\sqrt{(1-\delta^2)}$

We can arrive at $\omega_r^2 = \omega_n^2(1-\delta^2)$ $\quad \omega_r^2 = \omega_n^2 - \delta^2\omega_n^2$ $\quad \omega_r^2 = \omega_n^2 - \sigma^2$ $\quad \omega_n^2 = \omega_r^2 + \sigma^2\omega_r^2$

$$\frac{\theta_o}{\theta_i}(s) = \frac{\omega_r^2+\sigma\omega_r^2}{(s+\sigma)^2+\omega_r^2}$$

POLES

Poles are covered more fully in a later tutorial. They are the values of s that make the denominator zero. The denominator is a quadratic equation in s. Note that in general, the solution of s that satisfies the equation $(s^2 + 2\delta\omega_n s + \omega_n^2) = k$ are the roots of the equation. The poles are the roots when k = 0. To find the poles we must solve $(s^2 + 2\delta\omega_n s + \omega_n^2) = 0$. Using the quadratic equation the solution is:

$s = \frac{-b \mp \sqrt{b^2-4ac}}{2a}$ $\quad a = 1 \quad b = 2\delta\omega_n \quad c = \omega_n^2$ hence:

$$s = \frac{-2\,\delta\,\omega_n \mp \sqrt{(2\,\delta\,\omega_n)^2 - 4\omega_n^2}}{2} = -\delta\,\omega_n \mp \sqrt{(\delta\,\omega_n)^2 - \omega_n^2}$$

$$s = -\delta\omega_n \mp \omega_n\sqrt{\delta^2-1} \qquad s = -\delta\omega_n \mp j\omega_n\sqrt{1-\delta^2}$$

This is usually written as $\qquad s = -\sigma \mp j\omega_r$

It is apparent that the poles are the negative values of x and y earlier.

j is the complex operator ($\sqrt{-1}$) and it follows that these roots may be a complex number which could be plotted on an Argand Diagram.

If we denote the poles as p_1 and p_2 it follows that $p_1 = -\sigma + j\omega_r$ and $p_2 = -\sigma - j\omega_r$

The transfer function can now be expressed in polynomial form as $\frac{\theta_o}{\theta i}(s) = \frac{\omega_n^2 k}{(s-p_1)(s-p_2)}$

We have shown that the standard second order transfer function may be expressed in many ways.

$$\frac{\theta_o}{\theta_i}(s) = \frac{k}{T^2s^2+2T\delta s+1}$$

$$\frac{\theta_o}{\theta_i}(s) = \frac{\omega_n^2 k}{s^2+2\,\delta\,\omega_n s+\omega_n^2}$$

$$\frac{\theta_o}{\theta_i}(s) = \frac{\omega_n^2 k}{(s+x)(s+y)}$$

$$\frac{\theta_o}{\theta_i}(s) = \frac{\omega_n^2 k}{(s-p_1)(s-p_2)}$$

$$\frac{\theta_o}{\theta_i}(s) = \frac{\omega_n^2 k}{(s+\sigma)^2+\omega_r^2}$$

Any of these may be used to solve the dynamic response.

1.2 RESPONSE TO A STEP INPUT

The standard 2nd order transfer function represents many real systems. The derivation of this equation for some real systems is covered in tutorial 1. Finding the output time response for this type of system is much more difficult than anything covered so far. The simplest case is when $\delta = 0$. Let's use a worked example to examine this case.

WORKED EXAMPLE No.1

Find the time response of a standard second order system to a step input when $\delta = 0$ (undamped).

SOLUTION

In this example δ is made zero in order to simplify the solution but it should be remembered that this is a special case. Starting with the standard transfer function we have

$$\frac{\theta_o}{\theta_i} = \frac{1}{T^2 s^2 + 2\delta T s + 1}$$

Put $\delta = 0$ and make θ_o the subject. $\theta_o = \dfrac{\theta_i}{T^2 s^2 + 1}$

For a step input $\theta_i = H$ and $\theta_i(s) = H/s$ $\theta_o(s) = \dfrac{H/s}{T^2 s^2 + 1} = \dfrac{H}{s(T^2 s^2 + 1)}$

Rearrange into a form recognised in the table. $\theta_o(s) = \dfrac{H/T^2}{s(s^2 + 1/T^2)}$

If we substitute $1/T = \omega$ the following I recognised in the table.

$$\theta_o(s) = \frac{H\omega^2}{s(s^2 + \omega^2)}$$

From the table of transforms we find that the inverse transform which converts the output into a function of time gives the result

$$\theta_o(t) = H(1 - \cos \omega t)$$

This means that the output is a negative cosine curve as shown. The amplitude is H and the frequency of oscillation is $\omega = 1/T$ rad/s.

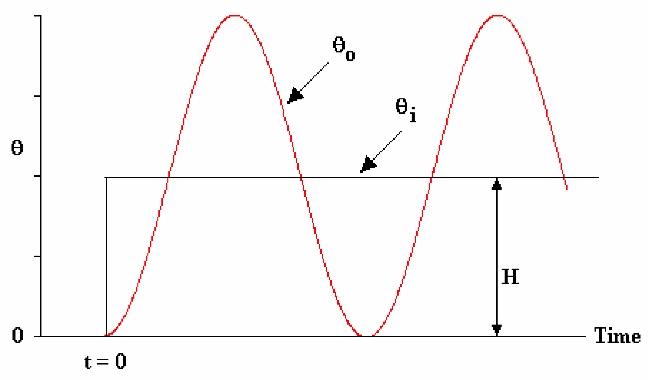


Figure 1

Remember that this is a special case and the 2nd order equation will be examined in greater detail later. If the system was a mass on a spring, the mass would oscillate up and down. If the system was an electrical L – C circuit, the current would oscillate back and forth.

Now consider the time response of a standard second order system with damping to a unit step input. We will use the standard form $\dfrac{\theta_o}{\theta_i}(s) = \dfrac{\omega_n^2}{(s+\sigma)^2 + \omega_r^2}$

For a unit step $\theta_i(s) = 1/s$ hence $\theta_o(s) = \dfrac{1}{s}\dfrac{\omega_r^2 + \sigma\omega_r^2}{(s+\sigma)^2 + \omega_r^2}$

From the table of Laplace transforms, the inverse is $\theta_o(t) = 1 - e^{-\sigma t}(\cos\omega_r t + \dfrac{\sigma}{\omega_r}\sin\omega_r t)$

This may be plotted for different values of δ to produce a normalised graph that may be used for solving specific problems. The settling level is 1.0. ω_n is the natural angular frequency of the system. The natural frequency is $f_n = \omega_n/2\pi$. This is the frequency at which the system would oscillate if there was no damping.

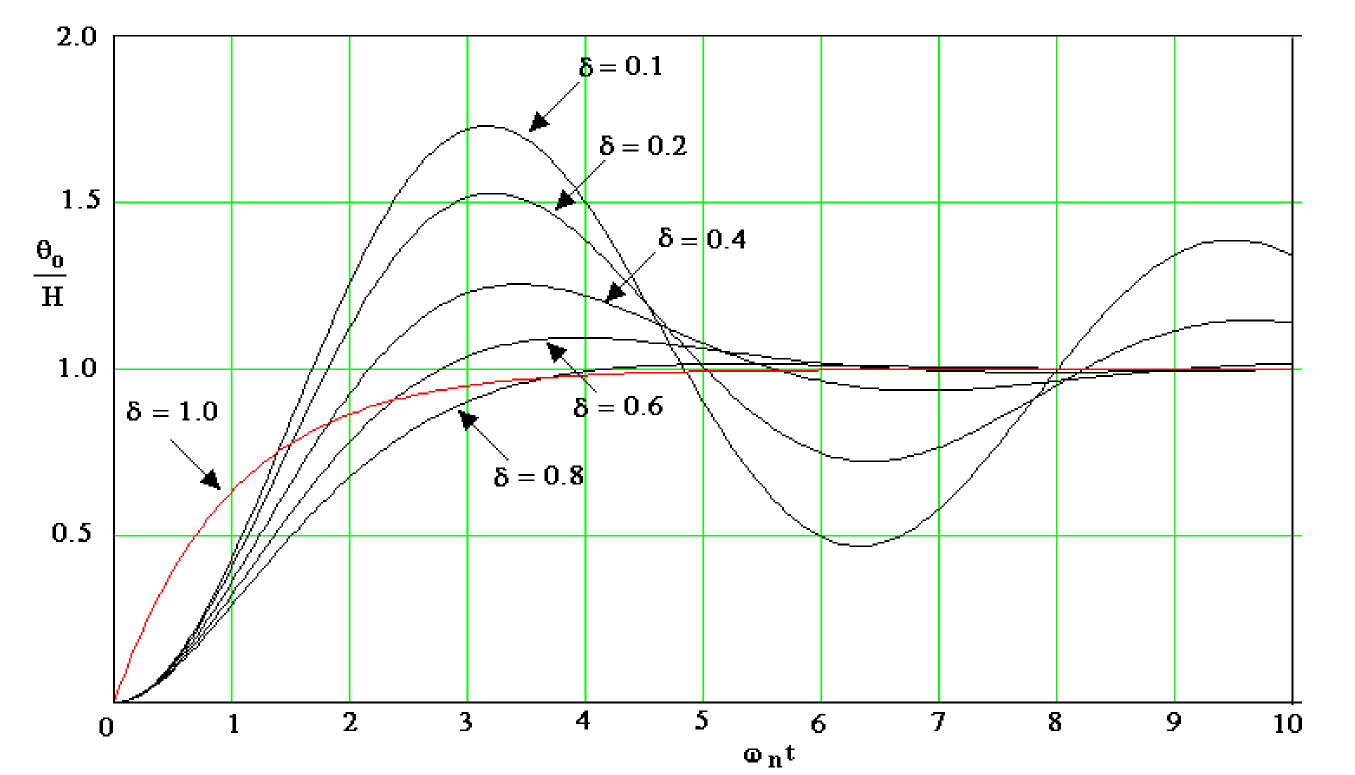


Figure 2

WORKED EXAMPLE No.2

A damped system with a natural frequency of 4 rad/s has a damping ratio $\delta = 0.4$. Find the time taken from the start of a step change to when the output overshoots by 20% for the first time.

SOLUTION

We are looking for a value of θ_o/H = 120% or 1.2 on the vertical scale and where this intersects with the graph for $\delta = 0.4$.

This gives us a value of $\omega_n t = 2.8$ on the horizontal scale. The time taken is hence $t = 2.8/4 = 0.7$ s.
Note that unfortunately the normalised response graph is never given in the EC Exam and other methods are needed to solve problems like this.

A fuller analysis of the step response is given in a later tutorial. The following is useful to know at this stage. It was shown earlier through use of the polynomial form that the step response is :

$$\theta_o(t) = 1 - e^{-\sigma t}\left(\cos\omega_r t + \frac{\sigma}{\omega_r}\sin\omega_r t\right) \text{ where } \sigma = \delta\omega_n \text{ and } \omega_r = \omega_n\sqrt{1-\delta^2}$$

Since it is a unit step the amplitude is $\theta_A(t) = -e^{-\sigma t}\left(\cos\omega_r t + \frac{\sigma}{\omega_r}\sin\omega_r t\right)$. The diagram compares the response and amplitude plot for the unit step.

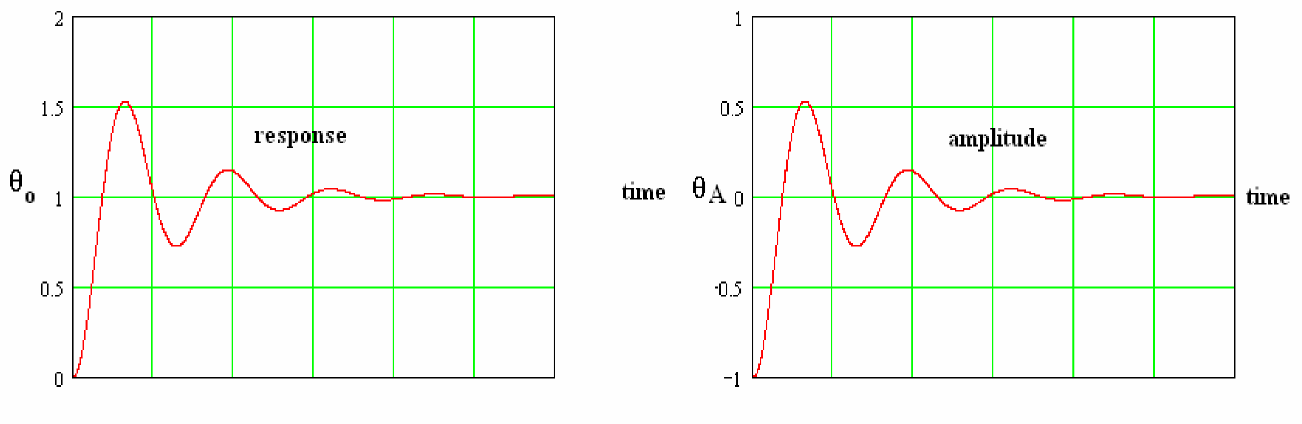


Figure 3

AMPLITUDE REDUCTION FACTOR

Consider two successive oscillations with amplitudes θ_1 and θ_2.

$$\theta_1 = -e^{-\sigma t}\left(\cos\omega_r t + \frac{\sigma}{\omega_r}\sin\omega_r t\right)$$

The second amplitude occurs T_p seconds later and $T_p = 2\pi/\omega_r$ is the periodic time (not to be confused with the time constant)

$$\theta_2 = -e^{-\sigma(t+T_p)}\left\{\cos\omega_r(t+T_p) + \frac{\sigma}{\omega_r}\sin\omega_r(t+T_p\right\}$$

$$\frac{\theta_1}{\theta_2} = \frac{-e^{-\sigma(t)}\left\{\cos\omega_r(t) + \frac{\sigma}{\omega_r}\sin\omega_r(t)\right\}}{-e^{-\sigma(t+T_p)}\left\{\cos\omega_r(t+T_p) + \frac{\sigma}{\omega_r}\sin\omega_r(t+T_p)\right\}}$$

It may be shown that $\dfrac{\left\{\cos\omega_r(t) + \frac{\sigma}{\omega_r}\sin\omega_r(t)\right\}}{\left\{\cos\omega_r(t+T_p) + \frac{\sigma}{\omega_r}\sin\omega_r(t+T_p)\right\}} = 1$

So the amplitude ratio of two successive oscillations is $\dfrac{\theta_1}{\theta_2} = \dfrac{-e^{-\sigma t}}{-e^{-\sigma(t+T_p)}}\;\dfrac{\theta_1}{\theta_2} = e^{-\sigma t}e^{\sigma(t+T_p)} = e^{\sigma T_p}$ Hence

$\ln\left(\dfrac{\theta_1}{\theta_2}\right) = \sigma T_p$ This expression is usually called the logarithmic decrement. Substituting for σ and T_p

$\ln\left(\dfrac{\theta_1}{\theta_2}\right) = \dfrac{2\pi\ \delta}{\sqrt{1-\delta^2}}$ This is also called the amplitude reduction factor.

WORKED EXAMPLE No.3

A recording of a damped oscillation is made and it is determined that the periodic time is 0.2 s and that the ratio of two successive amplitudes is 20 mm and 5 mm. Calculate the natural frequency and the damping ratio.

SOLUTION

$$\ln\left(\frac{\theta_1}{\theta_2}\right) = \frac{2\pi\ \delta}{\sqrt{1-\delta^2}} = \ln\left(\frac{20}{5}\right) = 1.386$$

$$\frac{(2\,\pi\ \delta)^2}{1-\delta^2} = 1.386^2 = 1.9218$$

$$(2\,\pi\ \delta)^2 = 1.9218\left(1-\delta^2\right) = 1.9218 - 1.9218\,\delta^2$$

$$39.478\,\delta^2 = 1.9218 - 1.9218\,\delta^2$$

$41.4\delta^2 = 1.9218$

$\delta = 0.215$

$f_r = 1/T_p = 1/0.2 = 5$ Hz $\qquad \omega_r = 2\pi f_r = 10\pi$ rad/s

$$\omega_r = \omega_n \sqrt{1-\delta^2}$$

$$\omega_n = \frac{\omega_r}{\sqrt{1-\delta^2}} = \frac{10\pi}{0.9765} = 32.17 \text{ rad/s}$$

1.3 RAMP INPUT

The standard transfer function is $\frac{\theta_o}{\theta_i}(s) = \frac{1}{T^2s^2 + 2\delta Ts + 1}$

The input is a ramp so $\theta_i(s) = c/s^2$ Substitute and $\theta_o(s) = \frac{c}{s^2(T^2s^2 + 2\delta Ts + 1)}$

The inverse transform using the computer software Mathcad™ is:

$$\theta_o(t) = -2\cdot c\cdot\delta\cdot T + t\cdot c - c\cdot T\cdot\exp\left(\frac{-\delta}{T}\cdot t\right)\cdot\frac{\sinh\left(\frac{1}{T}\cdot\sqrt{-1+\delta^2}\cdot t\right)}{\sqrt{-1+\delta^2}}$$

$$+2\cdot c\cdot T\cdot\delta^2\cdot\exp\left(\frac{-\delta}{T}\cdot t\right)\cdot\frac{\sinh\left(\frac{1}{T}\cdot\sqrt{-1+\delta^2}\cdot t\right)}{\sqrt{-1+\delta^2}} + 2\cdot c\cdot T\cdot\delta\cdot\exp\left(\frac{-\delta}{T}\cdot t\right)\cdot\cosh\left(\frac{1}{T}\cdot\sqrt{-1+\delta^2}\cdot t\right)$$

The plot of this function shows that the output is oscillatory and the level of oscillation depends on the damping ratio δ. When δ =1, the result is similar to that of the first order system.

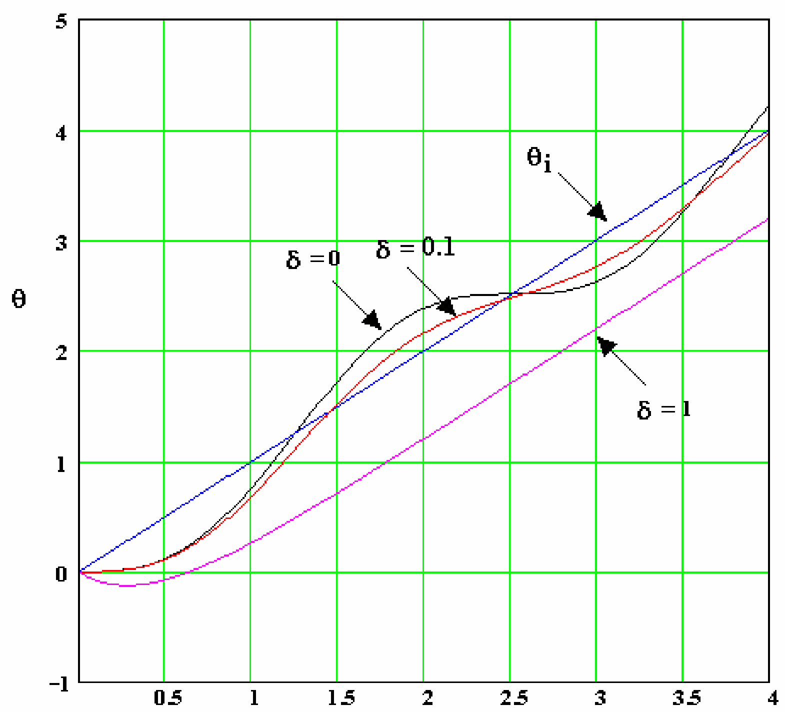


Figure 4

1.4 SINUSOIDAL INPUT

The standard transfer function is $\frac{\theta_o}{\theta_i}(s) = \frac{K}{T^2s^2 + 2\delta Ts + 1}$ K is the gain and δ is the damping ratio.

The input is sinusoidal so $\theta_i(t) = A\sin(\omega t)$ A is the amplitude

The Laplace transform is $\theta_i(s) = \frac{\omega}{\omega^2 + s^2}$

Substitute and $\theta_o(s) = \frac{K}{(T^2s^2 + 2\delta Ts + 1)}\frac{A\omega}{\omega^2 + s^2}$

The inverse transform using the computer software Mathcad™ is a very long complicated expression so it is not reproduced here but plotting the results with $\omega < 1/T$ yields the following.

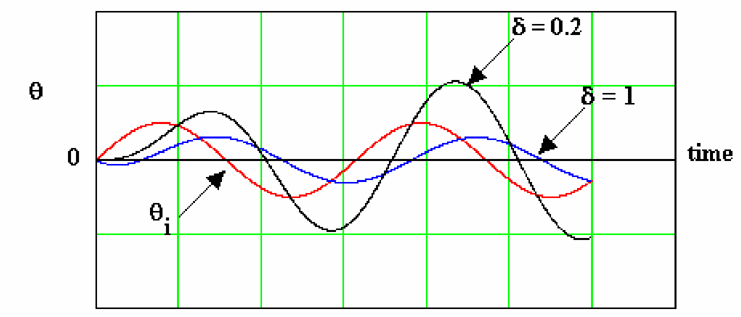


Figure5

During the transient stage, the output grows but settles down to constant amplitude out of phase with the input. If δ = 1 (critical damping) it reaches the steady state condition quicker.

If the plot is repeated with a higher frequency such that $\omega = 1/T$ we reach a condition called resonance and the plot is as shown.

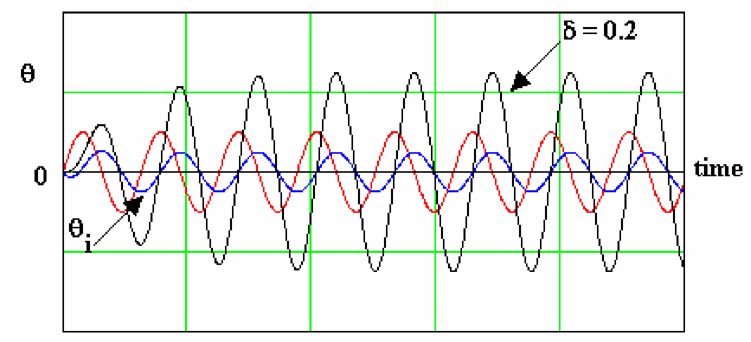


Figure 6

The lightly damped response grows rapidly to a much bigger amplitude than the input and the phase shift is ¼ cycle. If the damping is very small, the output amplitude would be enormous. If we raise the frequency further so that $\omega > 1/T$ the result is very different as shown.

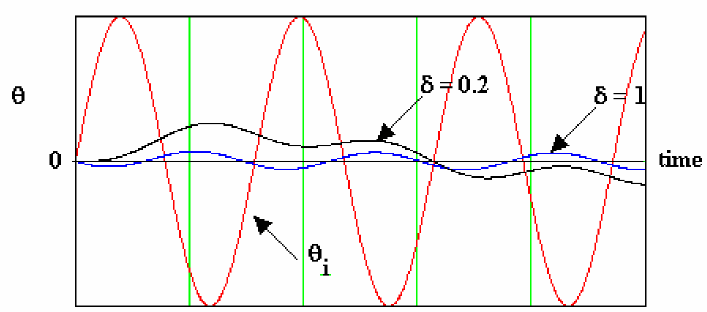


Figure 7

The output is much reduced and the phase shift is almost a half cycle.

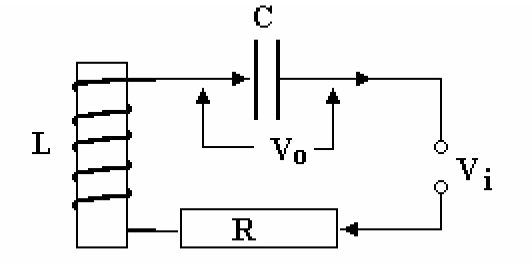
Clearly to understand this fully we need to examine the problem from a different angle and this is covered in the next tutorial.

SELF ASSESSMENT EXERCISE No.1

1. The voltage V_i in the L – C - R circuit shown is suddenly changed from 0 to 10 V. Calculate the natural frequency of oscillation and the critical value of R that will just stop oscillation. You should use the following information from tutorial 1.

The transfer function for the system is $G(s) = \frac{V_o}{V_i}(s) = \frac{1}{T^2 s^2 + 2\delta Ts + 1}$

The damping ratio $\delta = \frac{R}{\sqrt{4\frac{L}{C}}}$ The critical damping resistance is $R_c = \sqrt{4\frac{L}{C}}$ and $T = \sqrt{CL}$



C=20mF L = 5 μH R=0.020 Ω

Figure 8

(Answers $\delta = 0.632$, $Rc = 0.032\Omega$ and $f_n = 503.3$ Hz)

2. Using the answers from Q1, determine the time taken for the voltage to reach 10V for the first time. (ω_n =3163 rad/s produces t = 0.47 ms)

3. A single acting pneumatic cylinder has to push a mass M kg as shown. The cylinder is returned by a spring with stiffness K N/m. There is damping of k_d N s/m. The air pressure is p N/m^2 and the piston area is A m^2. The position of the mass is x metres.

Show that the transfer function for this system is $\frac{x}{p}(s) = \frac{A/M}{\left(s^2 + 2\delta\omega_n s + \omega_n^2\right)}$

ω_n is the natural frequency of the system $(K/M)^{1/2}$

δ is the damping ratio. $\delta = k_d/(4MK)^{1/2}$

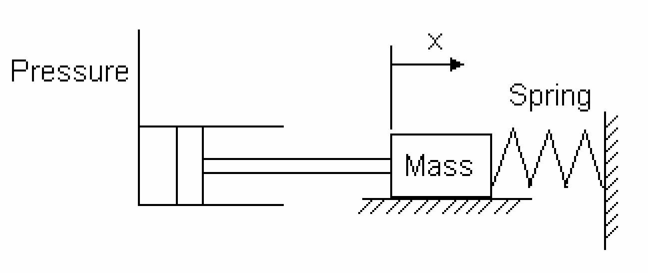


Figure 9

Given M = 50 kg, k_d = 80 N s/m, K = 2000 N/m and A = 0.2 m^2, determine the natural frequency and the damping ratio. (6.325 rad/s or 1.007 Hz and $\delta = 0.126$)

Determine the natural gain and the time constant. (1×10^{-4} and 0.158 s)

A step change is made in p. Using the normalised response graph, find the time when the output reaches the new value for the first time. (0.277s approx)

4. A standard second order system is subjected to a unit step disturbance and a recording is made of the time – output response. It is found that ratio of two successive amplitudes is 3.6 and the periodic time is 0.641 s. Determine the damping ratio and the natural frequency of the system.
   (0.2 and 10 rad/s

1.5 D.C. GAIN OF 2nd ORDER SYSTEMS

As with first order system, the D.C. gain is the magnitude of the transfer function when it contains no dynamic terms so we simply put s = 0.

$$\frac{\theta_o}{\theta_i}(s) = \frac{K}{T^2s^2 + 2\delta Ts + 1}$$

When the last figure on the bottom line is plus 1 the system has negative feed back and K is the D.C. gain of the system.

WORKED EXAMPLE No.4

A system has a transfer function $G(s) = 1/(4s^2 + 8s + 3)$. Find the D.C.gain.

SOLUTION

$G(s) = G(s) = 1/(4s^2 + 8s + 3)$ put s = 0 and the gain is 1/3 = 0.333 which in fact is attenuation.

SELF ASSESSMENT EXERCISE No.2

1. Find the d.c. gain of the following closed loop transfer functions.

$$G(s) = \frac{1000}{2s^2 + 50s + 200}$$ (5)

$$G(s) = \frac{4}{0.05s^2 + 0.2s + 0.4}$$ (10)

$$G(s) = \frac{500(s+2)}{s^2 + 10s + 1}$$ (1000)

1.6 STEADY STATE ERROR

The steady state error produced by a system may be found by applying the **FINAL VALUE THEOREM** (which is not explained here). This states that the final value of a time function G(t) as t→0 is given by the value of s G(s) as s→ 0.

WORKED EXAMPLE No.5

Find the final value of $G(s) = \frac{1}{s+4}$

SOLUTION

The final value in the time domain as t→∞ is $G(t) = s \text{ x} \frac{1}{s+4} = 0 \text{ when } s \to 0$

The final value of G(t) is zero

WORKED EXAMPLE No.6

Find the final value of $G(s) = \frac{1}{s(s+4)}$

SOLUTION

The final value in the time domain is $G(t) = s \text{ x} \frac{1}{s(s+4)} = \frac{1}{4} = 0.25 \text{ when } s \to 0$

The final value of G(t) is 0.25

The steady state error of a system may be found by applying the theorem to the transfer function representing the error.

Consider a system with a closed loop transfer function $G(s)_{cl}$. The error is:-

$$\theta_e = \theta_i - \theta_o \qquad \theta_o = G_{cl}\theta_i \qquad \theta_e = \theta_i - G_{cl}\theta_i = \theta_i[1 - G_{cl}]$$

Steady State Error $= \theta_e(t) = s\theta_e(s) = s\,\theta_i(s)[1 - G_{cl}]$ as s→0

Consider a system with an open loop transfer function $G(s)_{ol}$ and unit feedback. The closed loop transfer function is:

$$G(s)_{cl} = \frac{\theta_o}{\theta_i} = \frac{G_{ol}}{1 + G_{ol}}$$

Steady State Error $= \theta_e(t) = s\theta_e(s) = \frac{s\theta_i(s)}{1 + G_{ol}}$ as s→0

WORKED EXAMPLE No.7

Find the steady state error when a unit step input is applied to a system with an open loop transfer function of $G(s) = \frac{1}{(s+4)}$ with unit feedback.

SOLUTION

$\theta_e(t)\,(\text{steady state}) = \frac{s\theta_i(s)}{1+G}$ For a step input, $\theta_i(s) = \frac{1}{s}$ so $\theta_e(t) = \frac{1}{1+G}$

$\theta_e(t) = \frac{1}{1 + \left[\frac{1}{s+4}\right]}$ and as $s \to 0$ this becomes $= \frac{1}{1 + 1/4} = 0.8$

WORKED EXAMPLE No.8

Find the steady state error when a unit step input is applied to a system with a closed loop transfer function of $G(s) = \dfrac{3}{(s^2 + 2s + 1)}$

SOLUTION

$\theta_e(t)\,(\text{steady state}) = s\,\theta_i(s)\left[1 - G_{cl}\right]$ For a step input, $\theta_i(s) = \dfrac{1}{s}$ so $\theta_e(t) = 1 - \dfrac{3}{(s^2 + 2s + 1)}$ as $s \to 0$

This becomes $= 1 - \dfrac{3}{0 + 0 + 1} = -2$

This is logical since with a gain of the steady state output is 3 and the error is -2.

WORKED EXAMPLE No.9

Find the steady state error when a unit ramp input is applied to a system with an open loop transfer function of $G(s) = \dfrac{3}{s(s+1)(s+2)}$

SOLUTION

$\theta_e(t)\,(\text{steady state}) = s\,\theta_i(s)\left[\dfrac{1}{1 + G_{ol}}\right]$ For a unit ramp input, $\theta_i(s) = \dfrac{1}{s^2}$ so $\theta_e(t) = s \text{ x } \dfrac{1}{s^2}\left[\dfrac{1}{1 + G_{ol}}\right]$

$$\theta_e(t) = \frac{1}{s}\left[\frac{1}{1 + \frac{3}{s(s+1)(s+2)}}\right] = \left[\frac{1}{s + \frac{3}{(s+1)(s+2)}}\right] \text{ as } s \to 0 \quad \theta_e(t) = \frac{1}{0+2} = 0.5$$

WORKED EXAMPLE No.10

An open loop system with init feed back has an open loop transfer function of $G(s) = \dfrac{2}{s(s+1)(s+2)}$

Find the steady state error when $\theta_i(t) = 1 + 0.2t$

SOLUTION

$\theta_e(t)\,(\text{steady state}) = s\,\theta_i(s)\left[\dfrac{1}{1 + G_{ol}}\right]$ $\qquad \theta_i(s) = \dfrac{1}{s} + \dfrac{0.2}{s^2}$

$$\theta_e(t) = s\left(\frac{1}{s} + \frac{0.2}{s^2}\right)\left[\frac{1}{1 + G_{ol}}\right] = \left(1 + \frac{0.2}{s}\right)\left[\frac{1}{1 + G_{ol}}\right]$$

$$\theta_e(t) = \left(1 + \frac{0.2}{s}\right)\left[\frac{1}{1 + \frac{2}{s(s+1)(s+2)}}\right] = \left[\frac{s + 0.2}{s + \frac{2}{(s+1)(s+2)}}\right] = \left[\frac{s + 0.2}{s + \frac{2}{(s+1)(s+2)}}\right] \text{ as } s \to 0 \text{ we may write}$$

$$\theta_e(t) = \left[\frac{0.2}{1}\right] = 0.2$$

SELF ASSESSMENT EXERCISE No.3

1. Find the steady state error when a unit step is applied to a system with the following closed loop transfer functions.

$$G(s) = \frac{4}{0.05s^2 + 0.2s + 0.4} \qquad (-9)$$

$$G(s) = \frac{500}{s^2 + 10s + 100} \qquad (-4)$$

2. Find the steady state error when a unit step is applied to a system with the following open loop transfer functions with unit feedback.

$$G(s) = \frac{1}{s(s+1)(s+4)} \qquad (0)$$

$$G(s) = \frac{5}{(s+1)(s+4)} \qquad (4/9)$$

3. Find the steady state error when a unit ramp is applied to a system with the following open loop transfer functions with unit feedback.

$$G(s) = \frac{5}{(s+1)(s+4)} \qquad (\infty)$$

$$G(s) = \frac{1}{s(s+1)(s+4)} \qquad (4)$$

4. Find the steady state error when a system with the following open loop transfer function and unit feedback has an input $\theta_i(t) = 1 + 0.5t$

$$G(s) = \frac{5}{s(s+1)(s+3)} \qquad (0.3)$$

2. A DETAILED LOOK AT THE POLES OF THE SECOND ORDER SYSTEM

This section introduces you to the concept of root locus plots that are covered more fully in later tutorials. This part only applies to a standard second order system.

Earlier in this tutorial we examined the form of the transfer function $\frac{\theta_o}{\theta_i}(s) = \frac{\omega_n^2 k}{(s - p_1)(s - p_2)}$

p_1 and p_2 are the poles, that is, the value of s that make the denominator zero. We showed that the poles are given by $p_1 = -\sigma + j\omega_r$ and $p_2 = -\sigma - j\omega_r$ and these are complex numbers that may be plotted on an Argand diagram but now we call it the s plane. The coordinates are $-\sigma$ and $j\,\omega_r$

If we plot the poles for a range of values of δ we get the diagram shown. This is the root locus plot with respect to δ and it takes the form of a circle of radius ω_n.

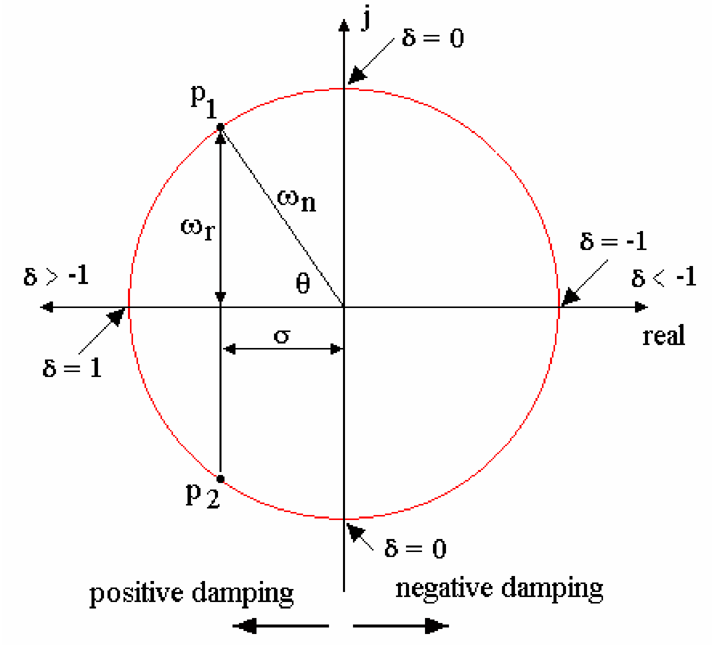


Figure 10

For a value of δ between 1 and 0 we get a typical pair p_1 and p_2.

Consider the angle θ. $\cos\theta = \sigma/\omega_n = \delta\omega_n/\omega_n = \delta$

Consider the radius $\text{radius} = \sqrt{(\sigma^2 + \omega_r^2)} = \sqrt{[(\delta\,\omega_n)^2 + \omega_n^2\,(1-\delta^2)]} = \omega_n$

It follows that the position of the poles depends upon the value of δ and ω_n. Note that when dealing with higher order systems in later tutorials, the complex pole can be represented in terms damping ratio δ and natural frequency ω_n.

The smaller the value of δ, the more oscillatory the response to a step input and the closer the resonant frequency is to the natural frequency. This can be summed up as follows. Consider the time response to a unit step input. The solution was shown earlier as

$$\theta_o(t) = 1 - e^{-at}(\cos\omega_r t + \frac{\sigma}{\omega_r}\sin\omega_r t)$$

Note that positive values of damping are stable and negative values are unstable. The diagram shows the unit step response of the system for various pole positions. This shows that greatest stability is when the poles are on the negative real axis and maximum instability is on the positive real axis. The response is more oscillatory when the poles have a large imaginary (j) term. This is an important point that is used in later tutorials where the choice of pole positions is considered in general.

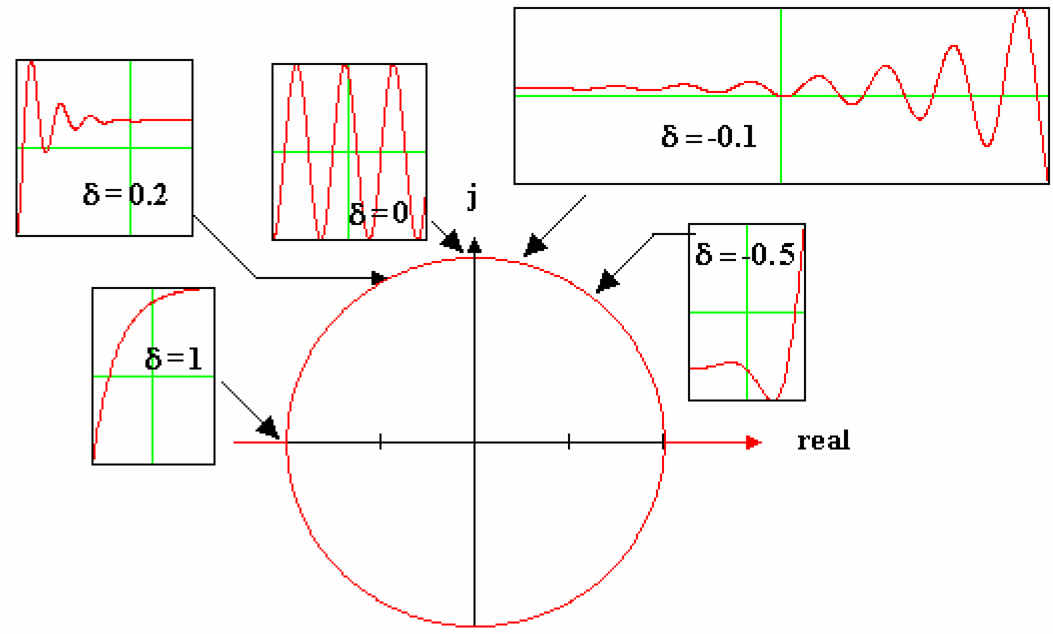


Figure 11

WORKED EXAMPLE No.11

A standard second order system has a natural frequency of 120 rad/s and a damping ratio of 0.4. Determine the position of the poles on the s plane.

SOLUTION

$\theta = \cos^{-1} \delta = \cos^{-1} 0.4 = 66.42^o$

$\sigma = \omega_n \delta = 120 \times 0.4 = 48$

$\omega_r = \omega_n \sin \theta = 110$ rad/s

The poles are at $-48 \pm j110$

SELF ASSESSMENT EXERCISE No.4

A standard second order system has a natural frequency of 50 rad/s and a damping ratio of 0.3. Determine the position of the poles on the s plane.

The poles are at $-47.7 \pm j50$