INSTRUMENTATION AND CONTROL

TUTORIAL 4 – SYSTEM RESPONSE

This tutorial is of interest to any student studying control systems and in particular the EC module D227 – Control System Engineering.

On completion of this tutorial, you should be able to do the following.

- Explain the output response of an ON/OFF control system.
- Explain and define the standard models for 1st and 2nd order systems.
- Explain and define the standard time dependant inputs to a system.
- Define a linear system and a linear time invariant system.
- Explain and calculate the response of a standard 1st order system to a step change in the input.
- Explain and calculate the response of a standard 1st order system to a ramp (velocity) change in the input.
- Explain and calculate the response of a standard 1st order system to a sinusoidal change in the input.
- Explain how to find the overall gain of a system.
- Use partial fractions to solve responses.

If you are not familiar with instrumentation used in control engineering, you should complete the tutorials on Instrumentation Systems.

In order to complete the theoretical part of this tutorial, you must be familiar with basic mechanical and electrical science.

You must also be familiar with the use of transfer functions and the Laplace Transform (see maths tutorials).

You must be able to convert polynomials into partial fractions.

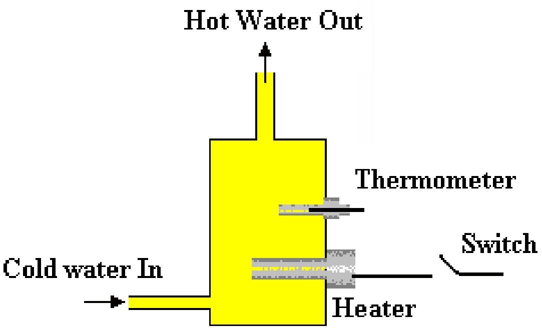
1. INTRODUCTION

A system may be defined as a set of connected things. We are concerned with engineering systems and we may define this more precisely as a set of connected elements designed to produce specified outputs when a given input is applied. System may include analogue elements or digital elements. First we will examine a special kind of simple system that uses ON/OFF control.

2. ON/OFF CONTROL

Consider a system made up of a tank of liquid, a heater and a switch.

When the switch is operated, power is supplied to the heater and the water gets hotter. The operation of the switch is the input action and the temperature of the water is the output action. The output temperature is related to the power by the following formula.



$$\text{POWER} = \frac{\text{Mass x Specific Heat x Temperature Change}}{\text{Time}} = \frac{mc\Delta\theta}{t}$$

Figure 1

We may represent our system as a simple block.

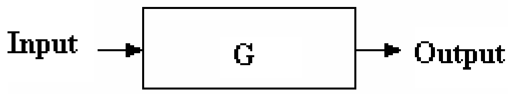


Figure 2

Often it is possible to represent the ratio of output/Input as a simple constant and give this constant the symbol G (it might be thought of as gain but this would not be strictly correct). In this example we have the following.

$$G = \frac{\text{Output}}{\text{Input}} = \frac{\Delta\theta}{P} = \frac{t}{mc}$$

The system described is known as an OPEN LOOP SYSTEM because the signal path does not form a closed loop and there is no control over the temperature. When the switch is closed, the temperature will keep rising with time.

Now consider an improvement to the system by using a thermostatic controller and a thermocouple to measure the output. The desired temperature θ_i (input) is set on the controller and the actual temperature θ_o is the output of the system. The controller compares the measured temperature with the set temperature. If the water is too cold, the heater is turned on. If the temperature is too hot it is switched off.

Suppose the temperature is low and correct and the set temperature is suddenly increased. This is called a step change. The thermostat will turn the heater on and the temperature will rise. Sometime later, the temperature reaches the correct value.

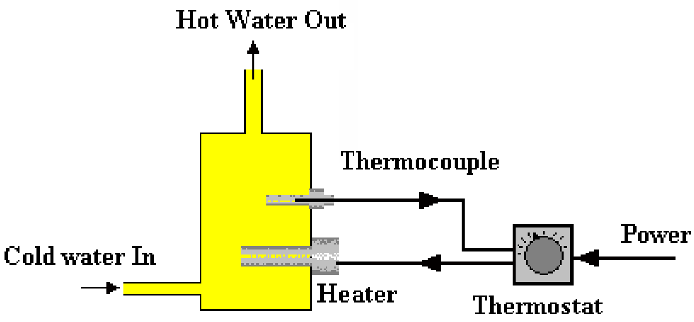


Figure 3

Ideally, the thermostat would switch off the heater at the precise moment that the correct temperature is reached.

In reality this cannot happen as heat will go on being put out after it is switched off and the thermostat has to detect that the temperature is too hot before it switches off. Consequently, the temperature rises above the correct value before being switched off and then the liquid starts to cool. It will cool to just below the correct value before the thermostat detects the error and switches the heater back on. The liquid is heated again and the process continues indefinitely producing a temperature – time graph as shown. This graph is called the SYSTEM TIME RESPONSE GRAPH.

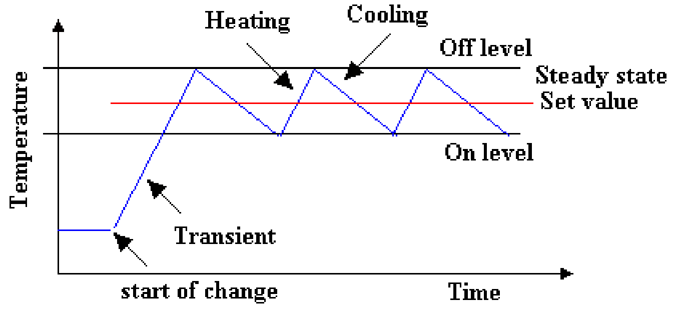


Figure 4

The block diagram of this system shows that the thermocouple turns the signal path into a closed loop so it is a CLOSED LOOP SYSTEM.

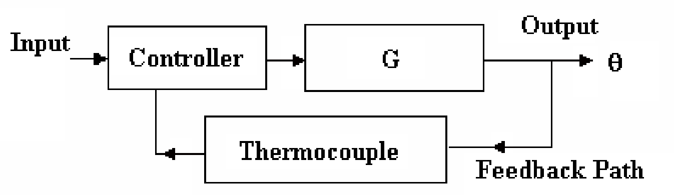


Figure 5

The problem with all ON/OFF control is that the output must cycle between an OFF value and an ON value which are just above and below the SET value. If you bring the ON and OFF levels closer together the system must switch on and off more quickly. It is impossible to get precise control with this method. This is adequate for many systems such as central heating where the exact temperature is not important. Systems that need precision control (such as the movement of a machine tool) require a more sophisticated method of regulation and these are what we must study in detail.

3. RESPONSE OF CONTINUOUS CONTROL SYSTEMS

We are going to discuss how to set about solving the way a given system behaves with respect to time. It should be noted, however, that time is not always the main variable but we will assume it is. We could, for example, be discussing how a robot manipulator moves in response to an input or how a valve on a pipe line moves in response to an input.

3.1 STANDARD MODELS

The first step is to derive a mathematical model along the lines shown in the previous tutorials. In those tutorials it was shown that the following models applied to several different systems that were analogues of each other. These are the standard first order and second order equations.

Standard 1st order equation. $\theta_i = T\, d\theta_o/dt + \theta_o$ ....(1)

Standard 2nd order equation. $\theta_i = T^2\, d^2\theta_o/dt^2 + 2\,\delta T\, d\theta_o/dt + \theta_o$ ....(2)

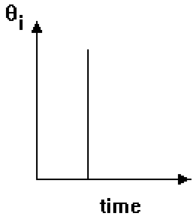
T is a time constant and δ is a damping ratio. These will be discussed in greater detail later. T and δ are examples of system parameters or constants.

Before you can solve how the output changes with time, you have to decide how the input changes with time. For example, you might make a sudden change to the input or you might be changing it over a period of time. Let's look at this next.

3.2 STANDARD INPUTS

Standard inputs are usually listed in the following order.

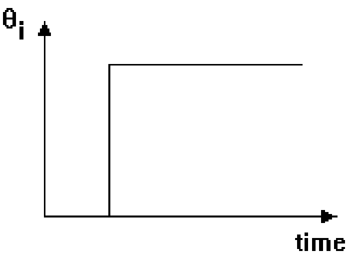
i. AN IMPULSE



This is an instantaneous change in θ_i lasting for zero length of time and returning to the initial value. This is mostly applied to digital systems where instantaneous values are sampled by digital to analogue converters. It is also widely used as a standard input to a system to compare the responses of different systems.

Figure 6

ii. A STEP CHANGE



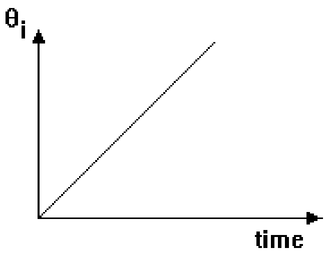
This is an instantaneous change in the input which then remains at the new value.

$\theta_i = H$ at all values of time after t= 0.

H is the change or height of the step.

Figure 7

iii. A RAMP OR VELOCITY CHANGE



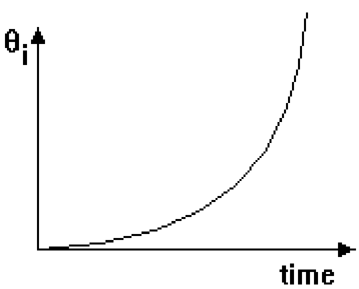
This is when the input changes at a constant rate. It is also called a velocity input.

$$d\theta_i/dt = c \quad \text{or} \quad \theta_i = ct$$

c is the rate of change (velocity).

Figure 8

iv. A PARABOLIC or ACCELERATION CHANGE



This is when $d^2\theta_i/dt^2 = a$ or $\theta_i = at^2/2$

This is also known an acceleration since the rate of rate of change is a constant a.

Figure 9

v. A SINUSOIDAL CHANGE

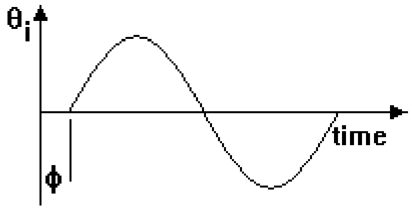


Figure 10

In this case the input changes with time sinusoidally.

$$\theta_i = A \sin(\omega t + \phi)$$

A is the amplitude and ϕ is the phase angle.

vi. AN EXPONENTIAL CHANGE

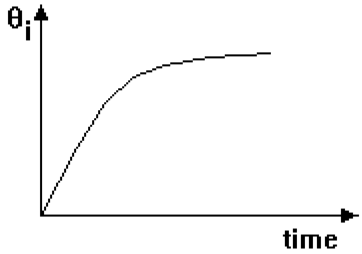


Figure 11

This is when the input changes exponentially with time.

$$\theta_i = A(1 - e^{-at})$$

There are several exponential forms e.g. representing growth and decay.

Once an equation has been formed to represent how the input to a system changes with time, the main task is to deduce how the output change with time so that you can see how accurate and how fast the response will be. It is this aspect which we will discuss in detail and in particular the use of LAPLACE TRANSFORMS to help us do it. Before this, however, let's look at some more definitions and terminology used to describe types of systems.

3.3 LINEAR SYSTEM

A property of a linear system is that of SUPERPOSITION. This means that if one input produces a given output and another input produces a second output, then both inputs together will produce the sum of the two outputs. A good example of this is the deflection of a spring. If a force F_1 is applied the resulting extension is x_1. If a force F_2 is applied the resulting extension is x_2. If a force $F_1 + F_2$ is applied, the resulting extension is $x_1 + x_2$.

Another property of a linear system is that of HOMOGENEITY. This means that if an input produces a given output, then multiplying the input will multiply the output by the same figure. In the case of the spring if a force F produces deflection x then a force of nF produces a deflection nx.

3.4 LINEAR TIME-INVARIANT SYSTEMS

Consider the mathematical model for a differential pressure flow meter. $Q = C (\Delta p)^{½}$. C is a constant for the meter and the power of ½ is a parameter, both of which define the equation. Q and p are variables. If the constants and parameters do not change over the period of time being studied, the system is a linear time-invariant system. T and δ in equations (1) and (2) are examples of constants for such systems.

For the moment we need to study how such equations are solved rather than how they are derived.

3.5. LAPLACE TRANSFORMS

In order to solve θ_o as a function of time we use a mathematical transformation known as LAPLACE TRANSFORMS. This changes the equation from a differential equation into an algebraic equation which can then be manipulated and converted back into a differential equation by inverse transforms. You should study the tutorial on this in the maths section unless you are already familiar with it.

First we replace 'd/dt' with the symbol 's' so for example $d\theta_o/dt$ becomes $s\theta_o$ and $d^2\theta_o/dt^2$ becomes $s^2\theta_o$.

You may go on to appreciate that θ_o on its own becomes $s^o\theta_o$ and that the integral of θ_o becomes $s^{-1}\theta_o$ and so on.

Conducting this change on equations (1) and (2) produces the algebraic equations as follows.

$$\theta_i = T\,s\theta_o + \theta_o$$

$$\theta_i = \theta_o(T\,s + 1)\text{....................}(3)$$

and

$$\theta_i = T^2\,s^2\theta_o + 2\,\delta T\,s\theta_o + \theta_o$$

$$\theta_i = \theta_o(\,T^2\,s^2 + 2\,\delta T\,s + 1)\text{..................}(4)$$

It is now possible to write the equations as a ratio and when this is done, the equation or model is called a TRANSFER FUNCTION. Equations 3 and 4 become

$$\frac{\theta_o}{\theta_i} = \frac{1}{Ts+1} \quad \text{and} \quad \frac{\theta_o}{\theta_i} = \frac{1}{T^2s^2 + 2\delta Ts + 1}$$

The transfer functions are now functions of s f(s) instead of functions of time f(t).

Before solution of the transfer function can proceed, we must change the input θ_i from a function of time f(t) into a function of s f(s). This is where the main transformation occurs. You do not have to transform the function mathematically but instead you will use standard tables. Here are two examples of how it is done (again you will find details in the maths section)

WORKED EXAMPLE No.1

Find the Laplace transform $\mathscr{L}\,H$ when H is a constant (a step change)

SOLUTION

$$\mathscr{L}(H) = \int_0^\infty e^{-st} f(t)\,dt = \int_0^\infty e^{-st} H\,dt = H\left[\frac{\left(-\,e^{-st}\right)}{s}\right]_0^\infty$$

$$f(s) = H\left[\frac{-\,0}{s} - \frac{-\,1}{s}\right] = \frac{H}{s}$$

$$\mathscr{L}(H) = f(s) = \frac{H}{s}$$

For a unit step H = 1 and the Laplace transform is 1/s

WORKED EXAMPLE No.2.

Find the Laplace transform of e^{-at}

SOLUTION

$$\mathscr{L}\left(e^{-at}\right) = \int_0^{\infty} f(t)\,dt = \int_0^{\infty} e^{-st}e^{-at}\,dt = \int_0^{\infty} e^{-(s+a)t}\,dt$$

$$\mathscr{L}\left(e^{-at}\right) = -\left[\frac{\left(e^{-(s+a)t}\right)}{s+a}\right]_0^{\infty}$$

$$\mathscr{L}\left(e^{-at}\right) = -\left[\frac{0}{s+a} - \frac{1}{s+a}\right]$$

$$\mathscr{L}\left(e^{-at}\right) = f(s) = \frac{1}{s+a}$$

TABLE OF COMMON LAPLACE TRANSFORMS

Note that the use of the letters for constants is arbitrary and that often the solution is found by interchanging 'a' with '1/T' and 'ω'. H and k are arbitrary constants.

	Time domain f(t)	Frequency domain f(s)	Description
1	$e^{-at} f(t)\,dt$	$f(s + a)$	
2	δt	1	Unit Impulse
3	H	$\frac{H}{s}$	Step H
4	ct	$\frac{c}{s^2}$	Ramp
5	H(t - T)	$H\frac{e^{-sT}}{s}$	Delayed Step
6		$\frac{1 - e^{-sT}}{s}$	Rectangular pulse
7	$k e^{-at}$	$\frac{k}{s + a}$	Exponential
8	$kt e^{-at}$	$\frac{k}{(s + a)^2}$	
9	$K(e^{-at} - e^{-bt})$	$\frac{k(b - a)}{(s + a)(s + b)}$	
9	k sin (ωt)	$\frac{k\omega}{s^2 + \omega^2}$	Sinusoidal
10	k ω t sin(ωt)	$\frac{2k\omega^2 s}{(s^2 + \omega^2)^2}$	
11	k cos (ωt)	$\frac{ks}{s^2 + \omega^2}$	Co sinusoidal
12	$k e^{-at}$ sin (ωt)	$\frac{k\omega}{(s + a)^2 + \omega^2}$	Damped sinusoidal
13	$k e^{-at}$ cos (ωt)	$\frac{s + a}{(s + a)^2 + \omega^2}$	Damped co sinusoidal
14	$k\left\{1 - e^{-\frac{t}{T}}\right\}$	$\frac{ka}{s(s + a)}$ a = 1/T	Exponential growth
15	$k\left\{t - \left(1 - e^{-\frac{t}{T}}\right)\right\}$	$\frac{ka}{s^2(s + a)}$	
16	k(1-cosωt)	$\frac{k\omega^2}{s(s^2 + \omega^2)}$	
17	k sin (ωt + ϕ)	$\frac{k\{\omega \cos\varphi + s \sin\varphi\}}{s^2 + \omega^2}$	
18	$1 - e^{-at}(\cos bt + \frac{a}{b}\sin bt)$	$\frac{a^2 + b^2}{s\left[(s + a)^2 + b^2\right]}$	Standard second order system step response

4. THE STANDARD 1ST ORDER SYSTEM

4.1 RESPONSE TO AN IMPULSE AND STEP

Any system with an impulse applied to the input with have an input of $\theta_i(s) = 1$. It follows that the output in the time domain will simply be the inverse Laplace transform of the transfer function.

WORKED EXAMPLE No.3

Compare the time response of a system with the transfer function $\frac{\theta_o}{\theta_i} = \frac{1}{Ts+1}$ to a unit impulse and a unit step input.

SOLUTION

UNIT IMPULSE

$\theta_o(s) = \frac{1\theta_i}{Ts+1}$ substitute $\theta_i(s) = 1$ for an impulse input.

$\theta_o(t) = \text{inverse Laplace transform of } \frac{1(1)}{Ts+1}$

rearrange into a recognisable transform and $\frac{1/T}{s+1/T}$ $\theta_o(t) = (1/T)e^{-t/T}$

UNIT STEP

$\theta_o(s) = \frac{1\theta_i}{Ts+1}$ substitute $\theta_i(s) = 1/s$ for an step input.

$\theta_o(t) = \text{inverse Laplace transform of } \frac{1}{s(Ts+1)}$

rearrange into a recognisable transform and $\frac{1/T}{s(s+1/T)}$ $\theta_o(t) = 1 - e^{-t/T}$

The diagram shows the two responses.

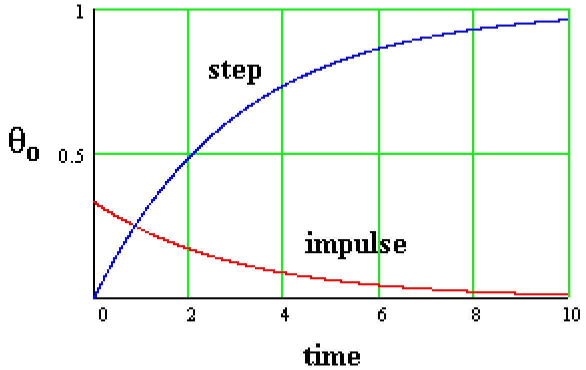


Figure 12

WORKED EXAMPLE No.4

A R- C circuit is shown in which R = 200 Ω and C = 15 μF. The voltage V_i is suddenly changed from 0 to 10 Volts. Determine the time constant and how long it takes V_o to reach 9.99V.

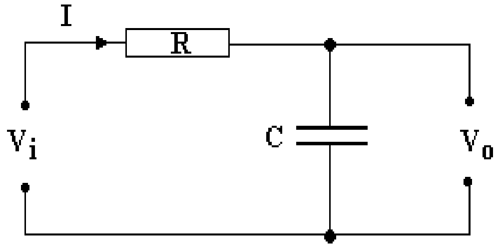


Figure 13

SOLUTION

The model for the R – C circuit shown was derived in tutorial 1 and shown to be $V_i = TdV_o/dt + V_o$

First replace dV_o/dt by s V_o $V_i = T s V_o + V_o$

Next rearrange into a transfer function $G(s) = V_o/V_i = 1/(Ts + 1)$

This may be represented diagrammatically as shown.

Figure 14

Next change the input from a function of time into a function of s by making a Laplace transformation. For a step input $V_i(t) = H$ the transform is $V_i(s) = H/s$. In this case H = 10 Volts.

$$V_o = \frac{V_i}{Ts + 1} = \frac{H}{s(Ts + 1)}$$

Next manipulate the equation into a recognisable Laplace transform as follows.

$$V_o = \frac{H/T}{s(s + 1/T)}$$

Looking in the table of transforms we see Ka/s(s+a) is the transform of $K(1 - e^{-at})$. This means that if we put a = 1/T and K = H the solution for V_o is $\mathbf{V_o = H(1 - e^{-t/T})}$

The time constant T = RC = 200 x 15 x 10^{-6} = 0.003 seconds

Put V_o = 9.99 V and H = 10 V

$$9.99 = 10\left(1 - e^{-\frac{t}{0.003}}\right) \qquad 0.999 = \left(1 - e^{-\frac{t}{0.003}}\right)$$

$$e^{-\frac{t}{0.003}} = 0.001 \qquad \ln(0.001) = \frac{-t}{0.003} = -6.9077 \qquad t = 0.0207\,s$$

4.1.1 FULL ANALYSIS OF THE RESPONSE

For any standard first order system with a step input, the output response is $\theta_o = H(1 - e^{-t/T})$. If we plot both input and output against time we get the result shown.

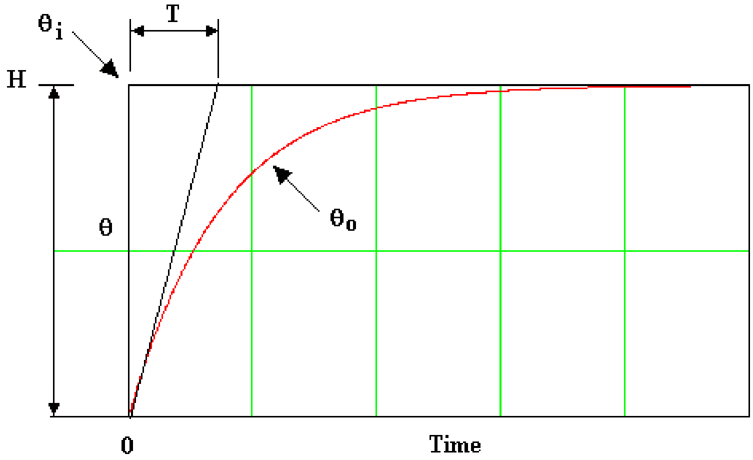


Figure 15

4.1.2 FIRST ORDER TIME CONSTANT

The time constant 'T' has a real meaning. Consider the response graph. The rate of change of θ_o is

$$\frac{d\theta_o}{dt} = H\left(\frac{1}{T}\right)e^{-\frac{t}{T}}$$

At the start of the change t = 0, so the initial rate of change is $\frac{d\theta_o}{dt} = H\left(\frac{1}{T}\right)e^{-0} = \frac{H}{T}$

The initial gradient (rate of change) is H/T. If the initial gradient is projected as shown, it will intercept θ_i at time t = T. This leads to a definition of T as the time it would take the output to match the input if it kept changing at the initial rate.

Another way of defining the time constant T is as follows.

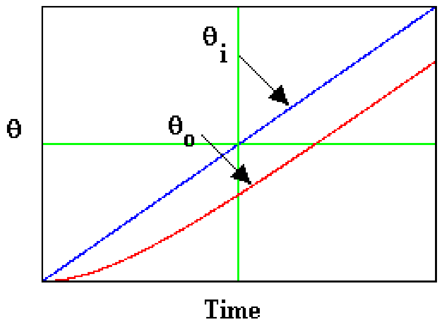
At time t = T the value of the output is $\theta_o = H\{1 - e^{-1}\} = 0.632H$

In other words, the output will reach 63.2% of the desired value after T seconds.

When t = 4T the output is $\theta_o = H(1 - e^{-t/T}) = H(1 - e^{-4}) = 0.999H$. This is as close to the correct value as we are likely to want so we normally assume that the system has reached the correct value after t = 4T seconds.

4.2 RESPONSE OF A STANDARD 1ST ORDER SYSTEM TO A RAMP INPUT

A ramp or velocity input occurs when $\theta_i(t) = ct$ and $\theta_i(s) = c/s^2$



$$\frac{\theta_o}{\theta_i}(s) = \frac{1}{(Ts+1)}$$

$$\theta_o(s) = \frac{\theta_i(s)}{(Ts+1)} = \frac{c}{s^2(Ts+1)} = \frac{c(1/T)}{s^2(s+1/T)}$$

If we replace 'c' with 'k' and '1/T' with 'a' we have

$\theta_o(s) = \dfrac{ca}{s^2(s+a)}$ and we can find this in the table of transforms. The inverse transform gives the solution

$$\theta_o(t) = c\{t - T(1 - e^{-t/T})\}$$

Figure 16

Plotting the output θ_o against time produces the result shown. The equation may be written as:

$$\theta_o(t) = c\,t - cT(1 - e^{-t/T})\}$$

At large values of time t the term ($e^{-t/T}$) becomes negligibly small and the output becomes:

$$\theta_o(t) = c\{t - T(1)\} = c(t - T)$$

The error becomes $\theta_e(t) = \theta_i - \theta_o = ct - ct + cT$ $\qquad \theta_e(t) = cT$

In other words, a constant error cT result after an initial transient stage and this is the steady state error.

WORKED EXAMPLE No.5

A position control system has a transfer function G(s) = 1/(0.2 s + 1). The input is changed at a constant rate of 5 degrees/s from the zero position. Calculate the error after 0.4 seconds and the steady state error.

SOLUTION

Comparing parameters it is apparent that T = 0.2 and c = 5 deg/s.

After 0.4 seconds $\theta_i = 5 \times 0.4 = 2$ degree.

$\theta_o = c\,t - cT(1 - e^{-t/T})\} = 5 \times 0.4 - 5 \times 0.2 \times (1 - e^{-0.4/0.2}) = 2 - 0.835 = 1.135$ degrees.

$\theta_e = 2 - 1.135 = 0.865$ degrees

The steady state error is cT where T = 0.2 and c = 5 degrees/s. $\theta_e = 5 \times 0.2 = 1$ degree.

4.3 RESPONSE OF A STANDARD 1ST ORDER SYSTEM TO A SINUSOIDAL INPUT

The standard first order system transfer function is $\frac{\theta_o}{\theta_i}(s) = \frac{k}{Ts + 1}$

In the steady state s = 0 and $\frac{\theta_o}{\theta_i}(s) = \frac{k}{1} = k$ so it follows that k is the steady state system gain.

Note that it must be +1 on the bottom line to make this so. T is a time constant.

For a sinusoidal input let $\theta_i(t) = A \sin(\omega t)$

The Laplace transform is $\theta_i(s) = \frac{\omega}{\omega^2 + s^2}$

The Output is $\theta_o(s) = \frac{\theta_i}{(Ts + 1)} = \frac{k}{(Ts + 1)} \frac{\omega}{(\omega^2 + s^2)}$

The inverse transform gives the output as a function of time.

$$\theta_o(t) = A \cdot K \cdot \frac{\sin(\omega \cdot t)}{(1 + T^2 \cdot \omega^2)} - A \cdot K \cdot \omega \cdot T \cdot \frac{\cos(\omega \cdot t)}{(1 + T^2 \cdot \omega^2)} + A \cdot K \cdot \omega \cdot T^2 \cdot \frac{\exp\left(\frac{-1}{T} \cdot t\right)}{(T + T^3 \cdot \omega^2)}$$

Plotting the input and output gives a typical result like this with A=1 and K=1).

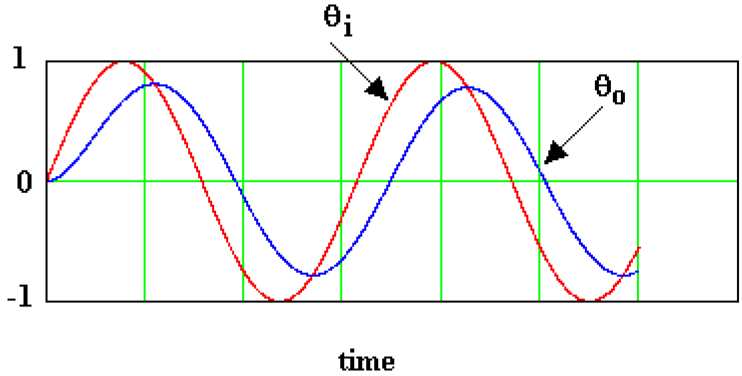


Figure 17

The plot shows that the output quickly settles down into steady state response with a sinusoidal form but with different amplitude and lagging the input with a phase shift. The transient stage only lasts for one cycle.

A more detailed analysis of the steady state response is covered in the next tutorial.

SELF ASSESSMENT EXERCISE No.1

1. Show the derivation of the transfer function for spring and damper system shown. Given that the damping coefficient k_d is 0.03 and the spring stiffness k is 4 kN/m, determine the time constant for the system. (Answer 7.5 μs)
If a force of F = 100 N is suddenly applied, calculate the value of x after T seconds. (Answer 16 mm)

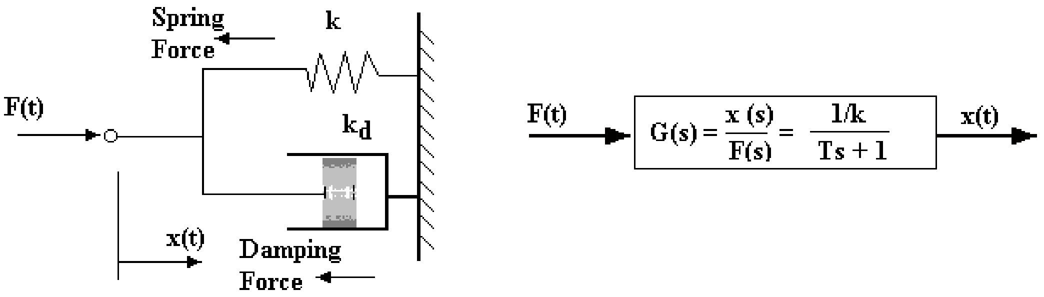


Figure 18

2. A block of metal has a mass of 0.5 kg, specific heat capacity 346 J/kg K and temperature of θ_1 = 20°C. It is dropped into a large tank of oil at θ_2 = 120°C and it is found that the temperature of the block takes 6 minutes to reach 119 °C.
Assume that the temperature of the block is changes by the law $\frac{\theta_1}{\theta_2}(s) = \frac{1}{(Ts + 1)}$
Show that the temperature of the block changes with time by the law $\theta = \theta_1 + (\theta_2 - \theta_1)(1-e^{-t/T})$

Determine the time constant T and hence the thermal resistance between the block and the oil. (You should see the tutorial 1 for details)

(Answer R=0.452 K/W)

3. A hydraulic motor has a nominal displacement of k_1 m^3/radian. The speed ω is controlled by a simple valve such that the pressure to the motor is k_2x where x is the input position of the valve.

The motor has a moment of inertia J kg m^2 and a damping coefficient of k_3 Nm s/radian.

Given that the torque developed by the motor is k_1p, show that the open loop transfer function relating output ω to input x is given by $\frac{\omega}{k_m x} = \frac{1}{Ts + 1}$

$T = \frac{J}{k_3}$ and $k_m = \frac{k_1 k_2}{k_3}$

The input is given a step change. Sketch the response of the output. Determine the % change in the output at t = T and t = 4T. (63.2% and 99.9%)

Show on the sketch the affect of increasing the moment of inertia.

4. A position control system has a transfer function G(s) = 1/(3 s + 1). The input is changed at a constant rate of 4 mm/s from the zero position. Calculate the error after 2 seconds and the steady state error. (2.161 mm and 12 mm)

4.3 GAIN OF 1st ORDER SYSTEMS

DC GAIN :- this is an electrical term and is the gain when there is no variation of the input in the time domain. When an input is constant all the derivative values of the transfer function are zero so the D.C. is the magnitude of G(s) when s = 0.

Hence the gain of the standard 1st order closed loop transfer function $\frac{\theta_o}{\theta_i}(s) = \frac{1}{Ts + 1}$ is unity.

When the transfer function is of the form $\frac{\theta_o}{\theta_i}(s) = \frac{K}{Ts + 1}$, the D.C. gain is K.

The D.C. gain is not always obvious at first glance e derivation. Consider the case $\frac{\theta_o}{\theta_i}(s) = \frac{A}{Ts + B}$

Put s = 0 and the D.C. gain is A/B. We could rearrange the transfer function to $\frac{\theta_o}{\theta_i}(s) = \frac{A/B}{(T/B)s + 1}$

This is now in the form $\frac{\theta_o}{\theta_i}(s) = \frac{K}{T_2 s + 1}$

T_2 is a new time constant resulting from the change. In the steady state s = 0 so K is the steady state gain of the system.

D.C. GAIN = A/B $T_2 = T/B$

The D.C. gain also affects the magnitude of a dynamic response

The response to a step change H will hence settle down at KH. The response to a ramp input will not settle down at a fixed value but $\theta_o(t) = Kc\{t - T(1 - e^{-t/T})\}$. The affect of introducing gain is shown on the response diagram.

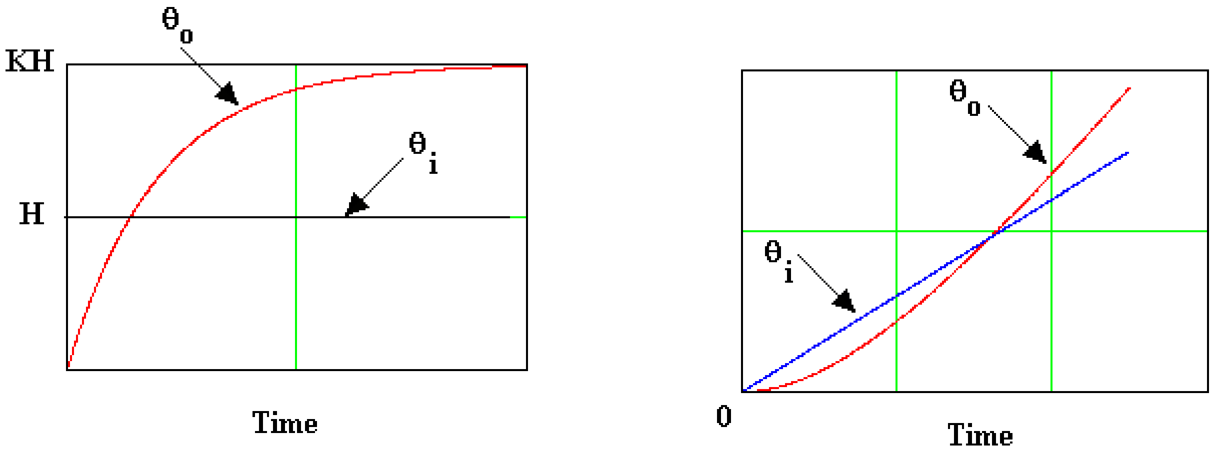


Figure 19

In the case of the step input θ_i = H, the output θ_o reaches a value of K H.

In the case of the ramp input θ_i = ct, the output passes and exceeds the input more and more as time progresses.

WORKED EXAMPLE No.6

The transfer function for a simple control system is $\frac{\theta_o}{\theta_i}(s) = \frac{10}{0.5s + 0.2}$

Solve and tabulate the values of θ_o against time for a step input of 2 units.

SOLUTION

Rearrange the transfer function. $\frac{\theta_o}{\theta_i}(s) = \frac{10/0.2}{(0.5/0.2)s + 1} = \frac{50}{2.5s + 1}$

From this we see the gain is 50 and the time constant is 2.5. For a step input of 2 units the transform gives $\theta_i(s) = 2/s$ Hence $\theta_o(s) = \frac{50(2)}{s(2.5s + 1)}$

The inverse transform is $\theta_o = 100(1 - e^{-t/2.5}) = 100(1 - e^{-0.4t})$

Plotting the result we get the response shown below.

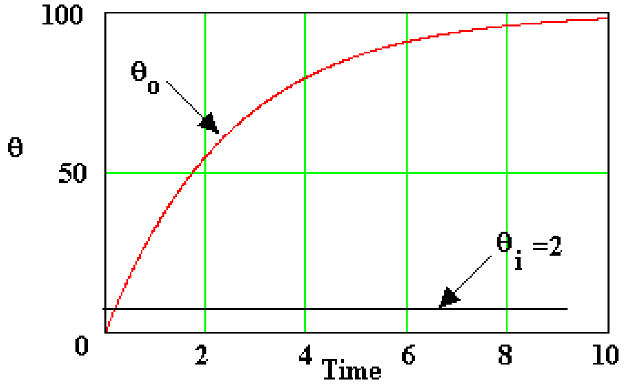


Figure 20

WORKED EXAMPLE No.7

The same system is subjected to a ramp input of 1 mm/s. Plot the output against time and determine the steady state error.

SOLUTION

$\theta_o(s) = \frac{50}{(2.5s + 1)}$ For a ramp input $\theta_i = ct$ and $c = 1$ mm/s. $\theta_i(s) = c/s^2$ hence

$\theta_o(s) = \frac{50c}{s^2(2.5s + 1)}$ The inverse transform of this is $\theta_o(s) = 50c\left\{t - 2.5\left(1 - e^{-\frac{t}{2.5}}\right)\right\}$

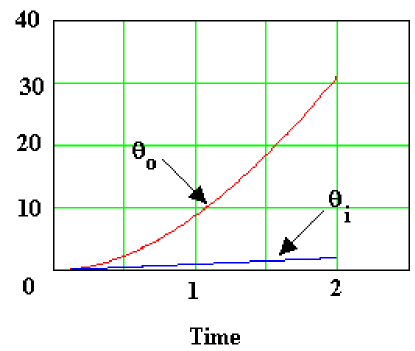


Figure 21

SELF ASSESSMENT EXERCISE No.2

1. Find the D.C. gain and time constant for the following transfer functions.

i. $G(s) = \dfrac{2}{0.2s + 0.5}$ (4 and 0.4)

ii. $G(s) = \dfrac{0.2}{0.05s + 0.1}$ (2 and 0.5)

iii. $G(s) = \dfrac{2}{3s + 1}$ (2 and 3)

iv. $G(s) = \dfrac{16}{8s + 4}$ (4 and 2)

2. The output speed of a motor (ω rad/s) is related to the angle of the input sensor (θ radian) by the transfer function $\dfrac{\omega}{\theta}(s) = \dfrac{k_m}{T_m s + 2}$

Where $k_m = 15\ s^{-1}$ and $T_m = 4$ s

Determine the D.C. gain and time constant of the system. (7.5 s^{-1} and 2 seconds)

5. USING PARTIAL FRACTIONS TO SOLVE RESPONSES

The response to most types of inputs can be solved by the use of partial fractions. Here are two examples using the standard first order transfer function.

First Order System $\frac{\theta_o}{\theta_i} = G(s) = \frac{1}{(Ts+1)} = \frac{1/T}{s+1/T}$

Put 1/T = p and $\frac{\theta_o}{\theta_i}(s) = \frac{p}{s+p}$

5.1 UNIT STEP INPUT

A unit step input in Laplace form is $\theta_i(s) = 1/s$ $\qquad \theta_o(s) = \frac{p}{s(s+p)}$

Change into partial fractions $\qquad \theta_o(s) = \frac{A_0}{s} + \frac{A_1}{s+p}$

Solve A_0 and A_1 in the normal way for partial fractions.

$$\frac{A_0}{s} + \frac{A_1}{s+p} = \frac{p}{s(s+p)}$$

$$\frac{A_0(s+p) + A_1 s}{s(s+p)} = \frac{p}{s(s+p)}$$

The numerators must equate so $A_0(s+p) + A_1 s = p$ and this is true for any value of s

Put s = -p and $A_0(0) - A_1 p = p$ so $\qquad A_1 = -1$

Now put s = 0 $A_0(p) + 0 = p$ so $\qquad A_0 = +1$

Substitute back into the solution $\qquad \theta_o(s) = \frac{1}{s} - \frac{1}{(s+p)}$

Now conduct an inverse Laplace transform. $\qquad \theta_o(t) = 1 - e^{-tp} = \left(1 - e^{-tp}\right)$

Put p = 1/T $\qquad \theta_o(t) = 1 - e^{-t/T}$

And this is the solution found by other methods in previous tutorials.

5.2 UNIT RAMP

$$\frac{\theta_o}{\theta_i}(s) = \frac{p}{s+p}$$

Partial fractions $\qquad \theta_o(s) = \frac{B_0}{s} + \frac{B_1}{s^2} + \frac{B_2}{s+p} = \frac{p}{s^2(s+p)}$

$$\theta_o(s) = \frac{s(s+p)B_0 + (s+p)B_1 + s^2 B_2}{s^2(s+p)} = \frac{p}{s^2(s+p)}$$

$$s(s+p)B_0 + (s+p)B_1 + s^2 B_2 = p$$

Equate the numerator and first let p = -s $\qquad s(0)B_0 + (0)B_1 + p^2 B_2 = p$

$$\text{put } s = -p \text{ and } B_2 = \frac{1}{p}$$

$\text{put } s = 0 \text{ and } B_1 = 1$

Next put s = 1 and substitute for B_1 and B_2 to solve B_0 and find $B_0 = -1/p$

$\theta_o(t) = -\frac{1}{p} + t + \frac{1}{p}e^{-pt}$ Let 1/T = p and the solution is

$$\theta_o(t) = -T + t + Te^{-\frac{t}{T}} = \theta_o(t) = t - T\left(1 - e^{-\frac{t}{T}}\right)$$

This is the solution found by other methods previously.

Let's extend this to a 3rd order system or higher. $\frac{\theta_o}{\theta_i} = G(s) = \frac{K}{(s+p_1)(s+p_2)(s+p_3)}$

UNIT STEP $\theta_i(s) = 1/s$ $\theta_o(s) = \frac{K}{s(s+p_1)(s+p_2)(s+p_3)}$

Change into partial fractions $\theta_o(s) = \frac{A_0}{s} + \frac{A_1}{s+p_1} + \frac{A_2}{s+p_2} + \frac{A_3}{s+p_3}$

Inverse Laplace Transform $\theta_o(s) = A_0 + A_1 e^{-pt} + A_2 e^{-pt} + A_3 e^{-pt}$

A_0, A_1, A_2 and A_3 etc must then be solved.

UNIT RAMP $\theta_i(s) = 1/s^2$ $\theta_o(s) = \frac{K}{s^2(s+p_1)(s+p_2)(s+p_3)}$

Change into partial fractions

$$\theta_o(s) = \frac{B_0}{s} + \frac{B_1}{s^2} + \frac{B_2}{s+p_1} + \frac{B_3}{s+p_2} + \frac{B_4}{s+p_3}$$

Inverse Laplace Transform $\theta_o(t) = B_0 + tB_1 + B_2 e^{-p_1 t} + B_3 e^{-p_2 t} + B_4 e^{-p_3 t}$

B_0, B_1, B_2 and B_3 etc must then be solved.

WORKED EXAMPLE No.8

A system has a transfer function $\frac{\theta_o}{\theta_i}(s) = \frac{1}{(s+1)(s+2)}$ Determine the time response to a unit step input.

SOLUTION

$\theta_o(s) = \frac{\theta_i}{(s+1)(s+2)} = \frac{1}{s(s+1)(s+2)}$ Convert to partial fractions

$\theta_o(s) = \frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2(s+2)}$ Inverse transforms $\theta_o(t) = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}$

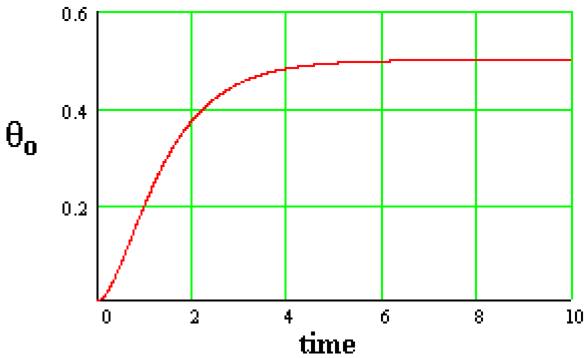


Figure 22

SELF ASSESSMENT EXERCISE No.3

1. A system has a transfer function $\frac{\theta_o}{\theta_i}(s) = \frac{1}{(s+2)(s+3)}$ Determine the output time response to a unit step input. Answer $\theta_o(t) = \frac{1}{6} - \frac{1}{2}e^{-2t} + \frac{1}{3}e^{-3t}$

2. A system has a transfer function $\frac{\theta_o}{\theta_i}(s) = \frac{1}{(s+1)(s+2)}$ Determine the output time response to a unit ramp input. $\theta_o(t) = \frac{t}{2} - \frac{3}{4} + e^{-t} + \frac{1}{4}e^{-2t}$