INSTRUMENTATION AND CONTROL
TUTORIAL 10

MATCHING RESPONSES WITH THE STANDARD 2^{nd} ORDER SYSTEM

This tutorial is of interest to any student studying Control System Engineering and is set at NVQ level 5 and 6

On completion of this tutorial, you should be able to do the following.

- Define the parameters that define the step response of second order system.
- Define the parameters that define the harmonic response of second order system.
- Produce parameters for higher order systems that give the closest resemblance to the second order responses.
- Solve exam standard problems on system response matching techniques.

In order to complete this tutorial you need to have studies all the previous tutorials in depth.

Contents

1. *Review the Standard Second Order Equation*

It was shown in an earlier tutorial that the standard second order transfer function can be presented in the following forms.

In terms of the time constant T

$$\frac{\theta_o}{\theta_i}(s) = \frac{k}{T^2s^2 + 2T\delta s + 1}$$

In terms of the natural frequency ω_n

$$\frac{\theta_o}{\theta_i}(s) = \frac{\omega_n^2 k}{s^2 + 2\delta\omega_n s + \omega_n^2} \quad \text{or} \quad \frac{k}{s^2/\omega_n^2 + 2\delta s/\omega_n + 1}$$

In terms of the poles (polynomial)

$$\frac{\theta_o}{\theta_i}(s) = \frac{\omega_n^2 k}{(s+x)(s+y)} \quad \text{or} \quad \frac{\omega_n^2 k}{(s-p_1)(s-p_2)}$$

In terms of two parameters σ and ω_r

$$\frac{\theta_o}{\theta_i}(s) = \frac{\omega_n^2 k}{(s+\sigma)^2 + \omega_r^2} = \frac{\omega_r^2 + \sigma\omega_r^2}{(s+\sigma)^2 + \omega_r^2}$$

Any of these may be chosen to find the step response or the harmonic response.

2. *Parameters of a Step Response for Standard 2nd Order System*

It was shown in an earlier tutorial that the time solution for the step response is

$$\theta_o(t) = 1 - e^{-\sigma t}\left\{\cos(\omega_r t) + \frac{\sigma}{\omega_r}\sin(\omega_r t)\right\}$$

A typical plot is shown below and illustrates the decaying oscillations of a damped system. **The step response is often used to define the behaviour of a system** and we need to be familiar with certain definitions for positive damping ratio δ. The diagram shows these definitions or parameters.

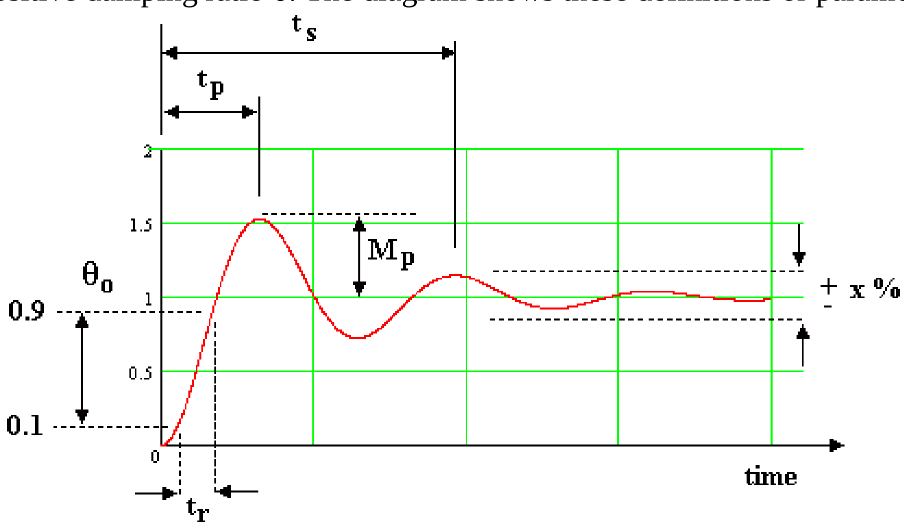


Figure 1

Settling Time	t_s is the time taken for the system to settle to a specified level ± x%
Rise Time	t_r is the time taken for the output to rise from 10% to 90% of the step.
Overshoot	M_p is the maximum overshoot.
Peak Time	t_p is the time taken to reach the peak value.

From the normalised response graph we can see that the rise time is very approximately the same for most values of damping and taking a mean value yields

$$\mathbf{t_r = \frac{1.8}{\omega_n}}$$

Now let's look at how to find t_p. At the first peak the gradient $d\theta_o/dt$ is zero. If we solve the response as a function of time and then differentiate we may find t_p. We must differentiate θ_o and equate to zero.

$$\frac{d\theta_o}{dt} = \frac{d\left[1 - e^{-\sigma t}\left\{\cos(\omega_r t) + \frac{\sigma}{\omega_r}\sin(\omega_r t)\right\}\right]}{dt} = 0$$

Without proof this comes turns out to be

$$\mathbf{t_p = \frac{\pi}{\omega_d}}$$

By substituting this back into the time equation we can also find that

$$\mathbf{M_p = e^{\frac{-\pi\delta}{\sqrt{1-\delta^2}}}}$$

This only applies to positive damping.

Finally let's look at the settling time. This is found by working out the time to reach a % of the step change. The decaying oscillation follows an exponential law $Ae^{-\delta\omega n t}$ but it can be shown that the deviation from the settling level for a unit step is

$$e^{-\delta\omega_n t} \text{ and so } \mathbf{t_s = -\frac{\ln(x\%)}{\delta\omega_n}}$$

Summary

$$\mathbf{t_r \approx \frac{1.8}{\omega_n} \qquad t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\delta^2}} \qquad M_p = e^{\frac{-\pi\delta}{\sqrt{1-\delta^2}}} \qquad t_s = -\frac{\ln(x\%)}{\delta\omega_n}}$$

We might measure these parameters in a test and deduce the effective damping ratio and or natural frequency.

WORKED EXAMPLE No. 1

A standard second order system has a damping ratio of 0.2 and a natural frequency of 10 rad/s. Calculate the rise time, the peak time, the peak value and the 2% settling time for a unit step input.

SOLUTION

$\delta = 0.2 \quad \omega_n = 10$

$$t_r \approx \frac{1.8}{\omega_n} = 0.18 \text{ s} \qquad t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\delta^2}} = 0.32 \text{ s}$$

$$M_p = e^{\frac{-\pi\delta}{\sqrt{1-\delta^2}}} = e^{-0.64} = 0.527 \text{ units}$$

$$t_s = -\frac{\ln(x\%)}{\delta\omega_n} 1.96 \text{ s}$$

SELF ASSESSMENT EXERCISE No. 1

1. A standard second order system has a damping ratio of 0.5 and a natural frequency of 40 rad/s. Calculate the damped resonant frequency, the rise time, the peak time, the peak value and the 2% settling time for a unit step input. (34.64 rad/s, 0.045, 0.091 s, 0.163 units and 0.196 s)

2. A standard second order system is subjected to a unit step input and the first overshoot is 30%. Calculate the damping ratio. (0.358)

   The ±2% settling time is found to be 1.5 seconds. Calculate the natural frequency of the system and the damped resonant frequency. (7.288 rad/s)

   What will be the expected time taken to reach the first peak? (0.462 s)

3. Parameters of a Harmonic Response For Standard 2nd Order System

When a second order system is subjected to a sinusoidal input, the output is sinusoidal with a change in magnitude and phase. This has been well covered in previous tutorials. The peak magnitude occurs at the damped resonant frequency and is given by:-

$$M_p = \frac{1}{2\delta\sqrt{1-\delta^2}}$$

The damped resonant frequency is given by

$$\omega_r = \omega_n\sqrt{1-\delta^2}$$

δ is the damping ratio and ω_n the natural frequency.
To show where this comes from we should examine the frequency response of a standard second order system.

$$\frac{\theta_o}{\theta_i}(s) = \frac{k}{\frac{s^2}{\omega_n^2} + \frac{2\delta s}{\omega_n} + 1} \qquad \frac{\theta_o}{\theta_i}(j\omega) = \frac{k}{\frac{-\omega^2}{\omega_n^2} + \frac{2j\delta\omega}{\omega_n} + 1}$$

$$\frac{\theta_o}{\theta_i}(j\omega) = \frac{k}{(1-\omega^2/\omega_n^2) + 2j\delta\omega/\omega_n} \times \frac{(1-\omega^2/\omega_n^2) - 2j\delta\omega/\omega_n}{(1-\omega^2/\omega_n^2) - 2j\delta\omega/\omega_n}$$

$$\frac{\theta_o}{\theta_i}(j\omega) = \frac{k\left(\frac{1}{\omega_n^2}\right) - \frac{2j\delta\omega}{\omega_n}}{\left(\frac{1-\omega^2}{\omega_n^2}\right) + \frac{2j\delta\omega}{\omega_n}} = k\frac{A - jB}{A^2 + B^2} \qquad \left|\frac{\theta_o}{\theta_i}\right| = k\sqrt{\left(\sqrt{\frac{A}{A^2+B^2}}\right)^2 \left(\sqrt{\frac{B}{A^2+B^2}}\right)^2}$$

$$\left|\frac{\theta_o}{\theta_i}\right| = k\sqrt{\left\{\frac{(\omega_n^2)^2}{(\omega_n^2 - \omega^2)^2 + (2\delta\omega\omega_n)^2}\right\}} = k\sqrt{\left\{\frac{1}{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\delta\frac{\omega}{\omega_n}\right)^2}\right\}}$$

$$\left|\frac{\theta_o}{\theta_i}\right| \text{db} = 20\log\left[k\sqrt{\left\{\frac{1}{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(2\delta\frac{\omega}{\omega_n}\right)^2}\right\}}\right] \quad \text{The phase angle is } \phi = \tan^{-1}\frac{2\delta\omega\omega_n}{\omega_n^2-\omega^2}$$

We can now plot the Bode diagram and the result is shown for various degrees of damping.

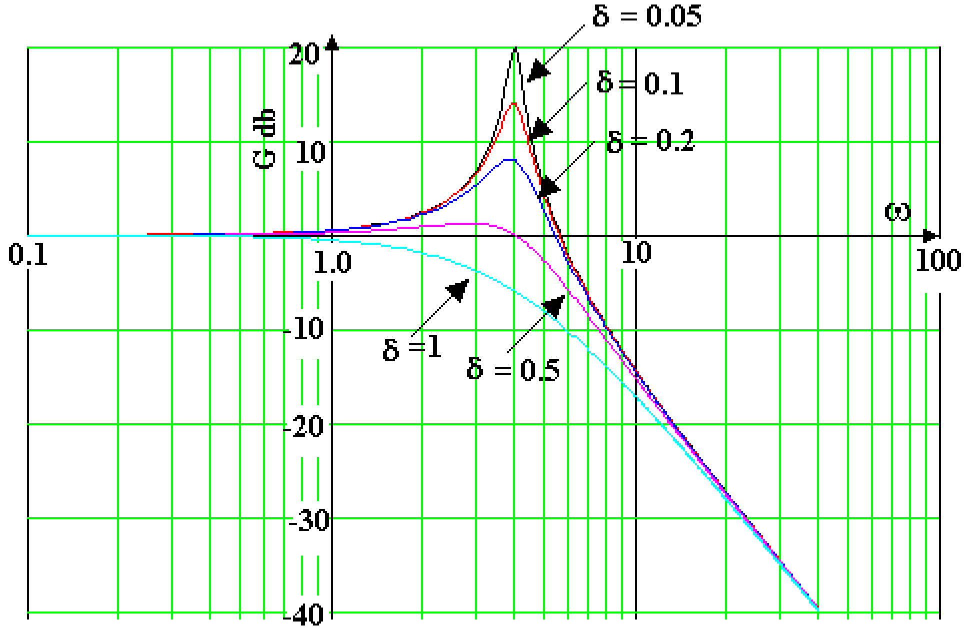


Figure 2

If we take a typical example, the parameters that define it are the bandwidth and the peak magnitude. The peak magnitude occurs at the resonant frequency and this is not the same as the natural frequency although they are usually close. The bandwidth is the frequency at -3 db

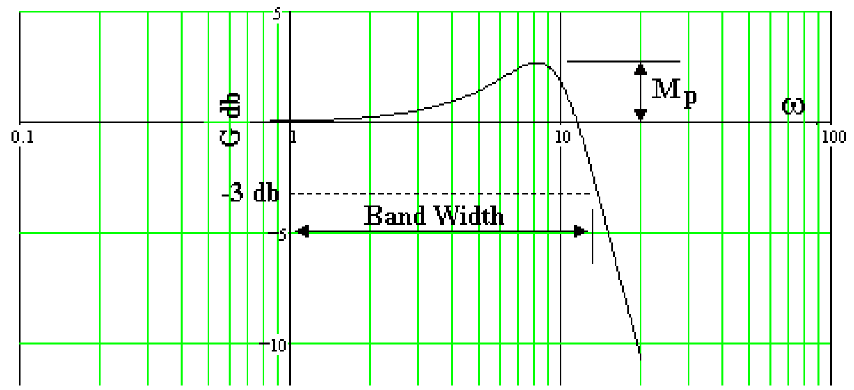


Figure 3

The PEAK M occurs when

$$\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(2\delta\frac{\omega}{\omega_n}\right)^2 \text{ is a maximum}$$

This can be found by max and min theory. Simplify this to

$$(1-r^2)^2+(2\delta r)^2$$

Differentiate with respect to r

$$\frac{d\{(1-r^2)^2+(2\delta r)^2\}}{dr} = 2(1-r^2)(-2r)+8\delta^2 r$$

Equate to zero and

Peak M occurs when

$$2(1-r^2)(-2r)+8\delta^2 r=0 \quad r=\sqrt{1-2\delta^2}$$

$$M_p=\frac{1}{2\delta\sqrt{1-\delta^2}}$$

The resonant frequency is

$$\frac{\omega}{\omega_n}=\sqrt{1-2\delta^2} \text{ so } \omega_r=\omega_n\sqrt{1-2\delta^2}$$

WORKED EXAMPLE No. 2

A second order system produces a peak magnitude of 2.2 at 5 Hz. Calculate the damping ratio and natural frequency. Write down the transfer function.

SOLUTION

$$M_p=2.2=\frac{1}{2\delta\sqrt{1-\delta^2}} \quad (1-\delta^2)\times 4.4^2\delta^2=1 \quad 19.36\delta^2-19.36\delta^4=1$$

Let $=\delta^2=x$ and solve x

$$19.36x-19.36x^2+1=0$$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{19.36\pm\sqrt{19.36^2-4\times 19.36\times 1}}{38.72}=\frac{19.36\pm 17.24}{38.72}=0.945 \text{ or } 0.0547$$

$\delta=\sqrt{x}$ = 0.972 or 0.234 the higher figure would give a negative resonant frequency so δ = 0.234

$$\omega_r=2\pi\times 5=31.42=\omega_n\sqrt{1-2\times 0.234^2}=0.9436\omega_n$$

$$\omega_n=33.296 \text{ rad/s}$$

$$\frac{\theta_o}{\theta_i}(s)=\frac{33.65^2}{s^2+2(0.234)(33.296)s+33.296^2}$$

4. *Matching Higher Order Systems to the Standard Response*

Many higher order transfer functions produce a step or harmonic response similar to that of a standard second order system. It is possible to produce a second order transfer function that approximates the system. There are many tools that can be used and these are demonstrated in the following examples.

WORKED EXAMPLE No. 3

A system has an open loop transfer function

$$G(s) = \frac{2}{s(s+1)+(s+2)}$$

It is used in a closed loop with unity feed back.

(a) Verify that the system is stable by applying the Routh-Hurwitz criterion.

(b) Plot the Nyquist diagram and verify the stability.

(c) Create a Nichols plot.

(d) Using results from the Nichols plot create a frequency response for the closed loop system and demonstrate that it resembles the plot for a typical standard second order system.

(e) Determine the damping ratio and natural frequency of the approximated second order system and find the ±2% settling time of a unit step response.

(a) **SOLUTION**

Routh Hurwitz criterion - First multiply out

$$G(s) = \frac{2}{s^3 + 3s^2 + 2s}$$

The closed loop transfer function for unity feedback is

$$G_{cl} = \frac{1}{G(s) + 1s}$$

Characteristic equation is

$$\frac{2}{s^3 + 3s^2 + 2s} + 1 = 0 \qquad s^3 + 3s^2 + 2s + 2 = 0$$

If the highest power of the characteristic equation is 3 the criteria may be simplified as follows.

Characteristic equation = ($as^3 + bs^2 + cs + d$) so a = 1 b = 3 c = 2 d = 2

If the next coefficient is negative the system is unstable and this is given by

$$R = c - \frac{ad}{b} = 2 - \frac{1 \times 2}{3} = 1.33$$

This is positive so the system is stable.

(b) **SOLUTION**

Evaluate the gain and phase angles for

$$G_1 = \frac{2}{s} \qquad G_2 = \frac{1}{s+1} \qquad G_3 = \frac{1}{s+2}$$

A suitable frequency range is 0.1 to 1.6 rad/s. In general if we have

$$\frac{K}{n+Ts} \text{ giving a gain of } G = \frac{K}{\sqrt{n^2+\omega^2T^2}} \text{ and phase angle } \phi = -\tan^{-1}\left(\frac{\omega T}{n}\right)$$

To find the overall result we use $G = G_1\, G_2\, G_3$ and $\phi = \phi_1 + \phi_2 + \phi_3$ A suitable frequency range must be found by experimenting with figures. Here is the result.

The Nyquist plot verifies that the system is stable since the -1 point is not enclosed. (This needs more results around $\omega = 0.5$)

ω	G_1	G_2	G_3	G	ϕ_1	ϕ_2	ϕ_3	ϕ
0.1	19.999	0.499	0.995	9.937	-89.427	-2.862	-5.711	-98
0.2	10	0.498	0.981	4.879	-89.714	-5.711	-11.31	-106.734
0.3	6.667	0.494	0.958	3.157	-89.809	-8.531	-16.699	-115.039
0.4	5	0.49	0.928	2.276	-89.857	-11.31	-21.801	-122.968
0.5	4	0.485	0.894	1.735	-89.885	-14.036	-26.565	-130.487
0.6	3.333	0.479	0.857	1.369	-89.905	-16.699	-30.964	-137.568
0.7	2.857	0.472	0.819	1.105	-89.918	-19.29	-34.992	-144.2
0.8	2.5	0.464	0.781	0.906	-89.928	-21.801	-38.66	-150.39
0.9	2.222	0.456	0.743	0.753	-89.936	-24.228	-41.987	-156.151
1	2	0.447	0.707	0.632	-89.943	-26.565	-45	-161.508
1.1	1.818	0.438	0.673	0.536	-89.948	-28.811	-47.726	-166.485
1.2	1.667	0.429	0.64	0.457	-89.952	-30.964	-50.194	-171.11
1.3	1.538	0.419	0.61	0.393	-89.956	-33.024	-52.431	-175.411
1.4	1.429	0.41	0.581	0.34	-89.959	-34.992	-54.462	-179.413
1.5	1.333	0.4	0.555	0.296	-89.962	-36.87	-56.31	-183.142
1.6	1.25	0.39	0.53	0.259	-89.964	-38.66	-57.995	-186.619

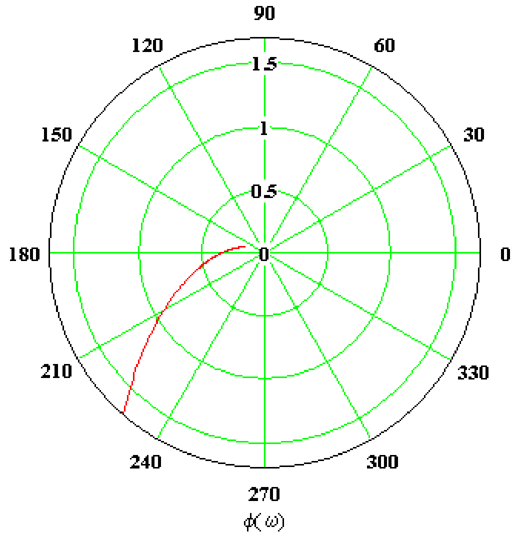


Figure 4

(c) **SOLUTION**

To produce a Nichols plot we need the gain in db so G db = 20 log G

ω	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2
ϕ	-107	-123	-138	-150	-162	-171	-179	-187	-193	-198
G	4.9	2.3	1.4	0.91	0.63	0.46	0.34	0.26	0.2	0.16
Gdb	13.8	7.1	2.7	-0.9	-4	-6.8	-9.4	-11.8	-14	-16

The Nichol's plot is as shown.

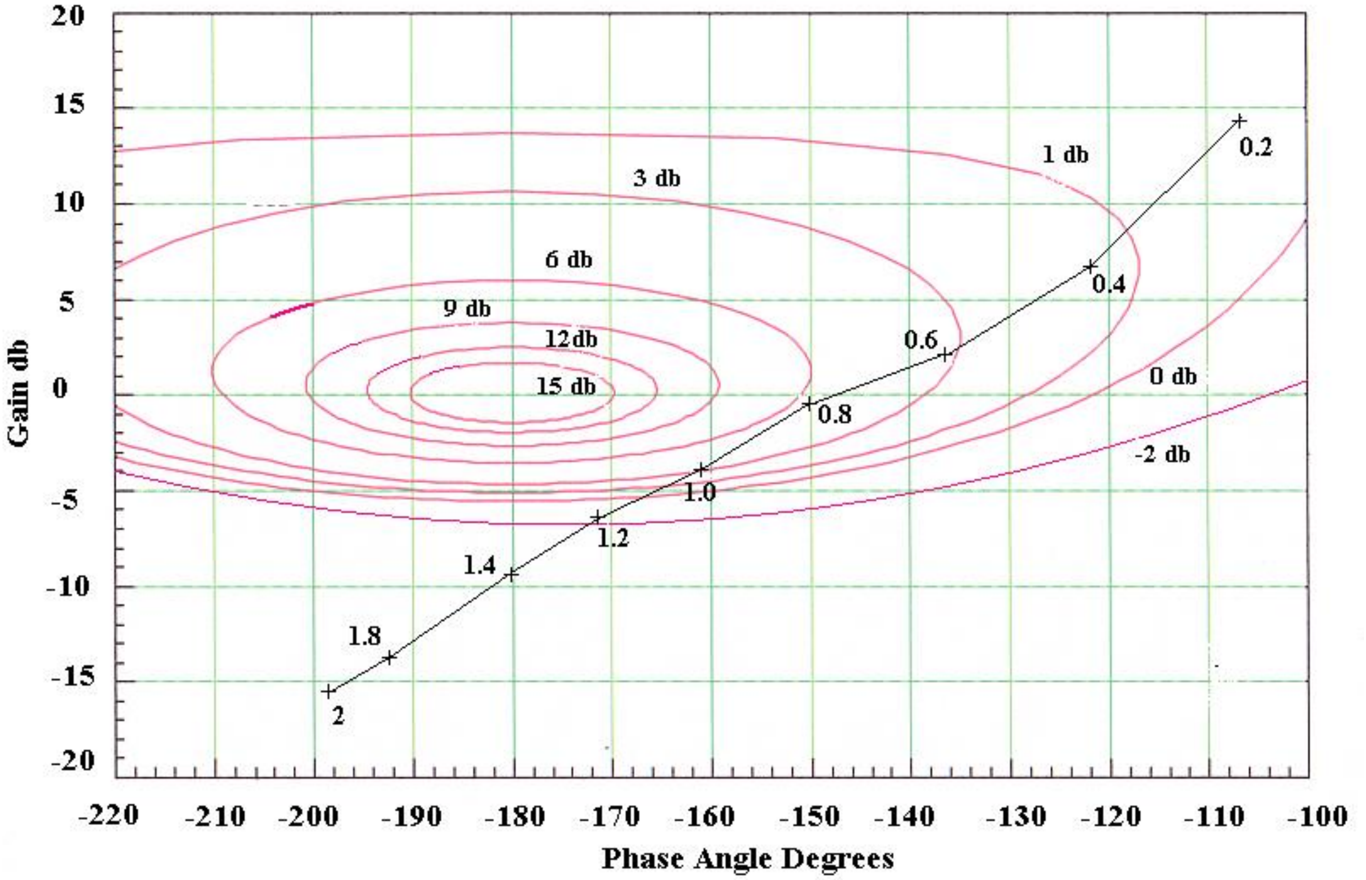


Figure 5

Now we can pick off values of closed loop gain at various frequencies and plot the frequency response. Here is the result. It is typical of a second order system.

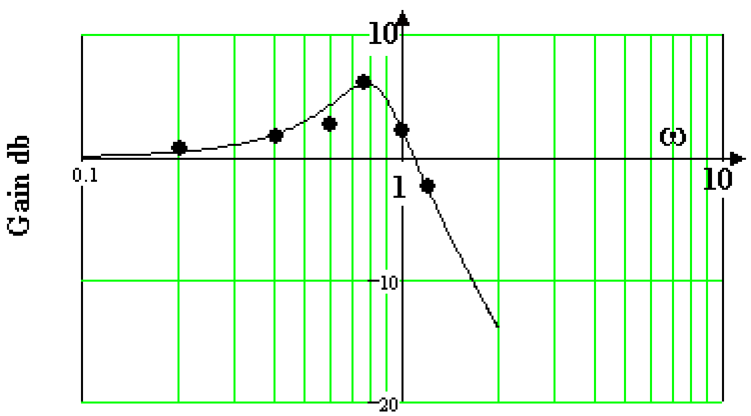


Figure 6

The peak magnification is at 0.8 rad/s and the closed loop gain is 6 db. 20 log M_p = 6 M_p = 2

$$M_p(\omega) = 2 = \frac{1}{2\delta\sqrt{1-\delta^2}} \qquad 4\delta\sqrt{1-\delta^2} = 1 \qquad 16\delta^2(1-\delta^2) = 1$$

Let $= \delta^2 = x$ and solve x $\qquad 16x(1-x) = 1 \qquad -16x^2 + 16x - 1 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-16 \pm \sqrt{16^2 - 4 \times 16 \times 1}}{-32} = \frac{-16 \pm 13.86}{-32} = 0.933 \text{ or } 0.067$$

$\delta = \sqrt{x} = 0.966$ or 0.259. The higher figure would not produce a peak as it would be critically damped.

The damping ratio is $\delta = 0.259$

Now find

$$\omega_r = \omega_n\sqrt{1-\delta^2} \qquad \omega_n = \frac{0.8}{\sqrt{1-0.259^2}} = 0.828 \text{ rad/s}$$

The standard second order transfer function is

$$G(s) = \frac{\omega_n^2}{s^2 + 2s\delta\omega_n + \omega_n^2}$$

Putting in the values found we get

$$G(s) = \frac{0.69}{s^2 + 0.43s + 0.69}$$

To prove the point, the solid line on the previous diagram is a true computer plot of this standard second order system and we can see it is a very good fit.

(d) SOLUTION

The settling time is found from the equation

$$e^{\delta\omega_n t_s} = \% \text{ overshoot} = 0.02$$

$$\ln(0.02) = -3.912 = \delta\omega_n t_s$$

$$t_s = \frac{3.912}{0.259 \times 0.828} = 18.3 \text{ s}$$

SELF ASSESSMENT EXERCISE No. 2

A system has an open loop transfer function

$$\frac{K}{s(\tau s + 1)}$$

It is used in a closed loop with unity feed back. A low frequency square wave input is applied and with gain K set to 20 and the system output indicates a maximum overshoot of 40% and requires 0.33 seconds to reach the peak value.

(a) Plot the open loop transfer function on a Nichols chart and establish the gain margin to limit the closed loop magnitude ratio to 4 db.

(b) Sketch the form of the closed loop frequency response on semi log paper.

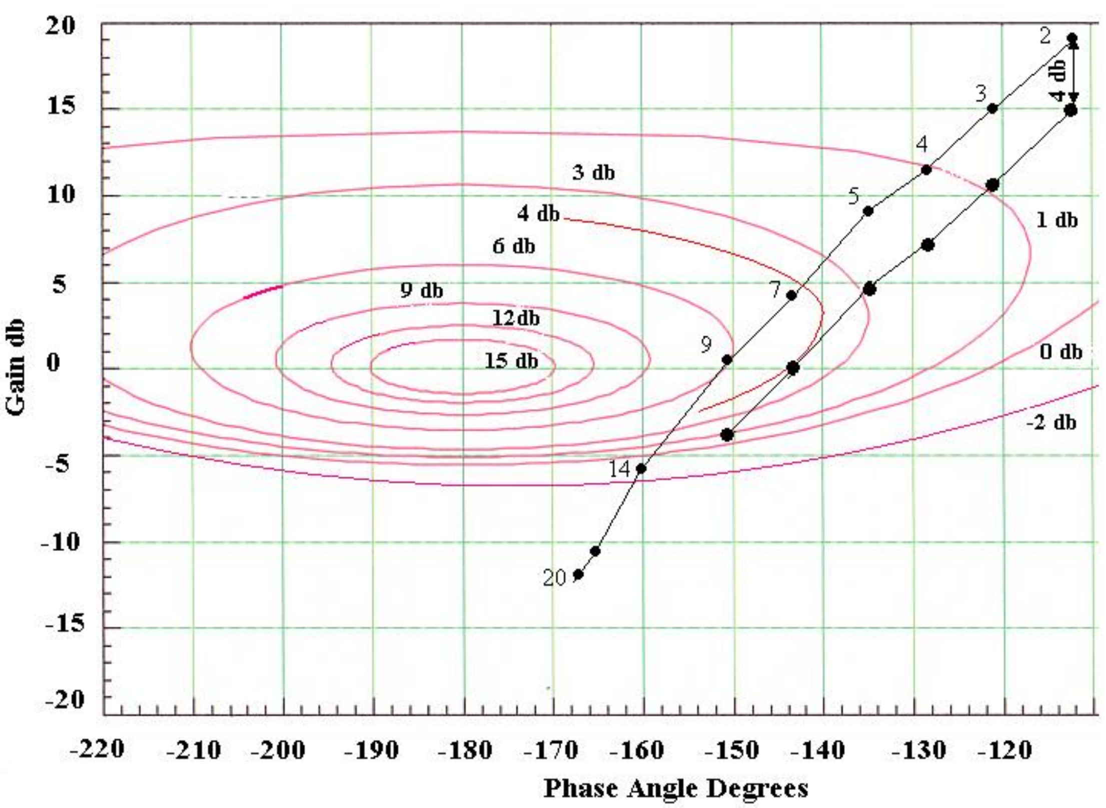
You are given the following information.
Time to reach peak value

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\delta^2}} \quad \text{The peak value is} \quad Mp_t = 1 + e^{\frac{-\delta\pi}{\sqrt{1-\delta^2}}}$$

ANSWERS – the following page is provided for you to check your work. You should not look at it before attempting the question

Answers should yield $\delta = 0.28$ $\omega_n = 9.917$ rad/s $\tau = 0.203$

The Nichols plot should be



ω	G db	ϕ
1	25.845	-101.475
2	19.337	-112.097
3	15.108	-121.341
4	11.78	-129.076
	8.966	-135.426
6	6.507	-140.613
7	4.32	-144.865
8	2.351	-148.377
9	0.563	-151.306
10	-1.073	-153.775
11	-2.579	-155.876
12	-3.973	-157.681
13	-5.27	-159.247
14	-6.481	-160.615
15	-7.618	-161.819
16	-8.687	-162.887
17	-9.697	-163.84
18	-10.654	-164.694
19	-11.562	-165.465
	-12.426	-166.163

Figure 7

In order to limit the peak closed loop gain to 4 db the plot must be moved down by 4 db. This is the change in the gain margin required.

The frequency response from he Nichols chart is shown with a computer plot of the second order system.

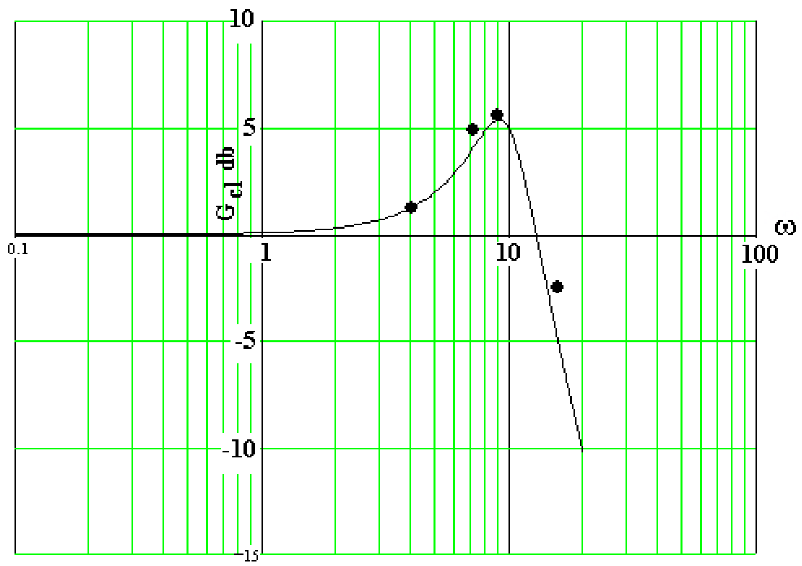


Figure 8