

INSTRUMENTATION AND CONTROL

TUTORIAL 12 – DIGITAL SYSTEMS

This tutorial is of interest to any student studying Control System Engineering. The level is set at NVQ Levels 5 and 6

On completion of this tutorial, you should be able to do the following.

- Explain basic principles of numerical processing and emulation.
- Explain basic principles of digital control.
- Explain how analogue signals are sampled and processed.
- Explain basic numerical techniques digital systems.
- Explain how proportional, Integral and Differential (PID) control is conducted numerically.
- Explain how dynamic compensation may be done digitally.
- Explain the principles of incremental control.
- Explain the use of the z transform.

If you are not familiar with instrumentation used in control engineering, you should complete the tutorials on Instrumentation Systems.

In order to complete the theoretical part of this tutorial, you must be familiar with basic mechanical and electrical science.

You must also be familiar with the use of transfer functions and the Laplace Transform (see maths tutorials).

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1. Introduction

Digital technology is increasingly taking over from the traditional analogue systems; particularly in the part of the system that does the control action. This involves turning the analogue signals into digital numbers with an Analogue to Digital Converter (A.D.C). The digital numbers produced are processed and then converted back into an analogue with a Digital to Analogue Converter (D.A.C.).

Consider how a simple DC motor may have its speed controlled by a computer. The system input may be directly from a key pad or from a voltage then converted into digital form. The analogue voltage from the tachometer is converted into a digital value. This is compared to the set value in the computer and the resulting error is processed and then converted back into an analogue signal. This is amplified and used to power the motor. In this way the speed of the motor may be regulated.

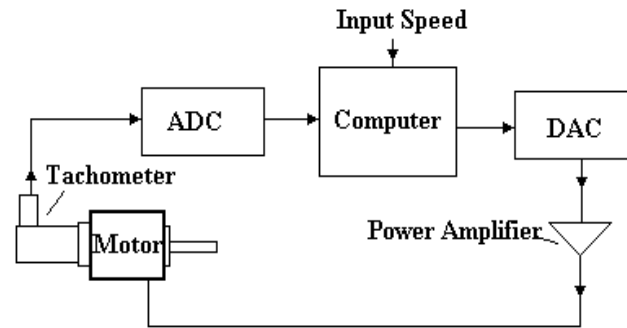


Figure 1

Digital processing requires computer technology and many programmable logic controllers (PLCs) are quite capable of performing the required functions. Otherwise, dedicated hardware is used.

The simplest example of processing is the summing or comparing of two signals to produce the error. An analogue device would be a simple differential amplifier to produce the difference in two voltages. The equivalent digital algorithm would take the digital numbers representing the input and output, subtract them and place the result into a register.

Another simple process is that which produces proportional action. A simple analogue device would be a linear amplifier or attenuator. The digital model would simply take the digital number and multiply it by a constant.

More complicated processing such as differentiation and integration can be done relatively easily with electronics (or with analogue computers) but a digital computer would require a numerical process to be designed so that it can simulate the operation. With suitable numerical methods, a computer can do the same functions as a dynamic compensator (lag and lead) or a three term controller. Most of this tutorial is about how to deduce a suitable algorithm to enable a numerical process to be carried out on the digital signal to produce a similar result as a given analogue processor.

Although it is only the control functions that are done by the computer in real systems, it is also useful to know some of the basic numerical processes used in computers in order to emulate a transfer function. The next section covers some of the basic process.

2. Basic Numerical Processes

2.1 Numerical Integration and Differentiation

Consider a continuous function of time $x = f(t)$ plotted on a graph and then divided up into strips of width T seconds. The time integral of the function is the area under the graph and this is the sum of the area of the strips. This is the basis of numerical integration and differentiation. Many methods have been developed to simplify the calculation of the area of each strip and allow the numerical process to be done. The diagram shows three that use straight lines to link each point. These are known as forward, backward and trapezoidal. There are more and you should refer to advanced maths text books for more information. Clearly the smaller the value of T the more accurate the integration will be and in the limit as $T \rightarrow 0$ we get the same result as integrating for all these methods.

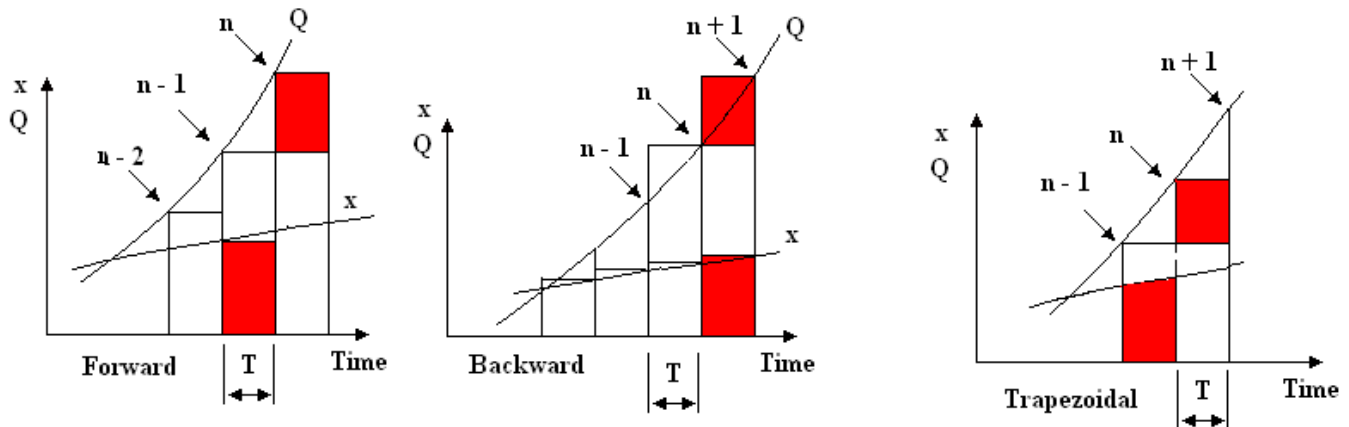


Figure 2

Our graph shows x plotted against time and divided into strips. Each vertical coordinate is numbered. 'n' indicates the present coordinate, $n-1$ the previous, $n-2$ the one before that and so on. $n+1$ means the next in time. These coordinates will become the digital samples, i.e. values taken at discrete intervals of T seconds. Let the sum of the area of the strips be denoted Q from $t = 0$ to $t = \text{last point}$ and this is also plotted for clarity.

Integration

When using the backward method Q_n is the last total so the new total will be

$$Q_{n+1} = Q_n + T(x)_{n+1}$$

When using the forward method Q_{n-1} is the last total so the new total will be

$$Q_n = Q_{n-1} + T(x)_{n-1}$$

Using the trapezoidal rule the area of the last strip is

$$T \left\{ \frac{(x)_n + (x)_{n-1}}{2} \right\} \text{ The new total is } Q_n = Q_{n-1} + T \left\{ \frac{(x)_n + (x)_{n-1}}{2} \right\}$$

Differentiation

This method is known as Euler's method.

$$\frac{dx}{dt} \rightarrow \frac{\Delta x}{T} = \frac{(x)_n + (x)_{n-1}}{T}$$

This becomes more accurate as $T \rightarrow 0$

When using the forward method we use

$$\frac{dx}{dt} \rightarrow \frac{\Delta x}{T} = \frac{(x)_{n+1} - (x)_n}{T}$$

The second order differentiation may also be approximated by taking two successive rates.

$$\frac{d^2x}{dt^2} = \frac{\frac{(x)_{n+1} - (x)_n}{T} - \frac{(x)_n - (x)_{n-1}}{T}}{T} = \frac{(x)_{n+1} - (x)_n - (x)_n + (x)_{n-1}}{T^2} = \frac{(x)_{n+1} - 2(x)_n + (x)_{n-1}}{T^2}$$

2.2 Difference Equations

Consider the equation $x = f(t)$. The discrete sample x_{n-1} is related to the last sample x_n by $x_n = x_{n-1} + \Delta x$. If it is possible to write down an equation for Δx based only on the previous sampled values, we have a difference equation. Here are some examples.

WORKED EXAMPLE No. 1

Use Euler's method to find the difference equation for the function

$$\frac{dx}{dt} + 2x = C$$

SOLUTION

Using Euler's forward equation

$$\frac{dx}{dt} = \frac{(x)_{n+1} - (x)_n}{T}$$

The equation is

$$\frac{(x)_{n+1} - (x)_n}{T} + 2(x)_n = C$$

This could be written as

$$\frac{\Delta(x)_n}{T} + 2(x)_n = C \quad \Delta(x)_n = CT - 2T(x)_n$$

This is the difference equation and allows us to predict the next value of x

$$x_{n+1} = x_n + \Delta x_n = x_n + CT - 2T x_n = x_n (1 - 2T) + CT$$

WORKED EXAMPLE No. 2

Use Euler's method to find the difference equation for the function

$$\tau \frac{dx}{dt} + x = C$$

SOLUTION

Using Euler's forward equation

$$\frac{dx}{dt} = \frac{(x)_{n+1} - (x)_n}{T}$$

The equation is

$$\tau \frac{(x)_{n+1} - (x)_n}{T} + (x)_n = \tau \frac{\Delta(x)_n}{T} + (x)_n = C$$

The difference equation may be written as

$$\Delta(x)_n = C \frac{T}{\tau} - \frac{T}{\tau} (x)_n$$

The next value is

$$(x)_{n+1} = (x)_n + \Delta(x)_n = (x)_n + C \frac{T}{\tau} - \frac{T}{\tau} (x)_n$$

$$(x)_{n+1} = (x)_n \left[\frac{T}{\tau} - 1 \right] + C \frac{T}{\tau}$$

WORKED EXAMPLE No. 3

Use Euler's method to find the difference equation for the function

$$A \frac{d^2x}{dt^2} + B \frac{dx}{dt} + Cx = 0$$

SOLUTION

We must substitute

$$\frac{d^2x}{dt^2} = \frac{(x)_{n+1} - 2(x)_n + (x)_{n-1}}{T^2} \quad \text{and} \quad \frac{dx}{dt} = \frac{(x)_{n+1} - (x)_n}{T}$$

$$A \frac{(x)_{n+1} - 2(x)_n + (x)_{n-1}}{T^2} + B \frac{(x)_{n+1} - (x)_n}{T} + C(x)_n = 0$$

$$A(x)_{n+1} - 2(x)_n + (x)_{n-1} + BT[(x)_{n+1} - (x)_n] + CT^2(x)_n = 0$$

$$A(x)_{n+1} + BT(x)_{n+1} - 2(x)_n + CT^2(x)_n - BT(x)_n + (x)_{n-1} = 0$$

$$(x)_{n+1}[A + BT] - (x)_n[2 - CT^2 + BT] + (x)_{n-1} = 0$$

$$(x)_{n+1} = \frac{(x)_n[2 - CT^2 + BT] + (x)_{n-1}}{[A + BT]}$$

SELF ASSESSMENT EXERCISE No. 1

1. Use Euler's method to find the difference equation for the function

$$\frac{dx}{dt} + x = 0$$

Answer

$$x_{n+1} = x_n\{1 - T\}$$

2. Use Euler's method to find the difference equation for the function

$$5 \frac{dx}{dt} + 2x - 3 = 0$$

Answer

$$x_{n+1} = x_n\{1 - 0.4T\} + 0.6T$$

3. Use Euler's method to find the difference equation for the function

$$\frac{d^2x}{dt^2} + 4 \frac{dx}{dt} + x = 0$$

Answer

$$x_{n+1} = \frac{x_n\{2 + 4T - T^2\} - x_{n-1}}{1 + 4T}$$

This work is all very well when dealing with known functions but in control where we need to process the error, the error is not a known function. The following shows how the work may be extended to this application.

3 Digitisation

3.1 Zero Order Hold

In order to turn an analogue signal into a digital signal the analogue signal must be sampled at regular intervals and converted into a series of digital numbers by using an ADC. The explanation of how these work is not covered here. Once a digital number has been created and stored in a register, it may be processed by the computer. The sampling of an analogue signal may be idealised as a switch closing for a very brief instant during which the digital value is obtained. This is repeated at regular intervals T seconds apart. The output is a series of pulses that are passed to the ADC and converted into a series of digital numbers.

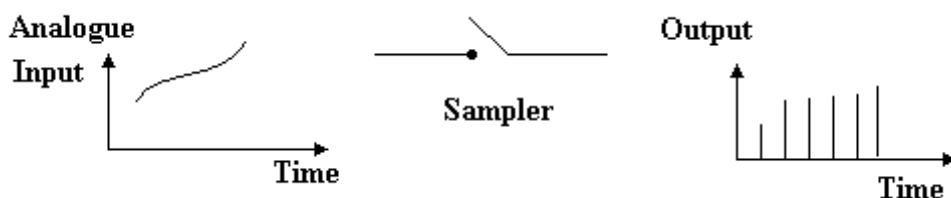


Figure 3

It is normal to use a unit to hold these values at the same level in between samples so the output is a staircase waveform and makes it easier for the ADC to read them. A unit that does this is called the zero order hold or ZOH.

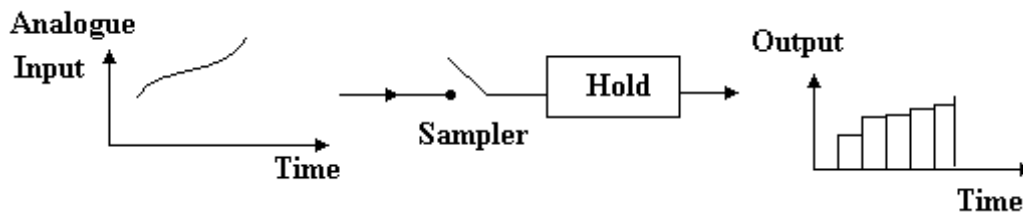


Figure 4

4. Digital Processing

A digital signal may be processed at any point. We are only going to examine processing the error signal by the control unit. The diagram shows a more complete block diagram of a digitally controlled system.

The error is produced by simply subtracting the digital numbers representing the input and output values. The error may then be processed to produce the required control action by applying arithmetical operations to the digital number.

With a suitable program, the error signal may be processed in any way such as changing the proportional constant, differentiating or integrating with respect to time.

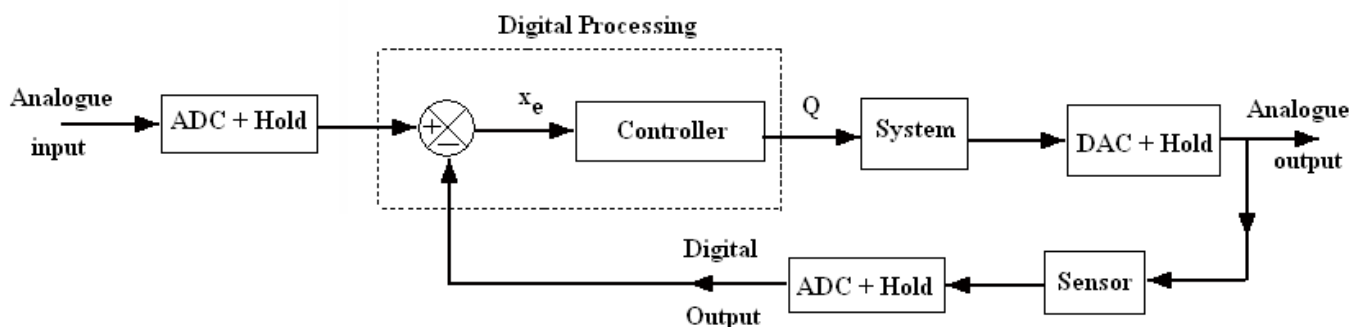


Figure 5

5 Incremental Calculations

Let's concentrate on the numerical solution for the output of a controller. The input is the error x_e and the output is Q . Suppose that x_e changes by an increment Δx_e . The output will change by an increment ΔQ . We need to find the difference equation for ΔQ . In this case we will not know the next sample $(x_e)_{n+1}$ and we cannot calculate them as in the previous section because the error is not a known function of time.



Figure 6

5.1 Proportional

The analogue equation is $Q = k_p x_e$

The incremental change is $\Delta Q = \Delta [k_p x_e] = k_p \Delta x_e$

The last two samples would be $(x_e)_n$ and $(x_e)_{n-1}$

It follows that

$$\Delta Q = k_p \{(x_e)_n - (x_e)_{n-1}\}$$

This may be easily computed and stored in a register. The proportional constant k_p is just a number stored in another register and used to multiply the error to produce the change required in the value of the output.

This is true regardless of the rules since we are using point values of x_e

The analogue equation is $Q = k_i \int x_e dt$ where k_i is a constant if required.

The incremental change is $\Delta Q = \Delta [k_i \int x_e dt] = k_i \Delta [\int x_e dt]$

Using the forward method, the incremental change to the output is $\Delta Q = k_i T (x_e)_{n-1}$

Using the backward method, the incremental change to the output is $\Delta Q = k_i T (x_e)_{n+1}$

Using the trapezoidal rule

$$\Delta Q = k_i T \left\{ \frac{(x_e)_n + (x_e)_{n-1}}{2} \right\}$$

5.3 Differential

The analogue equation is where k_d is a constant if required.

The incremental change is

$$\Delta Q = \Delta \left\{ k_d \frac{dx_e}{dt} \right\} = k_d \Delta \left\{ \frac{dx_e}{dt} \right\}$$

We use the current values $(x_e)_n$ and the previous $(x_e)_{n-1}$.

Using the forward method, (Euler's method)

$$\frac{dx_e}{dt} \rightarrow \frac{\Delta x_e}{T} = \frac{(x_e)_n - (x_e)_{n-1}}{T}$$

This becomes more accurate as $T \rightarrow 0$ The incremental change to the output is

$$\Delta Q = k_d \Delta \frac{(x_e)_n - (x_e)_{n-1}}{T}$$

This means that we need the change between successive calculations so we need the previous change as well.

$$\Delta Q = k_d \left[\frac{(x_e)_n - (x_e)_{n-1}}{T} - \frac{(x_e)_{n-1} - (x_e)_{n-2}}{T} \right]$$

$$\Delta Q = k_d \left[\frac{(x_e)_n - 2(x_e)_{n-1} + (x_e)_{n-2}}{T} \right]$$

5.4. Three Term Control Model

First let's revise the analogue model.

5.4.1 Analogue

This was covered in the previous tutorial and just restated here. 3 term control is widely used to enable a system to respond with respect to time in the best possible way. A basic block diagram of a closed loop system is shown.

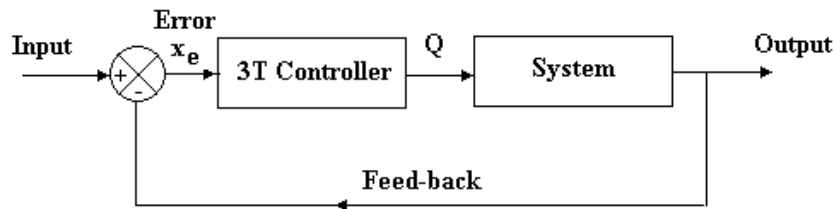


Figure 7

The output of the system is fed back negatively and compared with the input (set) value to produce an error. The system responds to the error such that the error is reduced to zero. The three terms used by the controller are Proportional, Integral and Differential. For this reason it is also called PID control.

The output of a PID controller is

$$Q(s) = k_d x_e \left[1 + \frac{1}{sT_i} + sT_d \right]$$

k_p is the proportional constant, T_i is the integral time and T_d the differential time.

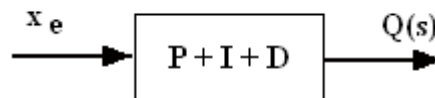


Figure 8

Now let's look at a the digital model

5.4.2 Digital

For all three control terms, the incremental change is the sum of the three difference equations derived earlier.

$$\Delta Q = k_p \left[(x_e)_n - (x_e)_{n-1} + k_i T (x_e)_n + k_d \frac{(x_e)_n - 2(x_e)_{n-1} + (x_e)_{n-2}}{T} \right]$$

The calculated incremental change must be added to last figure so the new output is

$$Q_n = Q_{n-1} + k_p \left[(x_e)_n - (x_e)_{n-1} + k_i T (x_e)_n + k_d \frac{(x_e)_n - 2(x_e)_{n-1} + (x_e)_{n-2}}{T} \right]$$

5.5. Dynamic Compensation

A dynamic controller uses a lead and lag term. The analogue model is

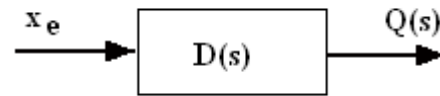
$$D(s) = \frac{Q(s)}{x_e(s)} = k_p \frac{s + A}{s + B}$$


Figure 9

If this is replaced with a digital controller, the difference equation we must use is found as follows.

$$D(s) = \frac{Q(s)}{x_e(s)} = k_p \frac{s + A}{s + B}$$

Rearrange as follows

$$(s + B)Q(s) = k_p(s + A)x_e(s)$$

$$sQ(s) + BQ(s) = k_p\{sx_e(s) + Ax_e(s)\}$$

Change into differential form

$$\frac{dQ}{dt} + BQ = k_p \left\{ \frac{dx_e}{dt} + Ax_e \right\}$$

Now use Euler's forwards method to digitize it and substitute

$$\frac{dQ}{dt} = \frac{\{(Q_n) - (Q_{n-1})\}}{T} \quad \text{and} \quad \frac{dx_e}{dt} = \frac{\{(x_e)_n - (x_e)_{n-1}\}}{T}$$

$$\frac{\{(Q_n) - (Q_{n-1})\}}{T} + BQ_{n-1} = k_p \frac{\{(x_e)_n - (x_e)_{n-1}\}}{T} + A(x_e)_{n-1}$$

Multiply through by T

$$(Q_n - Q_{n-1}) + TBQ_{n-1} = k_p\{(x_e)_n - (x_e)_{n-1}\} + TA(x_e)_{n-1}$$

Rearrange to make Q_n the subject

$$(Q_n - Q_{n-1}) = -TBQ_{n-1} + k_p\{(x_e)_n - (x_e)_{n-1}\} + TA(x_e)_{n-1}$$

$$(Q_n - Q_{n-1}) = T \left[-BQ_{n-1} + \left\{ k_p \left(\frac{(x_e)_n - (x_e)_{n-1}}{T} \right) + A(x_e)_{n-1} \right\} \right]$$

$$(Q_n) = Q_{n-1} + T \left[-BQ_{n-1} + \left\{ k_p \left(\frac{(x_e)_n - (x_e)_{n-1}}{T} \right) + A(x_e)_{n-1} \right\} \right]$$

This may be used to compute the new value of Q based on the last two samples.

6. z Transforms

6.1 Relationship of 'z' and 's'

It would be nice if we had some way of creating a numerical algorithm from the analogue transfer function and this is possible using the z transform. This is useful both for digitising the control function and for emulating system transfer functions. The basic process is first to turn the transfer function $G(s)$ into a transfer function $G(z)$ by using a transform called the 'z' transform. The theory behind this transform is not covered very thoroughly here and you should refer to advanced maths text books for this.

This is a good starting point for those who are not pure mathematicians but need an Engineer's approach.

The following is a brief explanation of how to interpret z transforms. Note that this can apply to any transfer function so we will drop the suffix 'e' on the input.

z^0 means sample number n (the last taken). Hence x_n becomes $z^0 x = x$ and Q_n becomes Q
 z^{-1} means sample number (n-1). Hence x_{n-1} becomes $z^{-1} x$ and Q_{n-1} becomes $z^{-1} Q$
 z^{-2} means sample number (n-2) Hence x_{n-2} becomes $z^{-2} x$ and Q_{n-2} becomes Qz^{-2}
 and so on.

With this, we can change the difference equations into an equation of z.
 Consider the analogue transfer function for an integrator.

$$Q = \int x dt \quad Q(s) = x(s)s^{-1} = \frac{x(s)}{s}$$

The equivalent incremental change is the area of the last strip $\Delta Q = T x_{n-1}$ (Euler method)

The new output is $Q_n = Q_{n-1} + T (x)_{n-1}$
 Change to z form $Q z^0 = Q z^{-1} + T z^{-1} x$
 Rearrange $Q\{1 - z^{-1}\} = T z^{-1} x$

$$Q(z) = \frac{Tz^{-1}x}{1 - z^{-1}} = \frac{Tx}{z - 1}$$

Compare

$$Q(z) = \frac{Tx(z)}{z - 1} \text{ and } Q(s) = \frac{x(s)}{s}$$

It would appear by comparison that

$$s \equiv \frac{z - 1}{T}$$

If we use the backwards method

The new output is $Q_{n+1} = Q_n + T (x)_{n+1}$
 Change to z form $Q z^1 = Q z^0 + T z^1 x$
 Rearrange

$$Q(z - 1) = Tz^1x = \frac{Tz^{-1}}{z - 1}$$

$$\text{Compare } Q(z) = \frac{Tz^{-1}}{z - 1} \text{ and } Q(s) = \frac{x(s)}{s}$$

It would appear by comparison that

$$s \equiv \frac{z - 1}{zT} = \frac{1 - z^{-1}}{T}$$

Let's see what happens if we use the trapezoidal rule instead.

$$Q_n = Q_{n-1} + T \left\{ \frac{x_n + x_{n-1}}{2} \right\}$$

Change to z form

$$Q = z^{-1}Q + \frac{T}{2} \{z^0x + z^{-1}x\}$$

Rearrange into a transfer function

$$Q = z^{-1}Q + \frac{Tx}{2} \{1 + z^{-1}\} \quad Q - z^{-1}Q = \frac{Tx}{2} \{1 + z^{-1}\}$$

$$Q(1 - z^{-1}) = \frac{Tx}{2} \{1 + z^{-1}\}$$

Multiply the top and bottom by z

$$\frac{Q}{x}(z) = \frac{T}{2} \left(\frac{1 + z^{-1}}{1 - z^{-1}} \right) = \frac{T}{2} \left\{ \frac{z + 1}{z - 1} \right\}$$

This is equivalent to the analogue equation

$$\frac{Q}{x}(s) = \frac{1}{s}$$

It would appear by comparison that

$$s = \frac{2}{T} \left\{ \frac{z - 1}{z + 1} \right\}$$

What we would really like to do is the reverse process, turning a transfer function of s into a difference equation for use in the computer programme. We have seen that the result is not the same for the three processes and great care should be taken when substituting for s to find the z form. You are advised to read this up in text books.

The following examples show how this is done.

WORKED EXAMPLE No. 4

Convert the transfer function into G(z)

$$G(s)s = \frac{Q}{x}(s) = \frac{6}{3s + 1}$$

Deduce the algorithm for a numerical process to find Q. Use Euler's forward rule. The sample time T is 0.1 seconds.

SOLUTION

Using the substitution from the Euler's rule

$$s \equiv \frac{z - 1}{T}$$

$$G(z) = \frac{6}{3 \left(\frac{z - 1}{T} \right) + 1} = \frac{6T}{3(z - 1) + T} = \frac{6T}{3z - 3 + T}$$

Make Q(z) the subject

$$Q(z) = \frac{6Tx(z)}{3z + (T - 3)} \quad Q(z)3z + (T - 3) = 6Tx(z)$$

Since the last sample taken is n, we must make zero the highest power and in this example we simply divide every term by z

Change this into sample number

$$3z^0Q + z^{-1}Q(T - 3) = 6Tz^{-1}x$$

$$3Q_n + Q_{n-1}(T - 3) = 6Tx_{n-1}$$

Now make Q_n the subject

$$Q_n = \frac{6Tx_{n-1} - Q_{n-1}(T - 3)}{3}$$

If the sample time is 0.1 seconds this becomes

$$Q_n = \frac{0.6x_{n-1} - Q_{n-1}(-2.9)}{3} = 0.2x_{n-1} + 0.967Q_{n-1}$$

The numerical calculation would be

$$3Q_n = 0.967Q_{n-1} + 0.2x_{n-1}$$

WORKED EXAMPLE No. 5

Convert the transfer function into $G(z)$

$$G(s)s = \frac{Q}{x}(s) = \frac{s + 10}{s + 5}$$

Deduce the algorithm for a numerical process to find Q . Use the trapezoidal rule. The sample time T is 0.1 seconds. Compare the result with that obtained from the formulae

$$Q_n = Q_{n-1} + T \left[-BQ_{n-1} + \frac{k_p \{x_n - x_{n-1}\}}{T} + Ax_{n-1} \right]$$

This was derived earlier using Euler's forward method.

SOLUTION

Using the substitution from the Trapezoidal rule

$$s = \frac{2(z - 1)}{T(z + 1)}$$

$$\frac{Q}{x}(z) = \frac{\frac{2(z - 1)}{T(z + 1)} + 10}{\frac{2(z - 1)}{T(z + 1)} + 5} = \frac{\frac{2(z - 1)}{0.1(z + 1)} + 10}{\frac{2(z - 1)}{0.1(z + 1)} + 5} = \frac{\frac{(z - 1)}{(z + 1)} + 0.5}{\frac{(z - 1)}{(z + 1)} + 0.25} = \frac{(z + 1)(z - 1) + 0.5(z + 1)}{(z + 1)(z - 1) + 0.25(z + 1)}$$

$$\frac{Q}{x}(z) = \frac{(z - 1) + 0.5(z + 1)}{(z - 1) + 0.25(z + 1)} = \frac{z - 1 + 0.5z + 0.5}{z - 1 + 0.25z + 0.25} = \frac{1.5z - 0.5}{1.25z - 0.75}$$

$$G(z) = \frac{1.5z - 0.5}{1.25z - 0.75} \text{ has been obtained from } G(s) = \frac{s + 10}{s + 5}$$

Make zero the highest power of z

$$\frac{Q}{X}(z) = \frac{1.5z^0 - 0.5z^{-1}}{1.25z^0 - 0.75z^{-1}}$$

$$1.25z^0 - 0.75z^{-1} = 1.5xz^0 - 0.5xz^{-1}$$

$$1.25Qz^0 = 0.75Qz^{-1} + 1.5xz^0 - 0.5xz^{-1}$$

$$Qz^0 = 0.6Qz^{-1} + 1.2xz^0 - 0.4xz^{-1}$$

Change this into sample numbers

$$Q_n = 0.6Q_{n-1} + 1.2x_n - 0.4x_{n-1}$$

This is the calculation to be performed by the computer based on two successive samples.

Based on Euler's method

$$Q_n = Q_{n-1} + T \left[-BQ_{n-1} + \frac{k_p \{x_n - x_{n-1}\}}{T} + Ax_{n-1} \right]$$

Put $T = 0.1$, $A = 10$, $B = 5$ and $k_p = 1$

$$Q_n = Q_{n-1} + 0.1 \left[-5Q_{n-1} + \frac{\{x_n - x_{n-1}\}}{0.1} + 10x_{n-1} \right]$$

$$Q_n = Q_{n-1} - 0.5Q_{n-1} + x_n - x_{n-1} + x_{n-1}$$

$$Q_n = 0.5Q_{n-1} + x_n$$

6.2 Final Value for a Unit Step Input

Eventually the value of Q and x may reach a constant value and so all the samples of Q and x will be the same. Taking the last, we find the final value is the same for both cases when x is constant.

$$Q_{n-2} = Q_{n-1} = Q_n = Q$$

$$x_{n-1} = x_n = x$$

$$Q_n = 0.5Q_{n-1} + x_n \quad Q_n = 0.6Q_{n-1} + 1.2x_n - 0.4x_{n-1}$$

$$Q = 0.5Q + x \quad Q = 0.6Q + 1.2x - 0.4x$$

$$Q = 2x \quad 0.4Q = 0.8x$$

$$Q = 2x$$

6.3 More on z Transforms

The problems highlighted previously result from using approximations in the difference equations. If we conducted rigorous mathematics to find the z transform we would obtain a table similar to the table for Laplace transforms. Here is an abbreviated version of that table.

	Time Domain	s Domain	z Domain
Unit Pulse		1	1 for the first sample and zero for all others
Unit Step	1	$\frac{1}{s}$	$\frac{z}{z-1} = \frac{1}{1-z^{-1}}$
Unit Ramp	t	$\frac{1}{s^2}$	$\frac{Tz}{(z-1)^2}$
Exponential	e^{-at}	$\frac{1}{s+a}$	$\frac{z}{z-e^{-aT}}$

6.4 Final Value Theorem

As with 's' transfer functions we can find the final value of 'z' transfer functions under specified conditions. The maths is not covered here.

The final value theorem for the transfer function

$$G(s) = \frac{\theta_o}{\theta_i} \text{ is } \theta_o = s\theta_i \text{ as } s \rightarrow 0$$

The equivalent for

$$G(z) = \frac{Q}{x} \text{ is } Q = (1 - z^{-1})x(z) \text{ as } z \rightarrow 1$$

For a unit step change

$$x(z) = \frac{1}{1 - z^{-1}} \text{ (see table of transforms)}$$

Hence the previous two examples produce the results

$$G(z) = \frac{Q}{x}(z) = \frac{1.5z^0 - 0.5z^{-1}}{1.25z^0 - 0.75z^{-1}}$$

$$Q = \frac{((1 - z^{-1}) \times 1.5z^0 - 0.5z^{-1})}{(1 - z^{-1}) \times (1.25z^0 - 0.75z^{-1})} = \frac{1.5z^0 - 0.5z^{-1}}{1.25z^0 - 0.75z^{-1}} \text{ put } z = 1 \quad Q = \frac{1.5 - 0.5}{1.25 - 0.75} = \frac{1}{0.5} = 2$$

This is the result obtained previously.

Note that the $1 - z^{-1}$ terms will always cancel for a unit step and all values of z^n are 1. It follows that the steady state value is simply the transfer function evaluated with $z = 1$.

WORKED EXAMPLE No. 6

A digital process has a transfer function

$$G(z) = \frac{y}{x}(z) = \frac{0.2z^1}{z^2 - 0.5z + 1}$$

Produce difference equations and evaluate the steady state response to a unit step.
Calculate the output value after 7 samples.

SOLUTION

First rearrange so that the highest power of z is 0. In this case we achieve this by dividing all terms by z^2 . The equation is then rearranged to make $y z^0$ the subject.

$$\frac{y}{x}(z) = \frac{0.2z^{-1}}{z^0 - 0.5z^{-1} + z^{-2}} = \frac{0.2z^{-1}}{1 - 0.5z^{-1} + z^{-2}}$$

$$y(1 - 0.5z^{-1} + z^{-2}) = 0.2xz^{-1} \quad y - 0.5yz^{-1} + yz^{-2} = 0.2xz^{-1}$$

$$y = 0.5yz^{-1} - yz^{-2} + 0.2xz^{-1}$$

Change the format to show the sample numbers

$$y_n = 0.5y_{n-1} - y_{n-1} + 0.2x_{n-1}$$

This is the difference equation.

Steady State Value

$$y_{n-2} = y_{n-1} = y_n = y \quad x_{n-1} = x_n = x$$

$$y = 0.5y - y + 0.2x \quad 1.5y = 0.2x$$

$$y = 0.1333 x = 0.1333$$

Using the final value theorem for a unit step change

$$x(z) = \frac{1}{1 - z^{-1}} \text{ (see table of transforms)}$$

$$G(z) = \frac{y}{x}(z) = \frac{0.2z^{-1}}{1 - 0.5z^{-1} + z^{-2}}$$

$$y(z) = \frac{x(z) \times 0.2z^{-1}}{1 - 0.5z^{-1} + z^{-2}} \text{ and the final value is } y(z) = (1 - z^{-1})x(z) \frac{0.2z^{-1}}{(1 - 0.5z^{-1} + z^{-2})}$$

$$y(z) = \frac{(1 - z^{-1})}{(1 - z^{-1})} \frac{0.2z^{-1}}{(1 - 0.5z^{-1} + z^{-2})}$$

Put $z = 1$

$$y(z) = \frac{0.2}{(1 - 0.5 + 1)} = 0.133$$

Evaluating each step is a case putting values in the difference equation. $y_{(n)} = 0.5y_{(n-1)} - y_{(n-2)} + 0.2x_{(n-1)}$
 All values of x are 1 for a unit step.

Sample	x_n	x_{n-1}	$y_{(n-1)}$	$y_{(n-2)}$	$0.5 y_{(n-1)}$	$-y_{(n-2)}$	$0.2 x_{(n-1)}$	$y_{(n)}$
n=0 (t=0)	1	0	0	0	0	0	0	0
n=1 (t=T)	1	1	0	0	0	0	0.2	0.2
n=2 (t=2T)	1	1	0.2	0	0.1	0	0.2	0.3
n=3 (t=3T)	1	1	0.3	0.2	0.15	-0.2	0.2	0.15
n=4 (t=4T)	1	1	0.15	0.3	0.075	-0.3	0.2	-0.025
n=5 (t=5T)	1	1	-0.025	0.15	-0.0125	-0.15	0.2	0.0375
n=6 (t=6T)	1	1	0.0375	-0.025	0.0187	0.025	0.2	0.243
n=7 (t=7T)	1	1	0.243	0.0375	0.1215	-0.0375	0.2	0.284

The next example illustrates how long division may be used to determine the outputs.

WORKED EXAMPLE No. 7

A module has a difference equation $y_n = ay_{n-1} + x_n$

Express this as

$$G(z) = \frac{y}{x}(z)$$

Determine the first 4 sampled outputs when the input is a unit pulse.

SOLUTION

$$y_n = ay_{n-1} + x_n$$

Change into z form

$$y z^0 = ay z^{-1} + x z^0 \quad y(z^0 - az^{-1}) = x z^0$$

$$G(z) = \frac{y}{x}(z) = \frac{z^0}{z^0 - az^{-1}} = \frac{1}{1 - az^{-1}}$$

Now divide out using long division.

$$\begin{array}{r}
 1 + a z^{-1} + a^2 z^{-2} + a^3 z^{-3} \dots\dots\dots \\
 1 - a z^{-1} \overline{) 1} \\
 \underline{1 - a z^{-1}} \\
 a z^{-1} + 0 \\
 a z^{-1} - a^2 z^{-2} \\
 \underline{a^2 z^{-2} + 0} \\
 a^2 z^{-2} - a^3 z^{-3} \\
 \underline{a^3 z^{-3} + 0}
 \end{array}$$

The result is a series

$$G(z) = \frac{y}{x}(z) = 1 + az^{-1} + a^2z^{-2} + a^3z^{-3} \dots$$

This form makes it easy to evaluate the output after every sample.

$$G(z) = 1 + az^{-1} + a^2z^{-2} + a^3z^{-3} \dots$$

Put back into sample form

$$y(z) = x + ax_{n-1} + a^2x_{n-2} + a^3x_{n-3} \dots$$

For a unit pulse $x(z) = 1$ for the starting value and is zero for all others after (see table of transforms)

The starting value	$y(0) = 1 + 0 + 0 \dots$	$= 1$
The next value is	$y(1) = 0 + a + 0 \dots$	$= a$
The next value is	$y(2) = 0 + 0 + a^2 \dots$	$= a^2$
The next value is	$y(3) = 0 + 0 + 0 + a^3 \dots$	$= a^3$

SELF ASSESSMENT EXERCISE No. 2

1. Find the first four values of y if x is a unit step and related to y by

$$\frac{y}{x}(z) = \frac{0.3z^1}{0.5z^2 - 0.4z + 0.8}$$

(Answers 0, 0.6, 1.08 and 0.504)

Find the steady state value of y for a step input of 1. ($y = 0.333$)

2. Find the first five values of y if x is a unit step and related to y by

$$\frac{y}{x}(z) = \frac{z^{-1}}{z^2 - 0.8z + 1}$$

(Answers 0, 1, 1.8, 1.44, 0.352)

Find the steady state value of y for a step input of 1. ($y = 0.833$)

3. An analogue system shown below is controlled by using unit feedback and a controller with a transfer function

$$D(s) = \frac{10}{s + 13}$$

The controller is to be replaced with a digital system using a zero hold sampler with DAC and ADC units.

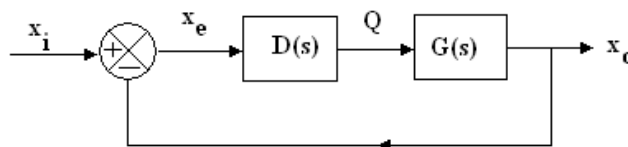


Figure 10

Sketch the block diagram of the digital system. Find the difference equation for the controller using Euler's forward rule.

The input and output of the system is zero when the input is changed to a value of 1. Given $T = 80 \mu s$ determine the digital output after 4 samples and the steady state value.

7 *Sampling Error*

This is a major topic and only a brief description is given below.

7.1 *Time Lag*

Note that the graph representing the output is not an analogue signal but a plot of the digital values against time. We should note that sampling produces a time lag in the signal of $T/2$ seconds and this could be important in some circumstances. The sampling rate is important and should be fast enough to produce a good model so it should be much faster than the frequency of any waveform being sampled.

7.2 *Aliasing*

Aliasing is a phenomenon produced when the sampling frequency is almost the same as the frequency of the signal being sampled or a multiple of it. This affect is seen on cinema films when the wheels of a cart appear to rotate forwards because the frame speed (sample rate) is slightly less than the wheel speed. Another name for this affect is stroboscopic. The same effect may be observed with a stroboscope when shone onto a rotating wheel. If the frequency is the same as the speed of rotation, the wheel appears to be static. If the frequency is changed from slightly above to slightly below the wheel speed, the wheel may appear to be running one way and then the other. If the stroboscope is set to multiples of the wheel speed, multiple images are produced. In a digital system (e.g. measuring speed), this affect produces errors. In most systems where the signal changes relatively slowly compared to the sampling rate, the effect is caused by high frequency noise (interference). It is necessary to filter out these high frequencies in the feedback signal with an *Anti-Aliasing Filter*.