STABILITY OF PNEUMATIC and HYDRAULIC VALVES

These three tutorials will not be found in any examination syllabus. They have been added to the web site for engineers seeking knowledge on why valve elements sometimes go unstable and what can be done to prevent it.

TUTORIAL 1 - ELECTRICAL ANALOGUES

Tutorial 1 extends the work on analogies covered in Control Tutorial No.1 where you will find more information. With these analogies we may apply electrical theory to pneumatic systems such as a dashpot which is discussed in depth. In the following work on pneumatics, the changes in density of the gas must be very small so the pressure changes must also be small. This makes the theory particularly useful for acoustics and the analysis of oscillations in pneumatic systems.

The analogous quantities throughout will be as follows.
Pressure (p) - Voltage (V)
Mass flow (\( \dot{m} \)) - Current(I or i)
Mass (m) - Charge (Q)
The analogy deals with three properties, Resistance, Capacitance and Inductance.

1. **RESISTANCE**

A fluid restriction may be approximated to an analogue of an electrical resistor in some analysis. A simple restriction is shown.

The flow rate is \( \dot{m} = \frac{dm}{dt} \) and this results from the pressure drop \( \Delta p = p_1 - p_2 \)

The relationship between \( \Delta p \) and \( \dot{m} \) is not linear so an exact analogy with the electrical resistance is not possible.

Ideally we want to have an analogy that fits Ohm's Law.

\[ \Delta p = R \cdot \dot{m} \quad \text{Pneumatic/Hydraulic} \]
\[ \Delta V = R \cdot I \quad \text{Electrical} \]

In general the relationship is as shown on the diagram. If the flow changes by a small quantity \( dm \) due to a small change in pressure \( dp \) then the new flow rate is \( \dot{m} = \text{bias flow} + \frac{dp}{R} \) where \( R \) is the gradient of the graph at the operating point. If the graph was linear then \( R \) would be the perfect analogy to electrical resistance. In the analysis of small perturbations in the flow, this analogy is quite useful.

Because the gas constant also has a symbol \( R \) we will use the alternative symbol \( G \) for pneumatic resistance when required.
2. **CAPACITANCE**

An electrical capacitor obeys the laws \( \frac{dQ}{dt} = i = C \frac{dV}{dt} \) and that \( C = \frac{\delta Q}{\delta V} \) and formula for the value of 
C in terms of the capacitor's parameters may be found in the electrical tutorials on the web site.

It follows that a pneumatic or hydraulic equivalent must obey the law \( \frac{dm}{dt} = C \frac{dp}{dt} \)

It will be shown below that this analogy is quite accurate and that for both hydraulic and pneumatic 
systems \( C = \frac{\delta m}{\delta p} = \frac{V}{a^2} \) a is the sonic velocity in the medium (liquid or gas).

If a volume \( V \) contains a mass \( m \), adding more mass \( \delta m \) will increase the pressure by \( \delta p \). It follows 
that pressure is an indicator of the mass.

**PNEUMATIC CAPACITANCE**

Consider a length of pipe containing gas such that the volume \( V = A \ell \). 
The mass is \( m \), the temperature is \( T \) and the pressure is \( p \). When the 
pressure rises adiabatically by an amount \( dp \), the temperature changes 
increases by \( dT \) and the mass by \( dm \). The gas law gives us:

\[
pV = mRT \quad \text{(R is the characteristic gas constant).}
\]

Differentiate with respect to time

\[
\frac{dp}{dt} = \frac{m}{V} \frac{dT}{dt} + \frac{RT}{V} \frac{dm}{dt} \quad \text{or} \quad \frac{dp}{dt} = \frac{pV}{T} \frac{dT}{dt} + \frac{RT}{V} \frac{dm}{dt} \quad \text{...............(2)}
\]

For an adiabatic process we know that \( pT^{\frac{\gamma}{\gamma-1}} = \text{constant} \)

Differentiate with respect to time \( \frac{\gamma-1}{\gamma} \frac{dp}{dt} = \frac{p}{T} \frac{dT}{dt} \)

Substitute into (2) and:

\[
\frac{dp}{dt} = \left( \frac{\gamma-1}{\gamma} \right) \frac{V}{R} \frac{dm}{dt} + RT \frac{dm}{dt}
\]

Rearrange and get

\[
\frac{dm}{dt} = \frac{V}{RT} \left( \frac{V}{R} \right) \frac{dp}{dt} = \frac{V}{RT} \left[ 1 - \left( \frac{\gamma-1}{\gamma} \right) \right] \frac{dp}{dt} = \frac{V}{RT} \left[ \frac{\gamma - \gamma + 1}{\gamma} \right] \frac{dp}{dt}
\]

\[
\frac{dm}{dt} = \left( \frac{V}{\gamma RT} \right) \frac{dp}{dt}
\]

Studies of gas flow reveal that the term \( \sqrt{\gamma RT} \) is the speed of sound 'a' hence

\[
\frac{dm}{dt} = \frac{V}{a^2} \frac{dp}{dt}
\]

Pneumatic capacitance is defined as \( C = \frac{V}{a^2} \) (m s\(^2\))

\[
\frac{dm}{dt} = C \frac{dp}{dt}
\]

The pneumatic capacitance per unit length of a pipe is \( \frac{C}{\ell} = \frac{V}{\ell a^2} = \frac{A\ell}{\ell a^2} = \frac{A}{a^2} \) (s\(^2\))
HYDRAULIC CAPACITANCE

Liquids are much more compact than gas so the capacitance is smaller and depends on the elasticity of the fluid and for high pressures, the elasticity of the pipe walls.

Consider a volume of liquid that is compressed by $\delta V$ due to a pressure change $\delta p$. Define the capacitance as $C = \frac{\delta m}{\delta p}$

The compressibility of the liquid depends on the bulk modulus defined as $K = \frac{V}{\delta p/\delta V}$

This is a well documented property of liquids.

Assuming constant density, $K = \rho V \delta p/\rho \delta V = m \delta p/\delta m$

$C = \frac{\delta m}{\delta p}$ then $C = \frac{m}{K} = \frac{\rho V}{K}$ It can be shown that $a^2 = \frac{K}{\rho}$ where $a$ is the sonic velocity in a liquid.

$C = \frac{\delta m}{\delta p} = \frac{V}{a^2} = \frac{m}{\rho a^2}$ exactly as for the pneumatic case.

If the elasticity of the pipe wall is considered we use the modified bulk modulus $K'$.

$K' = \left( \frac{D}{tE} + \frac{1}{K} \right)^{-1}$ and $E$ is the modulus of elasticity for the pipe material.

Expressed in differential form we have $\frac{dm}{dt} = C \frac{dp}{dt}$ as with the pneumatic case.

In terms of a pipe volume $A$ the capacitance per unit length is $\frac{C}{\ell} = \frac{A}{\ell a^2} = \frac{A}{a^2}$

Compare this with the electrical and pneumatic formulae for a capacitance.

$$\frac{dm}{dt} = C \frac{dp}{dt} \quad \text{Pneumatic and Hydraulic}$$

$$\frac{dQ}{dt} = i = C \frac{dV}{dt} \quad \text{Electrical}$$
3. **INERTANCE**

This derivation is the same for liquids or gases. Inertance is a property that can be given to a fluid flowing in a pipe so that it may be used in the analysis pressure and flow variations using the same kind of approach as used to electric transmission lines. Consider the length \( l \) of a section of pipe with fluid flowing through it.

\[
A = \text{cross sectional area} \\
l = \text{length} \\
v = \text{velocity} \\
\rho = \text{density} \\
p = \text{pressure} \\
m = \text{Mass} = \rho A l
\]

If a small change in pressure \( dp \) travels along the pipe, the mass will be accelerated. If \( dp \) is small, the change in density may be neglected.

Inertia force = \( m \frac{dv}{dt} = \rho A l \frac{dv}{dt} \) and the pressure force = \( A \frac{dp}{dt} \)

Equating

\[
A \frac{dp}{dt} = \rho A l \frac{dv}{dt} \\
\frac{dp}{dt} = \rho l \frac{dv}{dt} \quad \text{..................(1)}
\]

The mass flow rate is \( \rho A v = \frac{dm}{dt} \)

The rate of change of mass flow rate is \( \frac{d^2 m}{dt^2} = \rho A \frac{dv}{dt} \)

Rearrange and \( \frac{dv}{dt} = \frac{1}{\rho A} \frac{d^2 m}{dt^2} \)

Substitute into (1)

\[
\frac{dp}{dt} = \rho l \frac{dv}{dt} = \rho l \frac{1}{\rho A} \frac{d^2 m}{dt^2} = \frac{\ell}{A} \frac{d^2 m}{dt^2}
\]

Define inertance as \( L = \frac{\ell}{A} \) and \( \frac{dp}{dt} = L \frac{d^2 m}{dt^2} \)

Inertance is hence \( L = \frac{\ell}{A} \) (m\(^{-1}\)) or per unit length is \( \frac{L}{\ell} = \frac{1}{A} \) (m\(^{-2}\))

Compare this with the electrical formula.

\[ \frac{dV}{dt} = L \frac{d_i}{dt^2} \quad \text{.................. Pneumatic and Hydraulic} \]

\[ \frac{dV}{dt} = L \frac{d_i}{dt^2} \quad \text{.................. Electrical} \]

Hence we may use pneumatic inertance as an analogy of electrical inductance.

The next section shows how inertance may be applied to the flow through an orifice.
4. **ORIFICE INERTANCE (GASES ONLY)**

This is added for interested parties but not used in the following tutorials. We should know that it is possible to make a column of air resonate and produce sounds. Any musical instrument based on this is a form of acoustic resonator. Consider an acoustic resonator consisting of a volume and an orifice. This is analogous to a simple RCL circuit as shown.

![Diagram of an acoustic resonator](image)

When an alternating voltage is applied to the electrical circuit, it is found that it resonates at a frequency given by $\omega_r = \frac{1}{\sqrt{LC}}$ and $f_r = \frac{\omega_r}{2\pi}$. You will find the derivation elsewhere on the web site.

The pneumatic resonator is known as a Helmholtz Resonator and he showed that the resonant frequency is given by $\omega_r = \sqrt{\frac{d}{V}}$ where $a$ is the acoustic velocity ($a = \sqrt{\gamma RT}$). The volume may be any shape. If we substitute the pneumatic capacitance $C = \frac{V}{a^2}$ we get $\omega_r = \sqrt{\frac{d}{a^2 C}} = \sqrt{\frac{d}{C}}$

Comparing the electrical and pneumatic formulae we have:

$\omega_r = \frac{1}{\sqrt{LC}} = \sqrt{\frac{d}{C}}$

$\frac{1}{LC} = \frac{d}{C}$

$L_0 = \frac{1}{d}$ and this is the Orifice Inertance that may be used to analyse acoustic vibrations.

The resistance only serves to dampen the vibrations and has been ignored here.