CONTROL ENGINEERING
TUTORIAL 13

THE STABILITY OF PNEUMATIC and HYDRAULIC VALVES

This tutorial is aimed at engineers seeking knowledge on why valve elements sometimes go unstable and what can be done to prevent it.

Contents

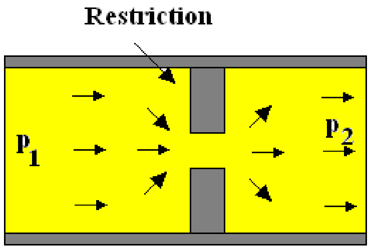
1. *Electrical Analogues*

Part 1 extends the work on analogies covered in Control Tutorial No.1 where you will find more information. With these analogies we may apply electrical theory to pneumatic systems such as a dashpot which is discussed in depth. In the following work on pneumatics, the changes in density of the gas must be very small so the pressure changes must also be small. This makes the theory particularly useful for acoustics and the analysis of oscillations in pneumatic systems.

The analogous quantities throughout will be as follows.
Pressure (p) - Voltage (V)
Mass flow ($\dot{m}$) - Current(I or i)
Mass (m) - Charge (Q)
The analogy deals with three properties, Resistance, Capacitance and Inductance.

1.1. Resistance

A fluid restriction may be approximated to an analogue of an electrical resistor in some analysis. A simple restriction is shown.

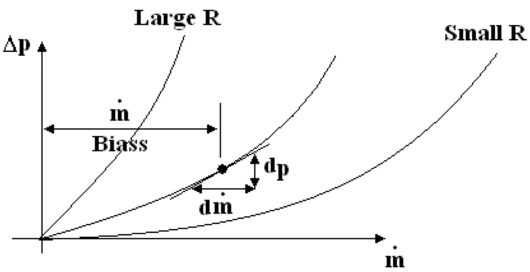


The flow rate and pressure drop are

$$\dot{m} = \frac{dm}{dt} \quad \text{and} \quad \Delta p = p_1 - p_2$$

The relationship between Δp and $\dot{m}$ is not linear so an exact analogy with the electrical resistance is not possible. Ideally we want to have an analogy that fits Ohm's Law.

$\Delta p = R\,\dot{m}$……………...Pneumatic/Hydraulic
$\Delta V = R\,I$ ……………….Electrical



In general the relationship is as shown on the diagram. If the flow changes by a small quantity $d\dot{m}$ due to a small change in pressure dp then the new flow rate is

$$\dot{m} = \text{bias} + \frac{dp}{R}$$

R is the gradient of the graph at the operating point. If the graph was linear then R would be the perfect analogy to electrical resistance. In the analysis of small perturbations in the flow, this analogy is quite useful.

Because the gas constant also has a symbol R we will use the alternative symbol G for pneumatic resistance when required.

1.2. Capacitance

An electrical capacitor obeys the laws

$$\frac{dQ}{dt} = i = C\frac{dV}{dt} \text{ and } C = \frac{\delta Q}{\delta V}$$

The formula for the value of C in terms of the capacitor's parameters may be found in the electrical tutorials on the web site. It follows that a pneumatic or hydraulic equivalent must obey the law

$$\frac{dm}{dt} = \dot{m} = C\frac{dp}{dt}$$

It will be shown below that this analogy is quite accurate and that for both hydraulic and pneumatic systems

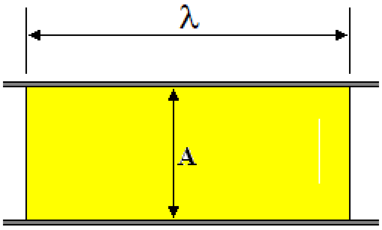
$$C = \frac{\delta m}{\delta p} = \frac{V}{a^2}$$

a is the sonic velocity in the medium (liquid or gas).

If a volume V contains a mass m, adding more mass δm will increase the pressure by δp. It follows that pressure is an indicator of the mass.

1.2.1 Pneumatic Capacitance

Consider a length of pipe containing gas such that the volume V = A ℓ. The mass is m, the temperature is T and the pressure is p. When the pressure rises adiabatically by an amount dp, the temperature changes increases by dT and the mass by dm.



The gas law gives us pV = mRT (R is the characteristic gas constant). Differentiate with respect to time (note V is constant)

$$V\frac{dp}{dt} = mR\frac{dT}{dt} + RT\frac{dm}{dt} = \frac{pV}{T}\frac{dT}{dt} + RT\frac{dm}{dt} \;......\; (1)$$

For an adiabatic process we know that

$$pT^{\frac{\gamma}{\gamma-1}} = \text{constant} \qquad p = CT^{\frac{\gamma-1}{\gamma}}$$

Differentiate with respect to time

$$\frac{\gamma-1}{\gamma}\frac{dp}{dt} = \frac{p}{T}\frac{dT}{dt}$$

Substitute into (1)

$$V\frac{dp}{dt} = \left(\frac{\gamma-1}{\gamma}\right)V\frac{dp}{dt} + RT\frac{dm}{dt}$$

Rearrange

$$\frac{dm}{dt} = \frac{V\frac{dp}{dt} - \left(\frac{\gamma-1}{\gamma}\right)V\frac{dp}{dt}}{RT} = \frac{V\frac{dp}{dt}\left(1-\frac{\gamma-1}{\gamma}\right)}{RT} = \frac{V\frac{dp}{dt}\left[\left(\frac{\gamma-\gamma+1}{\gamma}\right)\right]}{RT} = \frac{V}{\gamma RT}\frac{dp}{dt}$$

Studies of gas flow reveal that the term

$$\sqrt{\gamma RT} \text{ is the sonic velocity } 'a'$$

Hence

$$\frac{dm}{dt} = \frac{V}{a^2}\frac{dp}{dt}$$

Pneumatic capacitance is defined as

$$C = \frac{V}{a^2} \quad m\,s^2$$

$$\frac{dm}{dt} = C\frac{dp}{dt}$$

The pneumatic capacitance per unit length of a pipe of length λ and bore area A is

$$\frac{C}{\lambda} = \frac{V}{\lambda a^2} = \frac{A\lambda}{\lambda a^2} = \frac{A}{a^2} \quad s^{-2}$$

1.2.2 Hydraulic Capacitance

Liquids are much more compact than gas so the capacitance is smaller and depends on the elasticity of the fluid and for high pressures, the elasticity of the pipe walls.

Consider a volume of liquid that is compressed by δV due to a pressure change δp. Define the capacitance as

$$C = \frac{\delta m}{\delta p}$$

The compressibility of the liquid depends on the bulk modulus defined as

$$K = V\frac{\delta p}{\delta V}$$

This is a well documented property of liquids. Assuming constant density

$$K = \rho V\frac{\delta p}{\rho \delta V} = m\frac{\delta p}{\delta m}$$

$$C = \frac{\delta m}{\delta p} \qquad C = \frac{m}{K} = \frac{\rho V}{K}$$

It can be shown that

$$a^2 = \frac{K}{\rho}$$

$$C = \frac{\delta m}{\delta p} = \frac{V}{a^2} = \frac{m}{\rho a^2}$$

This is exactly the same as for the pneumatic case.

If the elasticity of the pipe wall is considered we use the modified bulk modulus K'.

$$K' = \left\{\frac{D}{tE} + \frac{1}{K}\right\}^{-1}$$

E is the modulus of elasticity for the pipe material.

Expressed in differential form we have the same as for the pneumatic case.

$$\frac{dm}{dt} = C\frac{dp}{dt}$$

The capacitance per unit length of a pipe of length λ and bore area A is

$$\frac{C}{\lambda} = \frac{V}{\lambda a^2} = \frac{A\lambda}{\lambda a^2} = \frac{A}{a^2} \quad s^{-2}$$

Compare this with the electrical and pneumatic formulae for a capacitance.

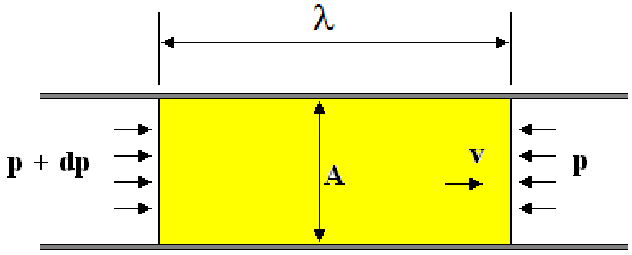
$$\frac{dm}{dt} = \dot{m} = C\frac{dp}{dt} \ldots\ldots \text{Pneumatic and hydraulic}$$

$$\frac{dQ}{dt} = i = C\frac{dV}{dt} \ldots\ldots \text{Electrical}$$

1.3. Inertance

This derivation is the same for liquids or gases. Inertance is a property that can be given to a fluid flowing in a pipe so that it may be used in the analysis pressure and flow variations using the same kind of approach as used to electric transmission lines. Consider the length λ of a section of pipe with fluid flowing through it.

A = cross sectional area
λ = length
v = velocity
ρ = density
p = pressure
m = Mass = ρA λ



If a small change in pressure dp travels along the pipe, the mass will be accelerated. If dp is small, the change in density may be neglected.

Inertia force

$$\text{Inertia Force} = m\frac{dv}{dt} = \rho A\lambda\frac{dv}{dt} \qquad \text{pressure force} = A\,dp$$

Equating

$$A\,dp = \rho A\lambda\frac{dv}{dt} \qquad dp = \rho\lambda\frac{dv}{dt} \ldots\ldots (1)$$

The mass flow rate is

$$\dot{m} = \rho A v = \frac{dm}{dt}$$

The rate of change of mass flow rate is

$$\frac{d^2m}{dt^2} = \rho A\frac{dv}{dt}$$

Rearrange

$$\frac{dv}{dt} = \frac{1}{\rho A}\frac{d^2m}{dt^2}$$

Substitute into (1)

$$dp = \rho\lambda\frac{dv}{dt} = \frac{\rho\lambda}{\rho A}\frac{d^2m}{dt^2} = \frac{\lambda}{A}\frac{d^2m}{dt^2}$$

Define Inertance as

$$L = \frac{\lambda}{A} \text{ and } dp = L\frac{d^2m}{dt^2}$$

Inertance is hence

$$L = \frac{\lambda}{A}\ m^{-1} \text{ or per unit length } = \frac{L}{\lambda} = \frac{1}{A}\ m^{-2}$$

Compare this with the electrical formula.

$$dp = L\frac{d^2m}{dt^2} \ldots\ldots \text{Pneumatic and Hydraulic}$$

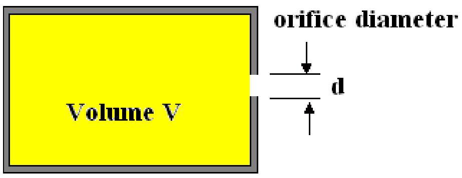
$$dV = L\frac{d^2i}{dt^2} \ldots\ldots \text{Electrical}$$

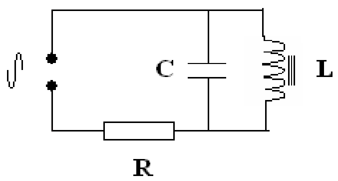
Hence we may use pneumatic inertance as an analogy of electrical inductance.

The next section shows how inertance may be applied to the flow through an orifice.

1.4. Orifice Inertance (Gases Only)

This is added for interested parties but not used in the following tutorials. We should know that it is possible to make a column of air resonate and produce sounds. Any musical instrument based on this is a form of acoustic resonator. Consider an acoustic resonator consisting of a volume and an orifice. This is analogous to a simple RCL circuit as shown.





When an alternating voltage is applied to the electrical circuit, it is found that it resonates at a frequency given by

$$\omega_r = \frac{1}{\sqrt{LC}} \text{ and } f_r = \frac{\omega_r}{2\pi}$$

You will find the derivation elsewhere on the web site.

The pneumatic resonator is known as a Helmholtz Resonator and he showed that the resonant frequency is given by

$$\omega_r = a\sqrt{\frac{d}{V}} \quad \text{and } a = \sqrt{\gamma RT}$$

'a' is the acoustic velocity and V the volume and this can be any shape. If we substitute the pneumatic capacitance

$$C = \frac{V}{a^2} \quad V = Ca^2 \quad \text{substitute and } \omega_r = a\sqrt{\frac{d}{a^2C}} = \sqrt{\frac{d}{C}}$$

Comparing the electrical and pneumatic formulae we have

$$\omega_r = \frac{1}{\sqrt{LC}} = \sqrt{\frac{d}{C}} \qquad \frac{1}{LC} = \frac{d}{C} \qquad L = \frac{1}{d}$$

L is now the orifice Inertance L_o

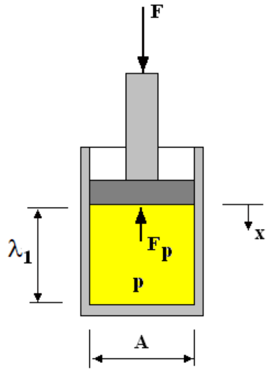
$$L_o = \frac{1}{d}$$

This may be used to analyse acoustic vibrations.

The resistance only serves to dampen the vibrations and has been ignored here.

2. Fluid Springs and Dashpots

This tutorial show how analogies may be used to derive the spring rate for a fluid column and the damping characteristics of a dashpot. These are important elements in any hydraulic or pneumatic system.



The analogous quantities throughout will be as follows.

Pressure (p) - Voltage (V)
Mass flow ($\dot{m}$) - Current(I or i)
Mass (m) - Charge (Q)

2.1. Pneumatic Spring

Examples of pneumatic springs are found in suspension systems and seats. Any linear pneumatic actuator will have a springiness that should be consider when analysing the possibility of oscillations due to the interaction of the mass and the spring.

The following shows the application of part 1 to a pneumatic spring. The diagram shows a volume of gas trapped in a cylinder by a piston. The gas pressure is 'p'. If the piston is moved a small distance 'x' the pressure rises as the gas is compressed. For rapid movement the compression is adiabatic.

$$p_1 V_1^\gamma = p_2 V_2^\gamma \qquad p_1 (A\lambda_1)^\gamma = p_2 (A\lambda_2)^\gamma \qquad p_1 (\lambda_1)^\gamma = p_2 (\lambda_2)^\gamma = p_2\,(\lambda_1 - x)^\gamma$$

$$p_2 = \frac{p_1 (\lambda_1)^\gamma}{(\lambda_1 - x)^\gamma}$$

The increase in the force due to the gas pressure is $F_p = A(p_2 - p_1)$

Substitute for p_2

$$F_p = A\left[\frac{p_1 {\lambda_1}^\gamma}{(\lambda_1 - x)^\gamma} - p_1\right] = Ap_1\left[\frac{{\lambda_1}^\gamma}{(\lambda_1 - x)^\gamma} - 1\right]$$

Differentiate with respect to x.

$$\frac{dF_p}{dx} = Ap_1\left[\left(\frac{\lambda_1}{\lambda_1 - x}\right)^{\gamma}\left(\frac{\gamma}{\lambda_1 - x}\right)\right]$$

This indicates that the spring rate depends on the position of the piston but when x = 0

$$\frac{dF_p}{dx} = Ap_1\left(\frac{\gamma}{\lambda_1}\right)$$

This is the pneumatic spring rate k_p at the start of the change but may be applied over a range if $x << \lambda_1$.

The expression is particularly useful when analysing vibrations with small amplitudes.

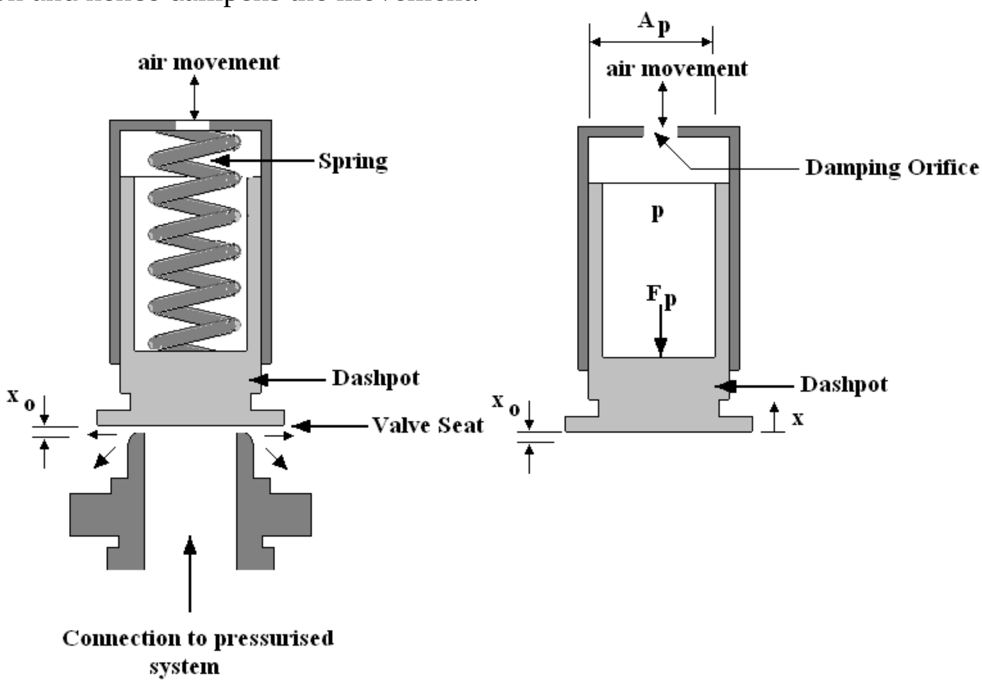
2.2. Hydraulic Spring

A hydraulic fluid is virtually incompressible and this depends on the bulk modulus K. Only at exceptionally high pressures does this become an issue (e.g. in some aircraft undercarriage designs the elasticity of the hydraulic fluid is used to produced a measure of springing). The elasticity of the pipes is more likely to be a factor in hydraulic circuits.

2.3. Pneumatic Dashpot

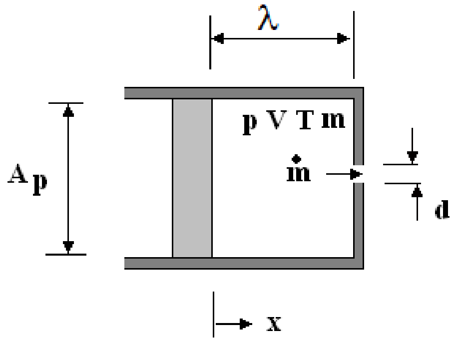
Pneumatic dashpots are used on many devices to damp out oscillations. The example shown here was used in conjunction with a pressure relief valve. It was found that the valve oscillated up and down at a high frequency when relieving air from some systems. This was due to the resonance of the connecting pipe and volume interacting with valve. It is interesting to analyse fully why the valve oscillated but in this section we will examine the damping characteristics of the dashpot. The purpose of the dashpot was to dampen these oscillations. In the original design there was no damping orifice and it was thought that the clearance gap between the piston and cylinder would produce damping. Research showed that the damper simply acted as a pneumatic spring that added to the steel spring simply determined the resonant frequency. The damping orifice made quite a difference.

Basically, when the piston moves up, air is pushed out of the chamber and when the piston moves down, air is sucked into the chamber. The pressure produced by the restriction and inertance always acts to oppose the motion of the piston and hence dampens the movement.



It is assumed that the changes in pressure are adiabatic. This is an accurate assumption for frequencies above 1 Hz. The pressure inside the dashpot is p and outside is atmospheric p_a. The pressure inside is equal with the pressure outside when the valve starts to oscillate starting from the rest position x_o. The pneumatic or pressure force acting on the piston is F_p.

Let us examine the case when the oscillations are small in amplitude. This approach is called a ***Small Perturbation Analysis***. Consider the simplified diagram.



The air inside the dashpot has a mass m, volume V, pressure p and temperature T. The characteristic gas law gives pV = mRT

Differentiate with respect to time.

$$V\frac{dp}{dt} + p\frac{dV}{dt} = mR\frac{dT}{dt} + RT\frac{dm}{dt}$$

$$V = A_p(\lambda - x) \ \text{ hence } \frac{dV}{dt} = -A_p\frac{dx}{dt}$$

It is reasonable to assume that the change in pressure is adiabatic so

$$pT^{\frac{\gamma}{\gamma-1}} = \text{constant}$$

$$\frac{dT}{dt} = \frac{\gamma-1}{\gamma}\left(\frac{T}{p}\right)\frac{dp}{dt}$$

$p = p_a + \delta p$ where δp is the increase in pressure relative to the outside. $dp = d(\delta p)$

The mass flow rate through the orifice is

$$\dot{m} = \frac{dm}{dt} = -\frac{\delta p}{G}$$

G is the orifice restriction or resistance. Combining the equations we have

$$V\frac{dp}{dt} + p\frac{dV}{dt} = mR\frac{dT}{dt} + RT\frac{dm}{dt}$$

$$V\frac{d(\delta p)}{dt} - A_p p\frac{dx}{dt} = mR\frac{\gamma-1}{\gamma}\left(\frac{T}{p}\right)\frac{dp}{dt} - RT\frac{\delta p}{G}$$

$$V\frac{d(\delta p)}{dt} - A_p p\frac{dx}{dt} - \frac{mRT}{p}\left(\frac{\gamma-1}{\gamma}\right)\frac{dp}{dt} = -RT\frac{\delta p}{G}$$

Note mRT = pV

$$V\frac{d(\delta p)}{dt} - A_p p\frac{dx}{dt} - V\left(\frac{\gamma-1}{\gamma}\right)\frac{dp}{dt} = -RT\frac{\delta p}{G}$$

Note dp = d(δp)

$$V\frac{d(\delta p)}{dt}\left[1 - \frac{\gamma-1}{\gamma}\right] - A_p p\frac{dx}{dt} = -RT\frac{\delta p}{G}$$

$$V\frac{d(\delta p)}{dt}\left[\frac{\gamma-\gamma+1}{\gamma}\right]-A_p p\frac{dx}{dt}=-RT\frac{\delta p}{G}$$

$$\frac{V}{\gamma}\frac{d(\delta p)}{dt}-A_p p\frac{dx}{dt}=-RT\frac{\delta p}{G}$$

$$\frac{V}{\gamma}\frac{d(\delta p)}{dt}+RT\frac{\delta p}{G}=A_p p\frac{dx}{dt}$$

It is convenient here to change to Laplace form.

$$\frac{V}{\gamma}s(\delta p)+RT\frac{\delta p}{G}=A_p p\,sx$$

$$\delta p\left[\frac{V}{\gamma}s+\frac{RT}{G}\right]=A_p p\,sx$$

$$\delta pA_p\left[\frac{V}{\gamma}s+\frac{RT}{G}\right]=A_p^2 p\,sx$$

Note $\delta p\ A_p\ = F_p$

$$F_p\left[\frac{V}{\gamma}s+\frac{RT}{G}\right]=A_p^2 p\,sx$$

$$\frac{F_p}{x}=\frac{A_p^2 p\,s}{\frac{V}{\gamma}s+\frac{RT}{G}}=\frac{\gamma GA_p^2 p\,s}{G\,V\,s+\gamma RT}$$

Pneumatic capacitance was defined as

$$C=\frac{V}{\gamma RT}$$

$$\frac{F_p}{x}=\frac{\gamma GA_p^2 p\,s}{G\,V\,s+\frac{V}{C}}=\frac{\gamma GA_p^2 p\,s}{V\left(G\ s+\frac{1}{C}\right)}=\frac{\gamma CGA_p^2 p\,s}{V(CG\ s+1)}$$

Define a time constant as $\tau = CG$

$$\frac{F_p}{x}=\frac{\gamma\tau A_p^2 p\,s}{V(\tau\,s+1)}$$

Note $V = A_p\,(\lambda - x)$

$$\frac{F_p}{x}=\frac{\gamma\tau A_p p\,s}{(\lambda-x)(\tau\,s+1)}$$

From the previous section we know that at the mean position the rate of change of force with distance is

$$k_p=\frac{dF_p}{dx}=\gamma\frac{A_p p}{\lambda}$$

If x is small compared with λ

$$\frac{F_p}{x}=\frac{k_p\tau\,s}{(\tau\,s+1)}$$

The pneumatic force is conveniently defined as

$$F_p=\frac{k_p\tau\,sx}{(\tau\,s+1)}$$

Making the Laplace substitution s becomes jω

$$F_p = \frac{k_p \tau\, j\omega x}{(\tau\, j\omega + 1)}$$

This shows that at high frequencies $F_p = k_p\, x$ and so behaves as a pneumatic spring with negligible damping. At low frequencies

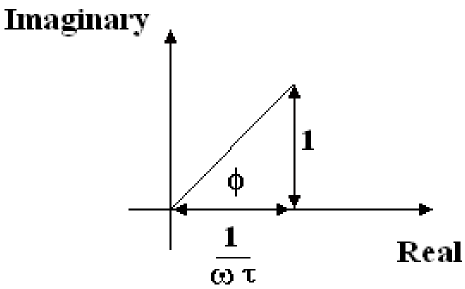
$$F_p = k_p \tau\, \omega x$$

In this case the dashpot behaves the same as a viscous damper where force is directly proportional to velocity. Substitute v = ω x

$$F_p = k_p \tau\, v = cv$$

c is the viscous damping coefficient and $c = k_p\, \tau$

$$\frac{F_p}{x} = \frac{k_p \tau\, j\omega}{(\tau\, j\omega + 1)} \text{ or } \frac{x}{F_p} = \frac{\tau\, j\omega + 1}{k_p \tau\, j\omega}$$



If this is turned into a complex number we have

$$\frac{x}{F_p} = \frac{1}{k_p}\left(1 - j\frac{1}{\tau\, j\omega}\right)$$

From the vector

$$\left|\frac{x}{F_p}\right| = \frac{\sqrt{1+\omega^2\tau^2}}{k_p \tau\, \omega} \text{ hence } \left|\frac{F_p}{x}\right| = \frac{k_p \tau\, \omega}{\sqrt{1+\omega^2\tau^2}}$$

The phase angle between F_p and x is

$$\phi = \tan^{-1}\left(\frac{1}{\tau\, \omega}\right)$$

Energy Dissipation

If the dashpot oscillates harmonically with an amplitude X the damping force is

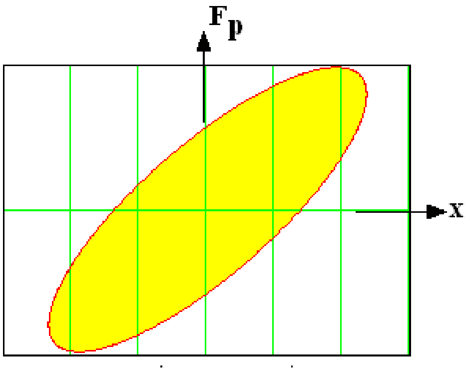
$$F_p = \frac{k_p \tau\, \omega}{\sqrt{1+\omega^2\tau^2}} xs = \frac{k_p \tau\, \omega}{\sqrt{1+\omega^2\tau^2}} X\sin(\omega\tau + \phi)$$

For a given set of parameters this may be written

$$F_p = KX\sin(\omega\tau + \phi)$$

The displacement is x = X sin(θ)

If we plot F_p against x for a given set of parameters we get a loop and the area within the loop is the work done against the pressure and hence the energy dissipated by the dashpot.



Enclosed Area = Energy Dissipated

For 1 cycle the energy dissipated is

$$E = \int_0^{2\pi} F dx = \int_0^{2\pi} KX\sin(\omega\tau + \phi)dx = \int_0^{2\pi} KX\sin(\theta + \phi)dx$$

$$x = X\sin(\theta) \text{ so } dx = X\cos(\theta)d\theta$$

$$E = KX^2 \int_0^{2\pi} \sin(\theta + \phi)\cos(\theta)d\theta$$

$$E = \frac{KX^2}{2}\left[\theta\sin(\phi) - \frac{1}{2}\cos(2\theta)\cos\phi - \sin(2\theta)\sin(2\phi)\right]_0^{2\pi}$$

$$E = KX^2\pi\sin(\phi)$$

From the vector we have

$$\sin\phi = \frac{1}{\sqrt{1 + \omega^2\tau^2}}$$

Substitute

$$K = \frac{k_p\tau\,\omega}{\sqrt{1 + \omega^2\tau^2}}$$

The energy dissipated for each cycle is

$$E = \frac{k_p\tau\,\pi\,\omega X^2}{1 + \omega^2\tau^2}$$

In a viscous damper the energy dissipated is $E = c\,\pi\,\omega\,X^2$

If we equate we can establish an equivalent viscous damping coefficient such that

$$c_e = \frac{k_p\tau}{1 + \omega^2\tau^2}$$

Maximum damping will occur at any given frequency when $\theta = 45^o$ and $\omega\,\tau = 1$ in which case

$$c_e = \frac{k_p\tau}{2} \text{ and } E = \frac{\pi k_p X^2}{2}$$

These are the design parameters for a dashpot to produce maximum damping at a given frequency.

CASE STUDY

A pneumatic dashpot similar to that shown in the previous diagram has the following parameters.

The volume of the air at the mean position is V = 10 460 mm^3 and the effective length was 25 mm.
The ambient conditions are p = 100 kPa, T = 288 K.
The gas constants for air are γ = 1.4 and R = 287 J/kg K.
The relationship between pressure drop and mass air flow through the orifice was measured and it was found that at low pressure values the pneumatic resistance was reasonably linear with a value of 23.8×10^6 N/ m s.

Determine the energy lost to damping and the equivalent damping coefficient when the dashpot is oscillated at 75 Hz with peak to peak amplitude of 2.35 mm.

First calculate the pneumatic capacitance of the dashpot.

$$C = \frac{V}{\gamma RT} = \frac{10\,460 \times 10^{-9}}{1.4 \times 287 \times 288} = 90.329 \times 10^{-12} \text{ m s}^2$$

The time constant is $\tau = GC = 23.8 \times 10^6 \times 90.32 \times 10^{12} = 2.151 \times 10^{-3}$ s

$X = 2.35/2 = 1.175$ mm $\lambda = 25$ mm hence $A = V/\lambda = 418.4 \times 10^{-6}\ m^2$

F = 75 Hz hence $\omega = 2\pi f = 471.239$ rad/s

$k_p = \gamma\, p\, A/\lambda = 1.4 \times x\ 100 \times 10^3 \times 418.4 \times 10^{-6} / 0.025 = 2\,343$ N/m

$\phi = \tan^{-1}(1/\omega\tau) = 44.6^o$

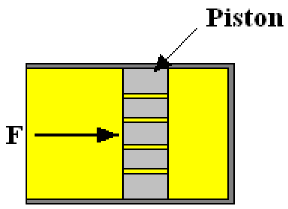
$$E = \frac{k_p \tau\, \pi\, \omega X^2}{1 + (\omega\tau)^2} = \frac{2\,343 \times 2.151 \times 10^{-3} \times \pi \times 471.239 \times 0.025^2}{1 + (471.239 \times 2.151 \times 10^{-3})^2} = 5.081 \times 10^{-3} \text{ J}$$

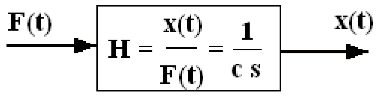
$$c_e = \frac{k_p \tau}{1 + (\omega\tau)^2} = \frac{2\,343 \times 2 \times 10^{-3}}{1 + (471.239 \times 2.151 \times 10^{-3})^2} = 2.486 \text{ N s/m}^2$$

Tests to determine the actual values gave a result quite close to the predicted values.

2.4. Hydraulic Dashpot

A typical hydraulic dashpot is a piston in a cylinder with holes allowing the liquid to move from one side of the piston to the other. Many variations are possible.





Without derivation, it can be shown that since the force required to shear a Newtonian fluid is directly proportional to the rate of shear, then the damping force produced by a hydraulic dashpot is directly proportional to the velocity of the piston. $F \propto v$. Velocity v is the first derivative of distance so $F \propto dx/dt$

The basic law of a dashpot is:

$$F(t) = c\frac{dx}{dt}$$

c is the damping coefficient (N s/m)

Changed into Laplace form $\quad F(s) = c\,s\,x$

Rearranged into a transfer function

$$H = \frac{x}{F}(s) = \frac{1}{cs}$$

When the piston is reciprocated harmonically with amplitude X the energy dissipated is

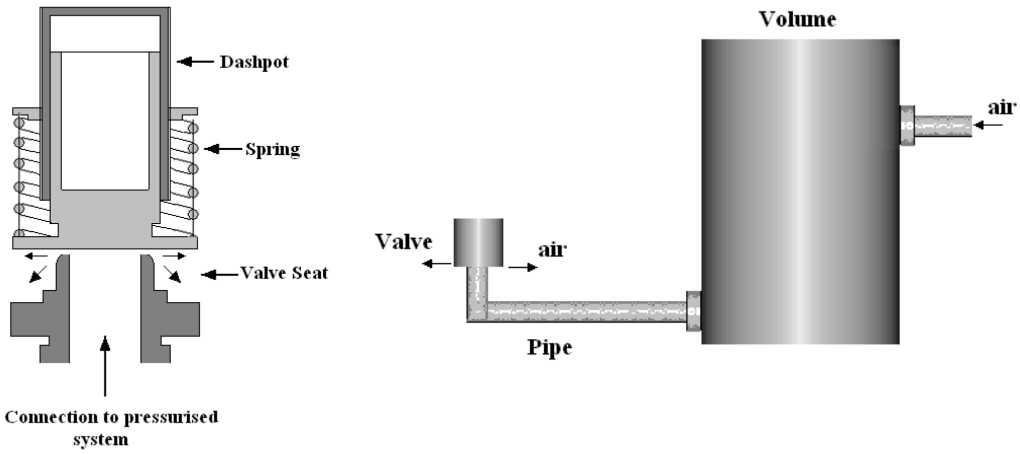
$$E = \frac{\pi k X^2}{2}$$

Instantaneous power dissipated is

$$P = \text{Force} \times \text{velocity}$$

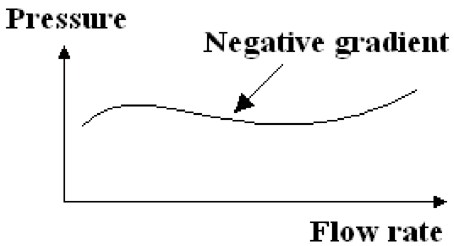
3. *Dynamic Stability of Valve and System*

The following case prompted the research outlined here but the general principles may be extended to a range of valves and situations. A simple spring loaded pressure limiting valve with a flat plate design was used to vent air from a vessel and was connected to the vessel by a pipe. It was found that the valve would go into violent oscillations at frequencies in excess of 50 Hz.



There are three forces acting on the valve plate: the force of the escaping air F_v, the force of the steel spring F_s and the pneumatic damping force due to the pressure of the air in the dashpot F_d. Each of these forces varies with distance x. These interact with the valve mass and the pipeline dynamics to produce instability. Perturbations in the pressure travel along the pipe at the sonic velocity and are reflected from the volume as rarefactions (negative pressure). If the pipe length is resonant at a frequency near to the resonant frequency of the valve, they will interact. (Note that at closed ends such as a hydraulic pump, the pressure perturbations are reflected as positive pressure).

Research revealed that on part of the operating characteristic the mass flow could increase with the inlet pressure decreasing. This gives the device a negative characteristic or restriction. A similar electronic device is a tunnel diode used to create microwave oscillations.



Research also revealed that the pneumatic dashpot behaved like a spring and produced negligible damping until a damping orifice was added to it. Also the steel spring, having a moving mass, produces a force that gets out of phase with movement and decreases with frequency so that it became ineffective. Much of the work uncovered has been explained in tutorials (1) and (2). The following gives some insight into the stability analysis.

3.1 Dynamic Stability

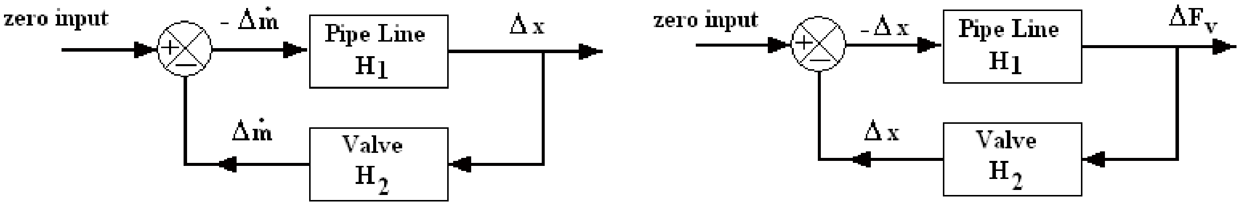
Valve oscillation has several causes. The main ones are the interaction between the valve mass and spring dynamics and the dynamics of the pipe system connected to it.

Water hammer is a phenomenon involving pressure fluctuations moving up and down pipes, usually due to the closure or opening of a pipe line valve. You will find information on this in the fluid mechanics tutorials. A related phenomenon occurs in hydraulic and pneumatic systems with interaction between the valve and the pipe dynamics.

When a valve moves, the flow rate through it changes and the pressure in the pipe changes. Pressure changes can travel along a pipe and are reflected from closed end as a pressure and from open ends as rarefaction. It becomes possible for the pressure and flow rate to get out of phase with each other and with the valve position and thus sustain the oscillation of the valve element.

An additional cause is that the possibility of a negative pressure - flow rate characteristic, a well known cause of instability.

3.2 Closed Loop Model



Two models are shown. One is based on the relationship between flow rate and valve position and the other is based on the relationship between valve movement and the fluid force acting on the valve element. If a transfer function is derived for the two blocks and a closed loop transfer function created, then a stability analysis can be made. This model may be applied to all manner of valves, hydraulic and pneumatic. The dynamics of the pipe line may be determined in many cases by applying electrical transmission line theory to it.

Generally, when the resonant frequency of the valve is close to the quarter wave resonant frequency of the pipe (or some multiple of the frequency) the valve will be unstable. There will be a 90^{o} phase shift between the flow and the pressure (like between current and voltage in an electrical system).

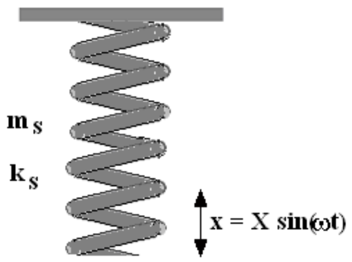
In the following a great simplification is used where the changes in the variables is assumed to be very small and so the relationships between them are linearised to the gradients of the functions at the operating point. This is called a 'Small Perturbation Analysis' and allows some insight to the onset of instability. Larger perturbations may make the system more or less stable.

3.3 Analysis of the Valve Dynamics

When a valve oscillates, the whole system becomes dynamic and surprising things can happen like what happens to an ordinary spring which we will examine first.

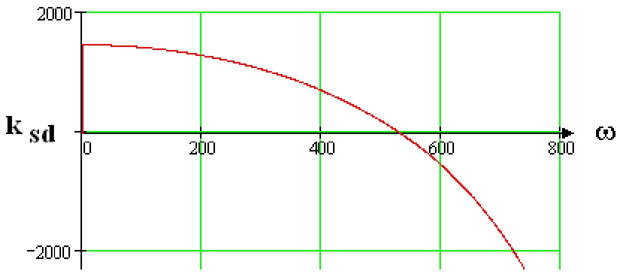
3.3.1 Dynamic Spring Rate

An ordinary spring has a spring rate $k_s = \Delta F/\Delta x$ but the spring coils have a mass and only one end moves so some of the mass moves faster than the rest.



When one end is moved harmonically the actual spring rate or dynamic spring rate has been shown to be

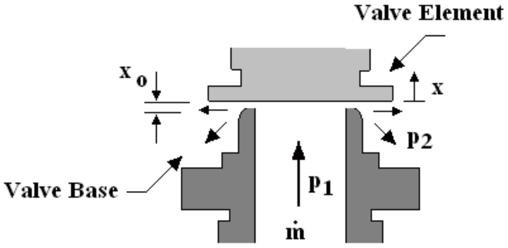
$$k_{sd} = k_s \omega \sqrt{\frac{M_s}{k_s}} \left[\tan\left(\omega \frac{M_s}{k_s} \right) \right]^{-1}$$



In a later case study
m_s = 1.46 g and k_s = 1.46 N/mm and the plot is shown. This shows that the spring rate can become zero or even negative if it is attached to the valve element.

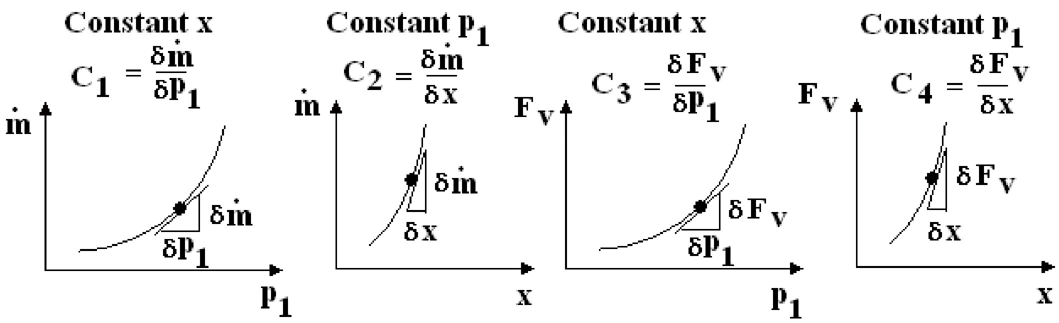
3.3.2 Small Perturbation Method

In this section the analysis is made by assuming that all the changes are small. This is called a small perturbation analysis.



We need to consider how the mass flow rate $\dot{m}$ and force F_v varies with pressure p_1 and opening x.
F_v is the fluid force acting on the valve element due to the pressure and momentum change of the fluid.

These are related by some function. At the operating point the gradients of the functions are C_1, C_2, C_3 and C_4. The graphs are only for illustrative purposes.



For a small change we consider that the relationship is linear. Clearly if the changes are large the result is different but this method is useful for determining the likelihood of instability occurring.

3.3.3 Valve Impedance and Transfer Functions

Many models use impedance to model the dynamics of the valve and this fits in with the electrical analogue. The following could apply to most types of hydraulic and pneumatic valves but a simple flat valve will be considered here. The mass flow rate $\dot{m}$ depends on the opening x_o and the pressure drop over the element $\Delta p = p_1 - p_2$.

We will consider p_2 as constant so

$$\delta \Delta p = \delta p_1.$$

3.3.4 Impedance

The impedance is defined as

$$Z_v = \frac{\delta p_1}{\delta \dot{m}}$$

The change in flow rate will be partly due to the change in opening and partly due to the change in pressure so

$$\delta \dot{m} = C_1 \delta p_1 + C_2 \delta x$$

$$C_1 = \frac{\delta \dot{m}}{\delta p_1} \qquad C_2 = \frac{\delta \dot{m}}{\delta x}$$

$$\frac{\delta p_1}{\delta \dot{m}} = \left(C_1 + C_2 \frac{\delta x}{\delta p_1}\right)^{-1} \quad \dots \dots (1)$$

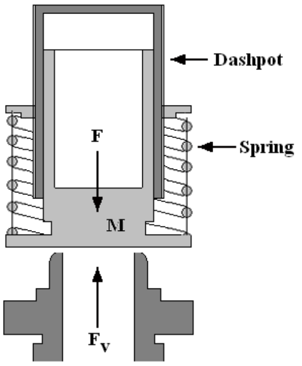
This shows that the impedance is negative when

$$C_2 \frac{\delta x}{\delta p_1} > C_1$$

The change in the force F_v will be partly due to the change in opening and partly due to the change in pressure so

$$\delta F_v = C_3 \delta p_1 + C_4 \delta x \dots \dots (2)$$

The force 'F_v' is opposed by various forces depending on the design. This could include the force due to the steel spring 'F_s', the damping force 'F_d' due to the dashpot and the inertia force 'F_i' due to the mass of the element 'M'.



Equating the forces we have

$$F_v = F_s + F_d + F_i$$

If these change slightly then

$$\delta F_v = \delta F_s + \delta F_d + \delta F_i$$

The inertia force is defined as

$$F_i = M \frac{d^2 x}{dt^2}$$

M is the moving mass.

Each of these forces varies with opening so we define the gradients of the functions as a spring rate such that

$$k_d = \frac{\delta F_d}{\delta x} \qquad k_s = \frac{\delta F_s}{\delta x} \qquad k_v = \frac{\delta F_v}{\delta x}$$

$$\delta F_v = k_s \delta x + k_d \delta x + M \frac{d^2(\delta x)}{dt^2} = \delta x (k_s + k_d) + M \frac{d^2(\delta x)}{dt^2}$$

In Laplace form this is

$$\delta F_v = \delta x (k_s + k_d + M s^2)$$

$$\delta F_v = \delta x(k_s + k_d) + Ms^2\delta x = \delta x(k_s + k_d + Ms^2) \ldots\ldots (3)$$

Equate (3) and (2)

$$C_3\delta p_1 + C_4\delta x = \delta x(k_s + k_d + Ms^2)$$

$$C_3\delta p_1 = \delta x(k_s + k_d + Ms^2) - C_4\delta x = \delta x(k_s + k_d + Ms^2 - C_4)$$

$$\frac{C_3}{(k_s + k_d + Ms^2 - C_4)} = \frac{\delta x}{\delta p_1}$$

Substitute into (1)

$$Z_v = \left(C_1 + \frac{C_2C_3}{(k_s + k_d + Ms^2 - C_4)}\right)^{-1}$$

If the valve oscillates harmonically with a small amplitude we may substitute s = jω

$$Z_v = \left(C_1 + \frac{C_2C_3}{(k_s + k_d - C_4 + M\omega^2)}\right)^{-1}$$

This is negative when

$$\omega > \left[\frac{k_s + k_d - C_4 + \frac{C_2C_3}{C_1}}{M}\right]^{1/2}$$

The resonant frequency is

$$\omega_r = \frac{1}{M}\left[k_s + k_d - \frac{\delta F_v}{\delta x}\right]$$

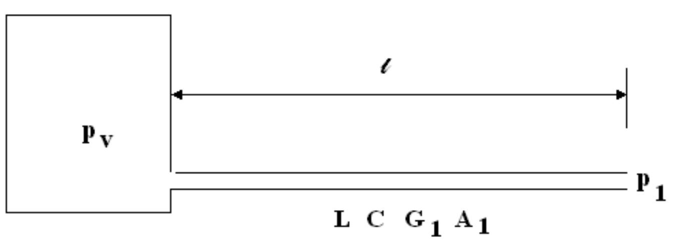
As discussed above, the steel spring rate a function of frequency and the dynamic rate k_{sd} should be used. This may give a negative value at the frequency of interest and can greatly increase the probability of negative valve impedance.

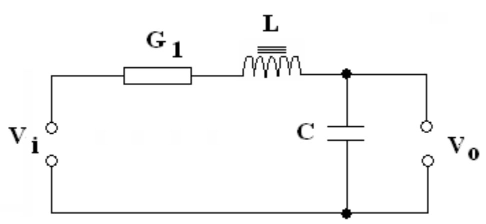
Negative impedance will occur when the net spring rate is negative.

Many valves have a characteristic that makes the restriction to flow decrease with movement such as poppet valves and spool valves. Variable area flow meters (Rotameters) are based on this principle and the floats can often oscillate up and down inside the tube due to this. Electronic devices such as tunnel diodes have a similar characteristic and are used to generate electronic oscillations.

3.4 Analysis of the Pipeline Dynamics

Consider a pipe connected to volume as shown. The pipe has a length ℓ and bore area A_1. The fluid in the pipe has inertance L, capacitance C and resistance G_1. The analogue circuit is shown. p_v is equivalent to V_i and p_1 is equivalent to V_o.





The circuit impedance viewed from the volume is

$$Z_p = G_1 + X_L + X_c$$

This is the ***Lumped Parameters***. It was shown in part 1 that for a gas

$$C = \frac{A\lambda}{a^2} \text{ and } L = \frac{\lambda}{A}$$

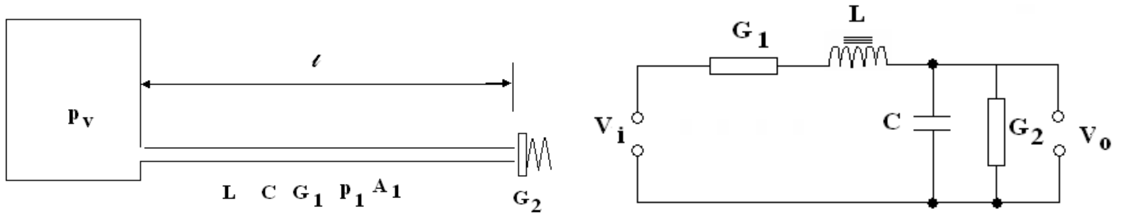
The resistance is less easy to express.

Well established electrical transmission line theory lets us make the analogy that the volume is the same as a short circuit in which case the impedance in terms of the ***Distributed Parameters*** is

$$Z_p = \frac{\Delta p_1}{\Delta \dot{m}} = -jZ_o \tan\left(\frac{\omega\lambda}{A}\right)$$

Consider that a simple valve is placed at the end of the pipe. In this case we will assume that the restriction of the valve at the operating point is G_2 but of course this could be a complex expression.

3.4.1 Method 1 Lumped Parameters Circuit



The circuit impedance viewed from the volume is

$$Z = G_1 + X_L - \frac{R_2 X_C}{X_C - G_2}$$

X_L is the inductive reactance and in Laplace form

$$X_L = sL$$

X_C is the capacitive reactance and in Laplace form

$$X_C = \frac{1}{sC}$$

$$Z = G_1 + sL - \frac{\frac{G_2}{sC}}{\frac{1}{sC} - G_2} = G_1 + sL - \frac{G_2}{1 - G_2 sC} = \frac{G_2 - G_1 + s(G_1G_2C - L) + s^2G_2LC}{G_2sC - 1}$$

The numerator is

$$s^2G_2LC + s(G_1G_2C - L) + G_2 - G_1$$

Apply the Routh-Hurwitz criteria and the requirements for stable operation are

$$G_1 > G_2 \quad \text{and} \quad G_1 > \frac{L}{G_2C}$$

For a pneumatic system the second criteria reduces to

$$G_1 > \frac{a^2}{G_2A_1^2}$$

The stabilising factors are a large capacitance and a suitable pipe restriction that depends on the friction coefficient of the pipe and the length.

In many instances, there will be an operating point of the valve where the mass flow rate increases even when the pressure p_1 reduces and this means there is an operating point with a negative G_2 and such systems are unstable.

3.4.2 Method 2 Closed Loop with Distributed Parameters

The volume at the supply end of the pipe is analogous to a closed circuit in electrical theory and well established transmission line theory gives the impedance of the line as viewed from that end. Making a pneumatic analogy we have the pipe impedance viewed from the volume as:

$$Z_p = \frac{Ap_1}{\Delta \dot{m}} = -jZ_o \tan\left(\frac{\omega\lambda}{a}\right)$$

Z_o is characteristic impedance of the pipe and by analogy is

$$Z_o = \sqrt{\frac{L}{C}} = \frac{a}{A}$$

L and C are the distributed parameters with negligible restriction (friction).
We had

$$\frac{\Delta p_1}{\Delta \dot{m}} = \left(C_1 + C_2 \frac{\delta x}{\delta p_1}\right)^{-1} \ldots\ldots (1)$$

$$\delta F_v = \delta x(k_s + k_d + Ms^2) \ldots\ldots (3)$$

Combine equations (1), (3) and

$$Z_p = \frac{\Delta p_1}{\Delta \dot{m}}$$

Eliminate δp_1 and we get the result

$$\frac{\delta F_v}{\Delta x} = \frac{C_3 C_2 Z_p}{1 - Z_p C_1} + C_4$$

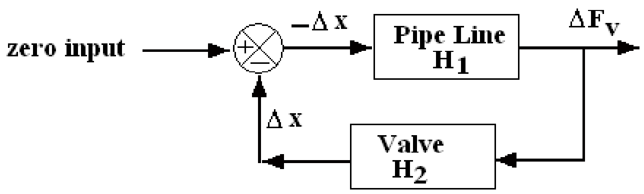
This is a transfer function H_2 relating the fluid forces on the valve element to the pressure in the pipe.

$$H = \frac{C_3 C_2 Z_p}{1 - Z_p C_1} + C_4$$

It was shown earlier that the relationship between the force and the pressure from examination of the pipeline is

$$H_1 = (k_s + k_d = M\omega^2)^{-1}$$

The closed loop block diagram is shown.



H_1 is the transfer function of the valve and H_2 the transfer function of the pipe line.
The open loop transfer function is:

$$H_{OL} = H_1 H_2 = (k_s + k_d = M\omega^2)^{-1} \frac{C_3 C_2 Z_p}{1 - Z_p C_1} + C_4$$

Remember in terms of distributed parameters that

$$Z_p = -jZ_o \tan\left(\frac{\omega\lambda}{a}\right)$$

When $\omega\ell/a = \pi/2, 3\pi/2$. …. this is infinity and when $\omega\ell/a = \pi, 2\pi, 3\pi$, …. it is zero.

If we could plot the Nyquist diagram the system would be unstable if it enclosed the -1 point. Since the constants C_3 and C_4 are difficult to determine this might be difficult to do but we do know that this would occur if H_1 or H_2 is negative and when

i. When $\omega\ell/a = \pi/2, 3\pi/2 \ldots$

$$\omega > \sqrt{\frac{k_s + k_d - C_4 + \frac{C_3 C_2}{C_1}}{M}}$$

ii. When $\omega\ell/a = \pi, 2\pi, 3\pi \ldots$

$$\omega > \sqrt{\frac{k_s + k_d - C_4}{M}}$$

This is a logical result.

Condition (i) corresponds to the pipe being resonant due to being ¼ wave length long or a multiple of this. At this condition C_3C_2/C_1 changes sign but C_4 being independent of pressure change will remains negative.

Condition (ii) corresponds to the pipe being resonant and ½ wave length long. At this condition the pressure variations in the pipe being reflected will cancel the variations being set up by the valve oscillation. The component C_3C_2/C_1 vanishes but C_4 will remains negative.