INSTRUMENTATION AND CONTROL

TUTORIAL 1 – CREATING MODELS OF ENGINEERING SYSTEMS

This tutorial is of interest to any student studying control systems and in particular the EC module D227 – Control System Engineering. The purpose of this tutorial is to introduce students to the basic elements of engineering systems and how to create a transfer function for them. The tutorial is mainly informative and consists of examples showing the derivation of models for real hardware systems. The self assessment material is based on basic general engineering knowledge.

On completion of this tutorial, you should be able to do the following.

- Derive the mathematical models of basic mechanical systems.
- Derive the mathematical models of basic fluid power systems.
- Derive the mathematical models of basic thermal systems.
- Derive the mathematical models of basic electrical systems.
- Recognise the similarity between models of different systems.
- Explain the standard first and second order transfer functions.
- Explain the link between open and closed loop transfer functions.

If you are not familiar with instrumentation used in control engineering, you should complete the tutorials on Instrumentation Systems.

In order to complete this tutorial, you must be familiar with basic mechanical and electrical science. You should also be familiar with the Laplace transform and a tutorial on this may be found in the maths section. You can also find tutorials on fluid power on this site.

Tutorial 2 in this series gives a detailed account of electric motor models and you may wish to study this first.

1. INTRODUCTION

Engineering systems is a very broad area ranging from control of a power station to control of a motor's speed. The student needs to have a broad base knowledge of engineering science in order to understand the various elements and see how many of them are mathematically the same (analogues of each other).

Different kinds of engineering systems often conform to similar laws and there are clear analogies between electrical, mechanical, thermal and fluid systems. The basic laws which we use most often concern Resistance R, Capacitance C, Inductance L and conservation laws. You do not need to study all these in detail and the appropriate law will be explained as required.

Here is a table showing the main analogue components. It is useful to note that capacitance is a zero order differential equation, resistance is a first order differential equation and Inductance/inertia/inertance is a second order differential equation.

MECHANICAL	ELECTRICAL	THERMAL	FLUID
Spring $x = C F = (1/k) F$	**Electrical Capacitor** $Q = C V$	**Thermal capacitor** $Q = C \Delta T$	**Fluid Capacitor** $M = C \times \Delta p$
Damper Force = k_d x velocity $F = k_d\, dx/dt$ Torque = k_d x Ang.vel	**Ohm's Law** $V = R I$ $V = R\, dQ/dt$	**Heat Transfer Laws** $\Delta T = R\, \Phi$ $\Delta T = R\, dQ/dt$	Fluid friction Laws do not conform to this pattern.
Newton's 2nd Law of motion Force = Mass x acceleration $F = M\, d^2x/dt^2$	**Law of Inductors** $V = L\ d^2q/dt^2$	**No equivalence**	**Fluid inertance** $\Delta p = L d^2v/dt^2$
D'Alembert's Principles $\sum$Force = 0 $\sum$Moment = 0	**Kirchoff's Laws** $\sum$current = 0	**Law of Conservation of Energy** $\sum$Energy = constant	**Law of Conservation of Mass** $\sum$Mass = constant

Let's look at the similarity of the various quantities used in these systems.

2. SIMILARITY OF ELEMENTS

CAPACITANCE

The symbol C will be used for electrical, thermal and fluid capacitance. Mechanical capacitance is equivalent to 1/k for mechanical systems where k is the spring stiffness.

RESISTANCE

The symbol R will be used for electrical and thermal resistance. Mechanical/hydraulic resistance is called the damping coefficient and has various symbols.

INDUCTANCE / INERTIA / INERTANCE

The symbol L will be used for electrical inductance and fluid inertance. In mechanical systems, mass M is the equivalent property for linear motion and moment of inertia I for angular motion.

OTHER EQUIVALENT PROPERTIES

Q is the symbol for electric charge and quantity of heat. This is equivalent to displacement in mechanical systems, these being distance (usually x) or angle (usually θ).

V is the symbol for electric voltage (potential difference or e.m.f) and is equivalent to temperature T for thermal systems, Force F for mechanical systems and pressure p for fluid systems.

v or u is the symbol for velocity in mechanical systems and this is equivalent to electric current (I or i) and heat flow rate Φ.

3 LAPLACE TRANSFORM and TRANSFER FUNCTIONS

Laplace is covered in detail in later tutorials and in the maths section. The purpose of this transform is to allow differential equations to be converted into a normal algebraic equation in which the quantity s is just a normal algebraic quantity. In this tutorial we should simply regard it as a shorthand method of writing differential coefficients such that:

$$\frac{d\theta}{dt} \text{ becomes } s\,\theta \qquad \frac{d^2\theta}{dt^2} \text{ becomes } s^2\,\theta \qquad \frac{d^n\theta}{dt^n} \text{ becomes } s^n\,\theta$$

4 TRANSFER FUNCTIONS

The models of systems are often written in the form of a ratio of Output/Input. If the models are turned into a function of s it is called a transfer function and this is usually denoted as G(s).

$$G(s) = \frac{\text{Output}}{\text{Input}}$$ The output and input are functions of s.

Now let's examine the mathematical models of some mechanical systems.

5. BASIC MODELS OF MECHANICAL SYSTEMS

5.1 GENERAL PROCEDURE

The general procedure for mechanical systems is as follows.

i. Adopt a suitable co-ordinate system with an appropriate sign convention. For linear motion, up is positive and left to right is positive. For rotation anticlockwise is positive and clockwise is negative. These may be ignored when convenient.
ii. Identify any disturbing forces acting on the system (inputs to the system).
iii. Identify displacements and/or velocities (outputs from the system).
iv. Draw a free body diagram for each mass showing all the forces and moments acting on it.
v. Apply Newton's 2nd Law to each free body diagram (F = Mass x Acceleration).
vi. Rearrange the equation(s) into a suitable form for solution by a convenient method.

Note that unless otherwise specified, ignore gravitational effects.

Let's now examine mechanical elements in detail.

5.2 LINEAR MECHANICAL SYSTEMS.

5.2.1 SPRING

The basic law of a mechanical spring is Force $\propto$ change in length. The diagram shows the model with mechanical symbols and as a block diagram.

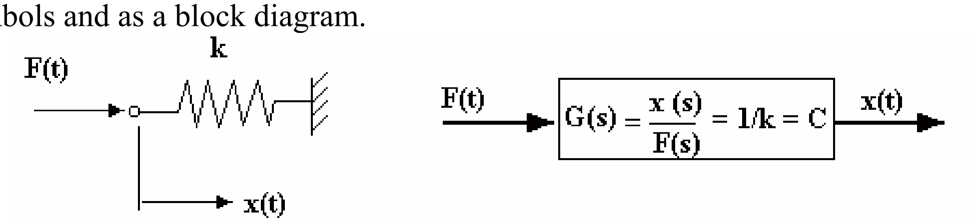


Figure 1

The relationship has no derivatives in it may be written as a function of t or s with no transform involved.
As a function of time we write F(t) = k x(t) where k is the spring stiffness.
As a function of s we write F(s) = kx(s)
This can be arranged as a transfer function such that $\frac{x}{F}(s) = 1/k = C$
C is the reciprocal of stiffness and it is called mechanical capacitance. The use of k is usually preferred in mechanics but C is used in systems as it is directly analogous to electrical capacitance.

5.2.2 DAMPER or DASHPOT

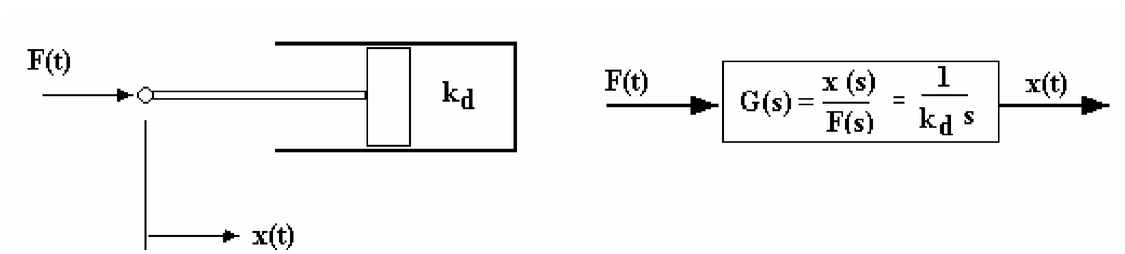


Figure 2

A damper may be idealised as a loosely fitting piston moving in a viscous fluid such that the force is directly proportional to velocity. $F \propto v$. Velocity v is the first derivative of distance so $F \propto dx/dt$

The basic law of a dashpot is: $F(t) = k_d \frac{dx}{dt}$ k_d is the damping coefficient.

Changed into Laplace form. $F = k_d s x$

Rearranged into a transfer function $\frac{x}{F}(s) = \frac{1}{k_d s}$

k_d is the damping coefficient with units of Force/Velocity or N s/m. The diagram shows the model with mechanical symbols and the control block.

5.2.3 MASS

When a mass is accelerated, the inertia has to be overcome and the inertia force is given by Newton's Second Law of Motion Force = Mass x Acceleration. Acceleration is the second derivative of x with time.

Basic Law $F(t) = M \frac{d^2x}{dt^2}$

Changed into Laplace form. $F = Ms^2 x$

Rearranged into a transfer function $\frac{x}{F}(s) = \frac{1}{Ms^2}$

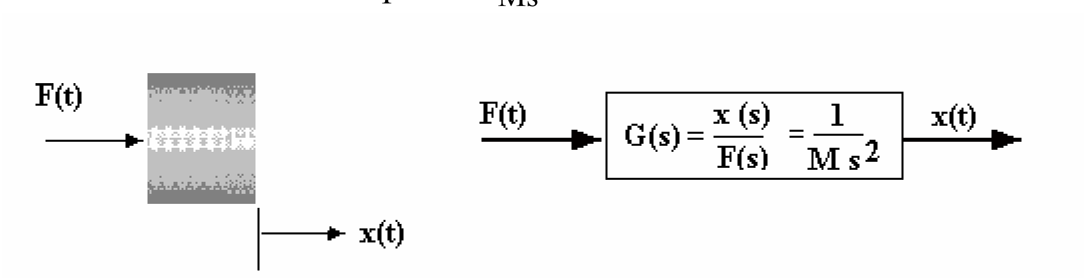


Figure 3

5.2.4 MASS - SPRING SYSTEM

For this spring - mass system, motion only occurs in one direction so the system has a single degree of freedom. It is normal for the direction of motion to be expressed as the x direction regardless of the actual direction. The free body diagram is as shown. The input is a disturbing force F which is a function of time F(t). This could, for example, be a sinusoidal force. The output is a motion x which is a function of time x(t). Let x be positive upwards.

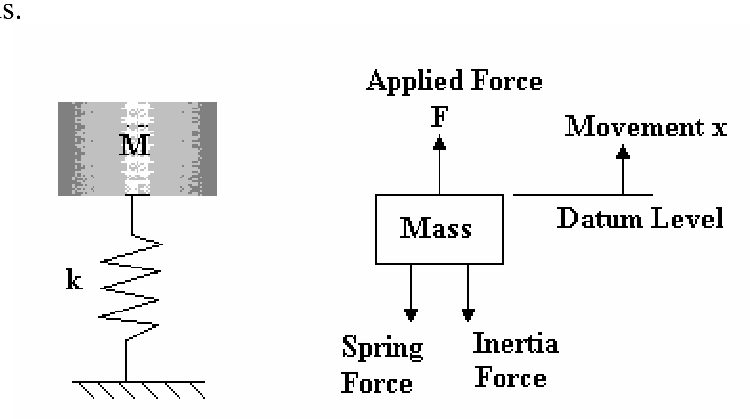


Figure 4

The input force is opposed by the spring force and the inertia force (which always opposes changes in the motion as stated in Newton's third law of motion).

Spring force = k x
Inertia force = M d^2x/dt^2

D'Alembert's Principle is that all the forces and moments on the body must add up to zero. In this case it means

$$F(t) - kx(t) - M\, d^2x/dt^2(t) = 0$$

or

$$F(t) = M\, d^2x/dt^2(t) + kx\,(t)$$

Changing to a function of s we have $F(s) = Ms^2 x + kx = x\,[Ms^2 + k]$

$$x(s) = \frac{F}{Ms^2 + k} = \frac{F(1/M)}{s^2 + k/M}$$

This may be shown as a transfer function. $G(s) = \frac{x(s)}{F(s)} = \frac{1/M}{s^2 + k/M}$

The block diagram for use in systems is as shown.

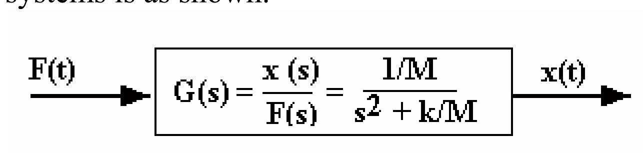


Figure 5

5.2.5 SPRING DAMPER

Force balance as a function of time. $F(t) = k\,x + k_d\, dx/dt$
Force balance as a function of s $F(s) = k\,x + k_d\, s\,x$
Rearrange into a transfer function. $\frac{x}{F}(s) = \frac{1/k}{(k_d/k)s + 1}$

The units of k_d/k are seconds and this is the time constant for the system $T = k_d/k$

$\frac{x}{F}(s) = \frac{1/k}{Ts + 1}$. This is the standard first order equation which we shall study many times in these tutorials.

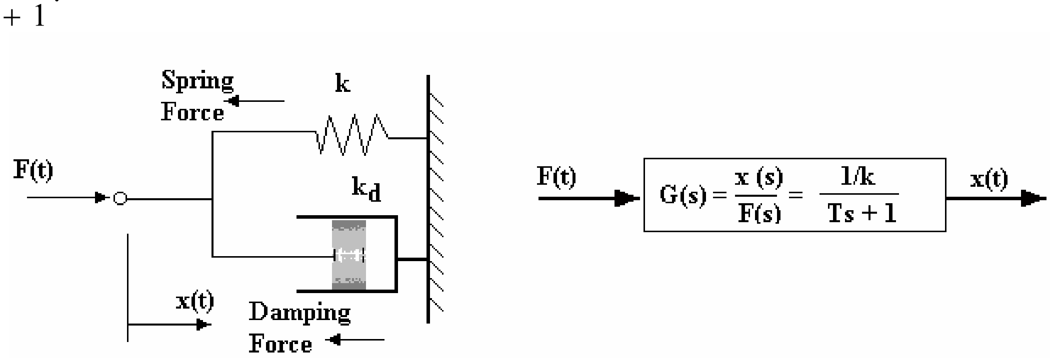


Figure 6

5.2.6 MASS -SPRING - DAMPER SYSTEM

The input is the force F and the output is the movement x, both being functions of time.

Spring force $F_s = kx$

Damping force $F_d = k_d\, dx/dt$

Inertia force $F_i = Md^2x/dt^2$

The three forces oppose motion so if the total force on the system is zero then $F = F_i + F_d + F_s$

$$F(t) = M\frac{d^2x}{dt^2} + k_d\frac{dx}{dt} + kx \qquad F(s) = Ms^2x + k_d sx + kx \qquad G(s) = \frac{x}{F}(s) = \frac{1/k}{s^2(M/k) + s(k_d/k) + 1}$$

If we examine the units of $(M/k)^{1/2}$ we find it is seconds and this is the second order time constant also with the symbol T. The transfer function may be written as

$$G(s) = \frac{x}{F}(s) = \frac{1/k}{T^2s^2 + 2\delta Ts + 1}$$

δ is the damping ratio defined as $\delta = k_d/C_c$ and C_c is the critical damping ratio defined as $(4Mk)^{1/2}$. The term $2\delta T$ is hence $2\frac{k_d}{C_c}T = \frac{2k_d\sqrt{M/k}}{\sqrt{4Mk}} = \frac{2k_d\sqrt{M}}{2\sqrt{M}\sqrt{k}\sqrt{k}} = \frac{k_d}{k}$ and so the forgoing is correct.

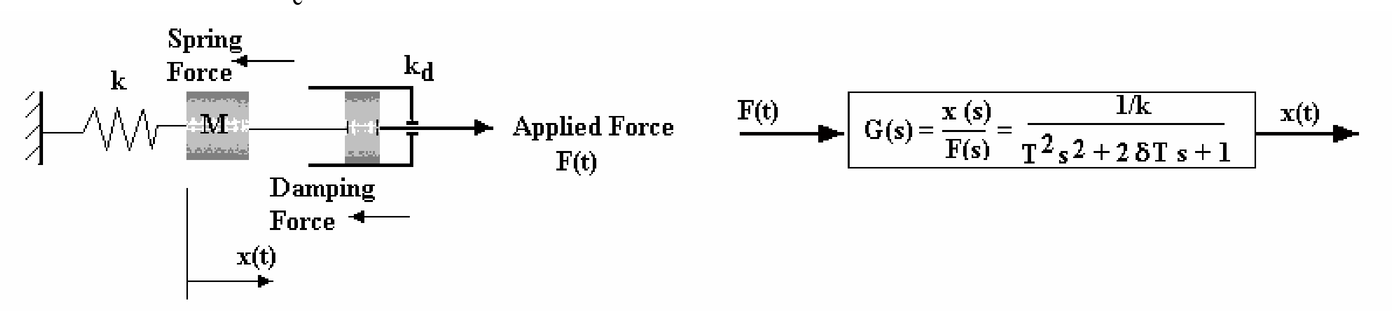


Figure 7

This is the standard 2nd order transfer function which will be analysed in detail later.

WORKED EXAMPLE No.1

A mass – spring –system has the following parameters.

Stiffness K = 800 N/m   Mass M = 3 kg   Damping Coefficient k_d = 20 Ns/m

i. Calculate the time constant, critical damping coefficient and the damping ratio.

ii. Derive the equation for the force required when the piston is accelerating.

iii. Use the equation to evaluate the static deflection when F = 12 N.

iv. Use the equation to evaluate the force needed to make the mass accelerate at 4 m/s^2 at the moment when the velocity is 0.5 m/s.

SOLUTION

i. $T = \sqrt{(M/k)} = \sqrt{(3/800)} = 0.0612$ seconds
$c_c = \sqrt{4MK} = \sqrt{(4 \times 3 \times 800)} = 97.97$ Ns/m
$\delta = k_d / c_c = 20/97.97 = 0.204$

ii. For a constant acceleration $s^2x = a$ (acceleration) and $sx = v$ (velocity)

$$\frac{x}{F}(s) = \frac{1/k}{T^2s^2 + 2\delta Ts + 1} \qquad F = kx\left(T^2s^2 + 2\delta Ts + 1\right)$$

$$F = 800x\left(0.0612^2 s^2 + 2 \times 0.204 \times 0.0612s + 1\right)$$

$$F = x\left(3s^2 + 20s + 800\right) = 0.00374s^2x + 20sx + 800x$$

$F = 3a + 20v + 800x$

iii. For a constant force and a static position there will be neither velocity nor acceleration so the s and s^2 terms are zero.

$$\frac{F}{x} = 800 \qquad x = \frac{12}{800} = 0.015 \text{ m or } 15\text{mm}$$

iv. For velocity = 0.5 m/s and a = 4 m/s^2
$F = 3a + 20v + 800x = 12 + 10 + 800x = 22 + 800x$

The deflection x would need to be evaluated from other methods $x = v^2/2a = 0.031$ m

F = 46.8 N

SELF ASSESSMENT EXERCISE No.1

1. A mass – spring –system has the following parameters.

Stiffness K = 1200 N/m Mass M = 15 kg Damping Coefficient k_d = 120 Ns/m

i. Calculate the time constant, critical damping coefficient and the damping ratio.
(0.112 s, 268.3 Ns/m and 0.447)

ii. If a constant force of 22 N is applied, what will be the static position of the mass?
(18 mm)

iii. Calculate the force needed to make the mass move with a constant acceleration of 12 m/s^2 at the point where the velocity is 1.2 m/s.
(396 N)

5.3 ROTARY MECHANICAL SYSTEMS

The following is the rotary equivalent of the previous work.

5.3.1 TORSION BAR

This is the equivalent of a mass and spring. A metal rod clamped at one end and twisted at the other end produces a torque opposing the twisting directly proportional to the angle of twist. The ratio T/θ is the torsion stiffness of the torsion spring and is denoted with a k.

T is torque (N m)
θ is the angle of twist (radian)
k is the torsional stiffness (N m/rad)

Balancing the torques we have $T(t) = k\theta(t)$

Change to Laplace form. $T(s) = k\theta(s)$

Write as a transfer function. $\frac{\theta}{T}(s) = \frac{1}{k}$

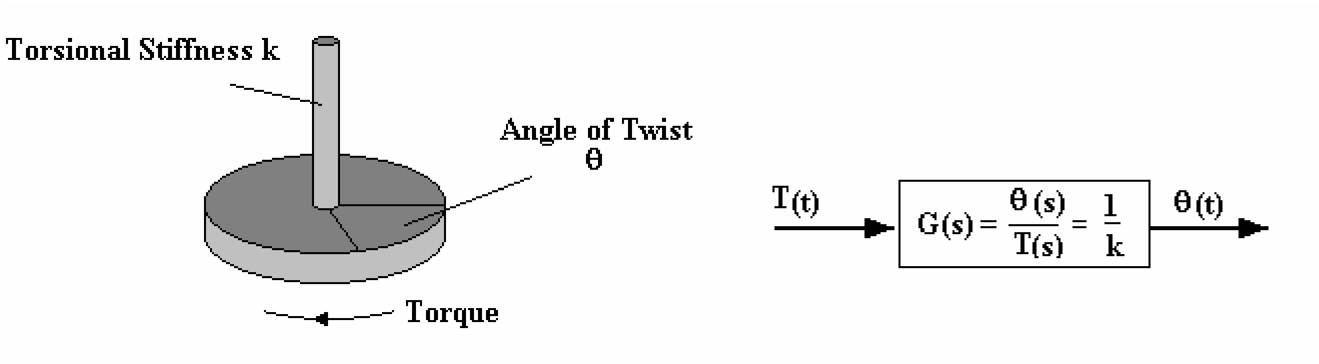


Figure 8

5.3.2 TORSION DAMPER

A torsion damper may be idealised as vanes rotating in a viscous fluid so that the torque required to rotate it is directly proportional to the angular velocity. k_d is the torsion damping coefficient in N m s/radian

$$T(t) = k_d \frac{d\theta}{dt} \qquad T(s) = k_d s\theta \qquad G(s) = \frac{\theta}{T}(s) = \frac{1}{k_d s}$$

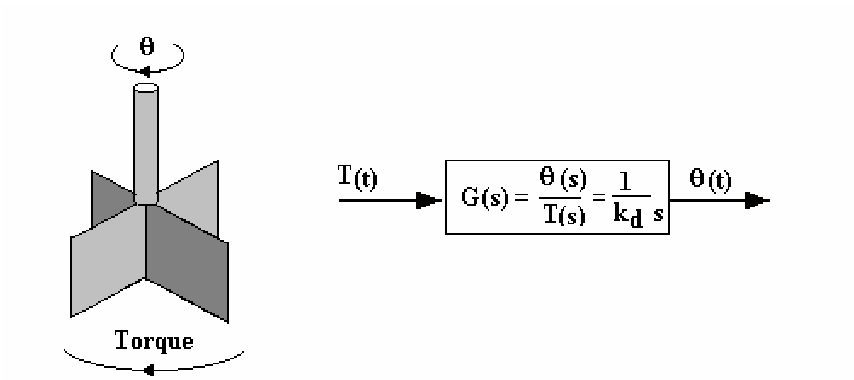


Figure 9

5.3.3 MOMENT OF INERTIA

Rotating masses oppose changes to the motion and Newton's 2nd law for rotating masses is $T = I\, d^2\theta/dt^2$
I is the moment of inertia in kg m^2.

$T(t) = I \dfrac{d^2\theta}{dt^2} \quad T(s) = I s^2 \theta \quad G(s) = \dfrac{\theta}{T}(s) = \dfrac{1}{Is^2}$. Note many text books also use J for moment of inertia.

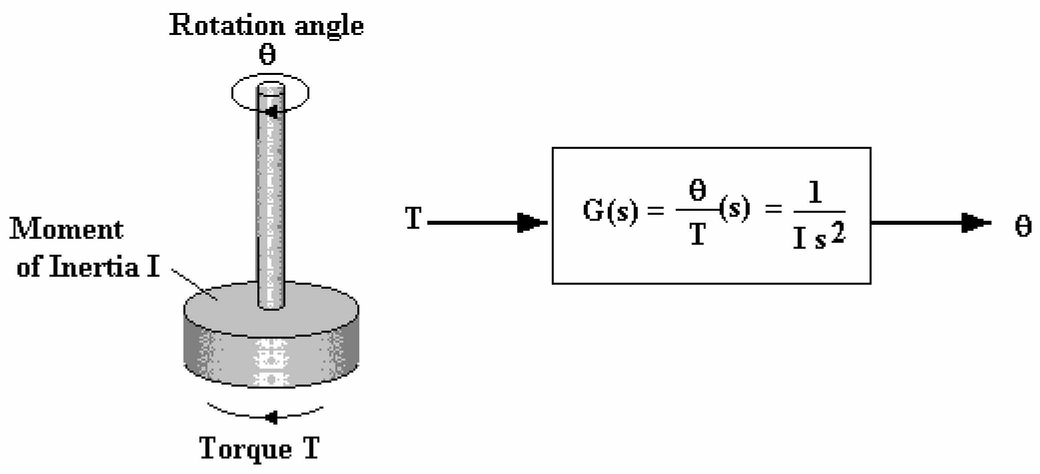


Figure 10

SELF ASSESSMENT EXERCISE No.2

Derive the transfer function for a mass on a torsion bar fitted with a damper and show it is another example of the second order transfer function. T is torque and J is moment of inertia.

$$G(s) = \frac{\theta}{T}(s) = \frac{1/k}{(J/k)s^2 + (Jk_d/k)s + 1} = \frac{1/k}{T^2 s^2 + 2\delta T s + 1}$$

5.3.4 GEARED SYSTEMS

When a mass is rotated through a gear system, the affect of the inertia is dramatically altered. Consider a motor coupled to a load through a speed changing device such as a gear box. There is damping (viscous friction) on the two bearings.

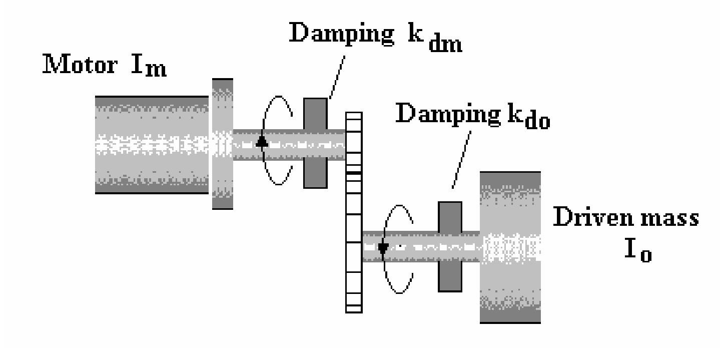


Figure 11

θ_m is the motor rotation and θ_o the output rotation. The gear ratio is $G_r = \theta_o/\theta_m$

Since this is a fixed number and is not a function of time, the speed and acceleration are also in the same ratio.

$d\theta_m/dt = \omega_m$ $\quad d\theta_o/dt = \omega_o$ $\quad G_r = \omega_o/\omega_m$ ω is the angular velocity

$d^2\theta_m/dt^2 = \alpha_m$ $\quad d^2\theta_o/dt^2 = \alpha_o$ $\quad G_r = \alpha_o/\alpha_m$ α is the angular acceleration.

The power transmitted by a shaft is given by Power = ωT. If there is no power lost, the output and input power must be equal so it follows that

$$\omega_m T_m = \omega_o T_o \text{ hence}$$
$$T_m = \omega_o T_o/\omega_m = G_r T_o$$

(In reality friction significantly affects the torque)

Consider the inertia torque due the inertia on the output shaft I_o.

$T_o = I_o\alpha_o = I_o\,\alpha_m \times G_r$ $\qquad T_m = T_o \times G_r = I_o\,\alpha_m \times G_r^2$

Now consider the damping torque on the output shaft.

$$T_o = k_{do}\,\omega_o = k_{do}\,\omega_m G_r \qquad T_m = T_o \times G_r = k_{do}\,\omega_m\,G_r^2$$

Now consider that there is an inertia and damping torque on the motor shaft and on the output shaft. The total torque produced on the motor shaft is

$$T_m = I_m\alpha_m + k_{dm}\,\omega_m + G_r T_o$$
$$T_o = I_o\alpha_o + k_{do}\,\omega_o$$
$$T_m = I_m\alpha_m + k_{dm}\,\omega_m + G_r\{I_o\alpha_o + k_{do}\,\omega_o\}$$
$$T_m = I_m\alpha_m + k_{dm}\,\omega_m + G_r^2 I_o\,\alpha_m + G_r^2 k_{do}\,\omega_m$$
$$T_m = \alpha_m (I_m + G_r^2 I_o) + \omega_m (k_{dm} + G_r^2 k_{do})$$

$(I_m + G_r^2 I_o)$ is the effective moment of inertia I_e and $(k_{dm} + G_r^2 k_{do})$ is the effective damping coefficient k_{de}.

The equation may be written as $\quad T_m = \alpha_m (I_e) + \omega_m (k_{de})$

In calculus form this becomes $\quad T_m = d^2\theta/dt^2 (I_e) + d\theta/dt\,(k_{de})$

Changing this into a function of s we have $T_m(s) = s^2\theta(I_e) + s\theta\,(k_{de}) = s\theta\{sI_e + k_{de}\}$

The output is the motor angle and the input is the motor torque so the geared system may be presented as a transfer function thus. $\theta\,(s)/T_m\,(s) = (1/I_e)/s\{s + K_{de}/I_e\}$

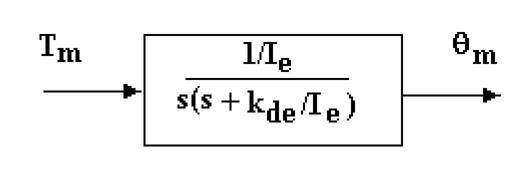


Figure 12

WORKED EXAMPLE No.2

A DC Servo motor has a moment of inertia of 0.5 kg m^2. It is coupled to an aerial rotator through a gear reduction ratio of 10. The driven mass has a moment of inertia of 1.2 kg m^2. The damping on the motor is 0.1 N m s/rad and on the rotator bearings it is 0.05 N m s/rad.

Write down the transfer function θ/T_m in the simplest form. Calculate the torque required from the motor to

i. Turn the aerial at a constant rate of 0.02 rad/s.

ii. Accelerate the rotator at 0.005 rad/s^2 at the start when $\omega = 0$

SOLUTION

i. $I_e = (I_m + G_r^2\, I_o) = (0.5 + 10^2 \times 1.2) = 120.5$ kg m^2.
$K_{de} = (k_{dm} + G_r^2\, k_{do}) = (0.1 + 10^2 \times 0.05) = 5.1$ N m s/rad.
$\theta\,(s)/T_m\,(s) = (1/I_e)/s\{s + K_{de}/I_e\}$

$$\frac{\theta}{T_m}(s) = \frac{1/I_e}{s\left(s + k_{de}/I_e\right)}$$

$$T_m = I_e\alpha + K_{de}\omega$$

$$T_m = 120.5\alpha + 5.1\omega$$

If the rotator is moving at constant speed α (acceleration) is zero. Hence:
$T_m = 5.1\omega = 5.1 \times 0.02 = 0.102$ Nm

ii. When accelerating at 0.005 rad/s^2 the motor acceleration is 10 times larger at 0.05 rad/s^2.

$T_m = 120.5\alpha + 5.1\omega = 60.25$ Nm when $\omega = 0$

SELF ASSESSMENT EXERCISE No.3

A DC Servo motor has a moment of inertia of 12 kg m^2. It is coupled to an aerial rotator through a gear reduction ratio of 4. The driven mass has a moment of inertia of 15 kg m^2. The damping on the motor is 0.2 N m s/rad and on the rotator bearings it is 0.4 N m s/rad.

Calculate the torque required from the motor to

i. Turn the aerial at a constant rate of 0.5 rad/s. (3.3 N)

ii. Accelerate the rotator at 0.02 rad/s^2 at the start when $\omega = 0$ (20.16 Nm)

6. THERMAL SYSTEM MODELS

6.1 HEATING and COOLING

Consider a mass M kg at temperature θ_1. The mass is placed in a hot environment at temperature θ_2 and heat Q is transferred into the mass causing its temperature to rise. The system could be for example, a resistance thermometer, and we want to know how long it takes for the sensor to warm up to the same temperature as the liquid.

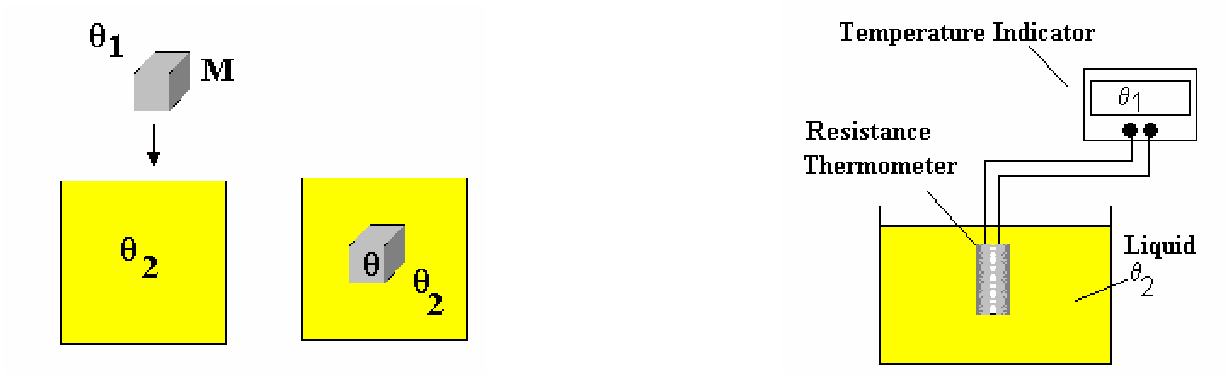


Figure 13

The laws of heat transfer tell us that the temperature rise is directly proportional to the heat added so:

$$dQ = Mc\, d\theta_1 = C\, d\theta_1$$

c is the specific heat capacity. C = Mc is the thermal capacitance in Joules/Kelvin.

Divide both sides by dt and: $$\frac{dQ}{dt} = \Phi = C\frac{d\theta_1}{dt}$$

The rate of heat transfer into the mass is $\Phi = C\, d\theta_1/dt$ and the rate is governed by the thermal resistance between the liquid and the mass. This obeys a law similar to Ohm's Law so that:

$\Phi = (\theta_2 - \theta_1)/R$ R is the thermal resistance in Kelvin per Watt.

Equating for Φ we have $$C\frac{d\theta_1}{dt} = \frac{\theta_1 - \theta_2}{R} \qquad \frac{d\theta_1}{dt} = \frac{\theta_1 - \theta_2}{RC}$$

$$\frac{d\theta_1}{dt} + \frac{\theta_1}{RC} = \frac{\theta_2}{RC}$$

In all systems, the product of the resistance and capacitance is the time constant T so we have

$$\frac{d\theta_1}{dt} + \frac{\theta_1}{T} = \frac{\theta_2}{T}$$

Changing from a function of time into a function of s we have

$$s\theta_1 + \frac{\theta_1}{T} = \frac{\theta_2}{T} \qquad \theta_1(Ts + 1) = \theta_2 \qquad \frac{\theta_1}{\theta_2}(s) = \frac{1}{(Ts + 1)}$$

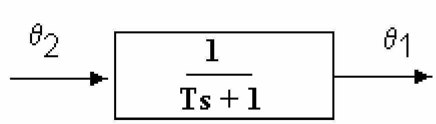


Figure 14

Note that this transfer function is the same standard first order equations derived for the spring - damper system and thermal capacitance C is equivalent to 1/k and resistance R is equivalent to k_d.

6.2 INDUSTRIAL HEATING SYSTEM

The diagram shows a schematic of an industrial process for controlling the temperature of a tank of liquid. The pneumatic controller will not be explained here but it has an input temperature set by adjustment of the control. The temperature of the liquid is measured with a suitable device and turned into a standard signal in the range 0.2 – 1 bar. This is connected to the controller and pressure sensing devices produce another air signal (0.2 – 1 bar) depending on the error. This is sent to a valve that is opened pneumatically working on the standard range. The overall result is that if the liquid is too cool, steam is allowed through to heat the liquid.

If the valve opened instead of closing and a cooling fluid was used instead of steam, the system control is by cooling.

The control equipment could just as likely be all electronic. Pneumatics are used in dangerous environments such as heating up oil tanks.

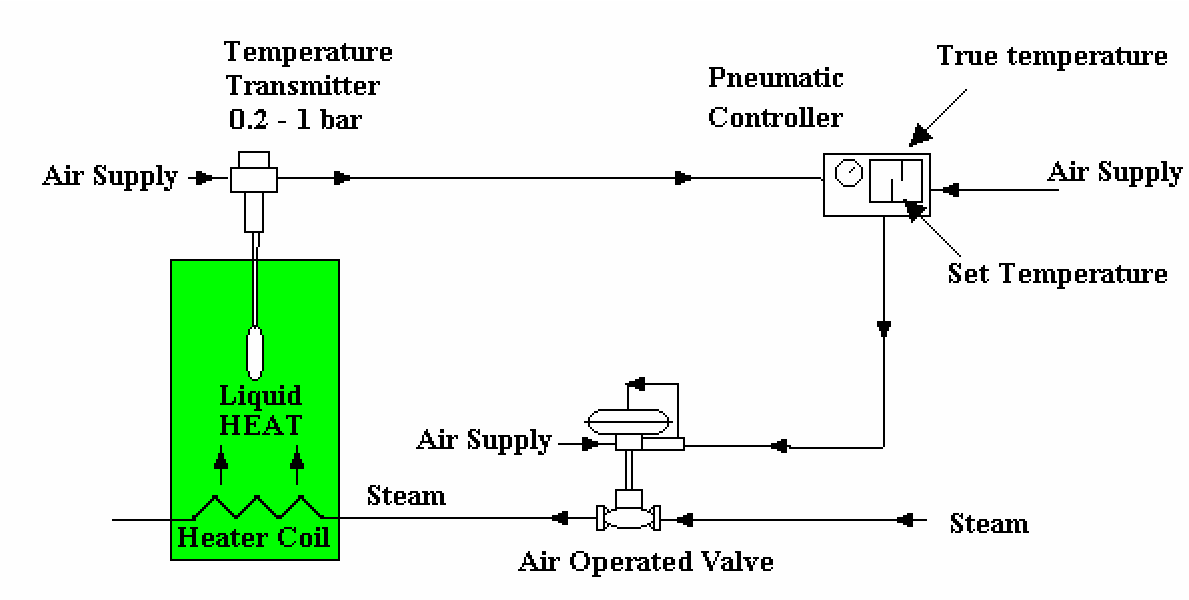


Figure 15

The model for the above system will not be derived here but it will be more complicated than simply $\frac{\theta_1}{\theta_2}(s) = \frac{1}{(Ts+1)}$ because the controller has the facility to do more than proportional control. (Three term control is covered in later tutorials)

WORKED EXAMPLE No.3

A simple thermal heating system has a transfer function $\frac{\theta_o}{\theta_i}(s) = \frac{1}{(Ts+1)}$

The temperature of the system at any time is θ_o and this is at 20°C when the set temperature θ_i is changed from 20 °C to 100 °C. The time constant 'T' is 4 seconds. Deduce the formulae for how the system temperature changes with time and sketch the graph.

SOLUTION

$$\frac{\theta_o}{\theta_i} = \frac{1}{Ts + 1} \qquad \theta_i = Ts\theta_o + \theta_o$$

θ_i is a constant (100°C) at all values of time after t = 0 (the start of the change).

$$\theta_i - \theta_o = Ts\,\theta_o = T\frac{d\theta_o}{dt}$$

Let $\theta_i - \theta_o = x$

Differentiate and $-d\theta_o = dx$ The equation becomes $x = T\dfrac{d\theta_o}{dt} = -T\dfrac{dx}{dt}$

Rearrange and $-\dfrac{dt}{T} = \dfrac{dx}{x}$

Integrate without limits $-\dfrac{t}{T} = \ln(x) + A$

Substitute for x $-\dfrac{t}{T} = \ln(\theta_i - \theta_o) + A$

When t = 0, θ_o = starting temperature = θ_1 Hence

$-\dfrac{t}{T} = 0 = \ln(\theta_i - \theta_1) + A \qquad A = -\ln(\theta_i - \theta_1) \qquad \theta_i - \theta_1 =$ change in temperature $\Delta\theta$

Substitute for A and $-\dfrac{t}{T} = \ln(\theta_i - \theta_o) - \ln(\Delta\theta) = \ln\dfrac{(\theta_i - \theta_o)}{\Delta\theta}$

Take anti logs and $e^{-\frac{t}{T}} = \dfrac{(\theta_i - \theta_o)}{\Delta\theta}$

$$\Delta\theta\, e^{-\frac{t}{T}} = (\theta_i - \theta_o)$$

$$\theta_o = \theta_i - \Delta\theta\, e^{-\frac{t}{T}}$$

Put in the values T = 4 $\quad \Delta\theta = 100 - 20 = 80 \quad \theta_i = 100$

$\theta_o = 100 - 80e^{-t/4}$ Evaluating and plotting produces the result below. It is an exponential growth.

t (s)	θ_o °C
0	20
2	51.478
4	70.57
6	82.15
8	89.173
10	93.433
12	96.017
14	97.584
16	98.535
18	99.111
20	99.461

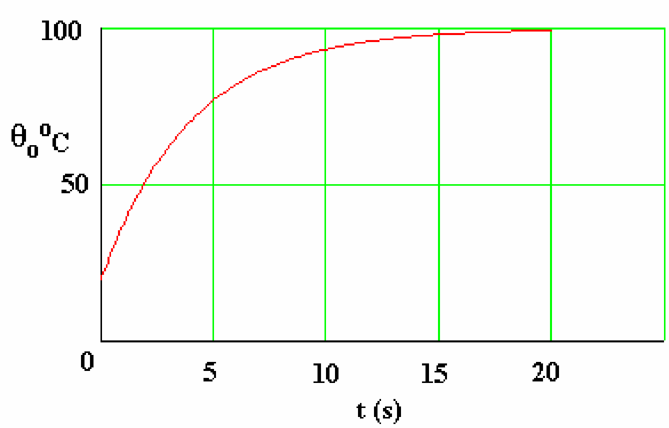


Figure 16

7. HYDRAULIC SYSTEM MODELS

The basic theory for hydraulic and pneumatic components may be found in the tutorials on fluid power.

7.1 HYDRAULIC MOTOR

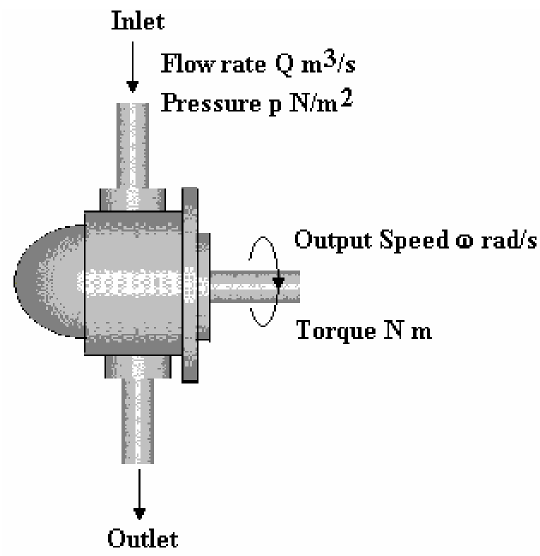


Figure 17

The following is the derivation of a model for use in control theory. The formula relating flow rate Q and speed of rotation ω is $Q = k_q\omega = k_q \frac{d\theta}{dt}$

k_q is a constant known as the nominal displacement with units of m^3 per radian. θ is the angle of rotation in radian. Written as a function of s this becomes $Q = k_q s\theta$

If we take the flow rate as the input and the angle of rotation as the output the transfer function is: $G(s) = \frac{\theta}{Q} = \frac{1}{k_q s}$

The formula that relates system pressure p to the output torque T is $T = k_q p$

If pressure is the input and torque the output then

$G(s) = \frac{T}{p} = k_q$ This is a further definition of the constant k_q.

WORKED EXAMPLE No.4

A hydraulic motor has a nominal displacement of 8 cm^3/radian. Calculate the torque produced at a pressure of 90 bar.

SOLUTION

$T = p\,k_q = 90 \times 10^5\ (N/m^2) \times 8 \times 10^{-6}\ (m^3/rad) = 72$ Nm

7.2 HYDRAULIC CYLINDER

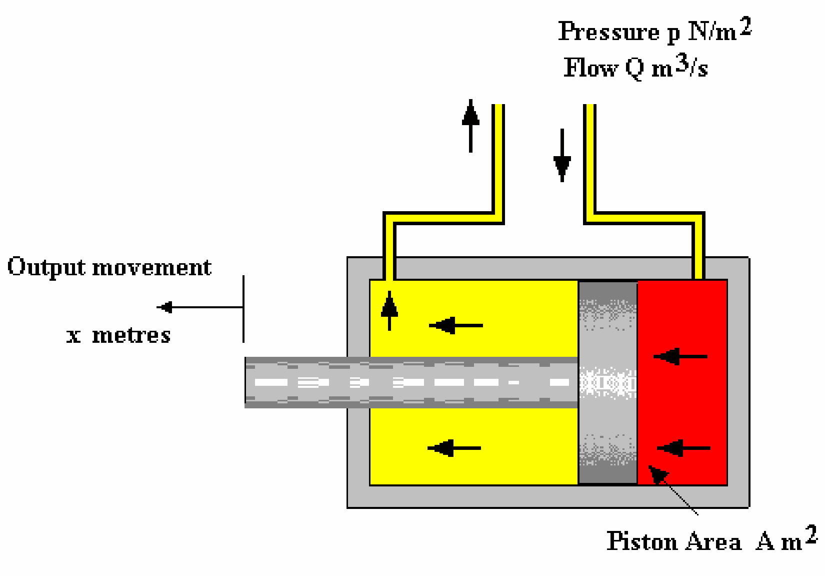


Figure 18

The flow rate and movement are related by the law Q = A dx/dt. Expressed as a transfer function with x being the output and Q the input we have: $G(s) = \frac{x}{Q} = \frac{1}{As}$

Force and pressure are related by the law F = pA. The transfer function with p as the input and F as the output is: $G(s) = \frac{F}{p} = A$

7.3 MODEL FOR A FLOW METERING VALVE AND ACTUATOR

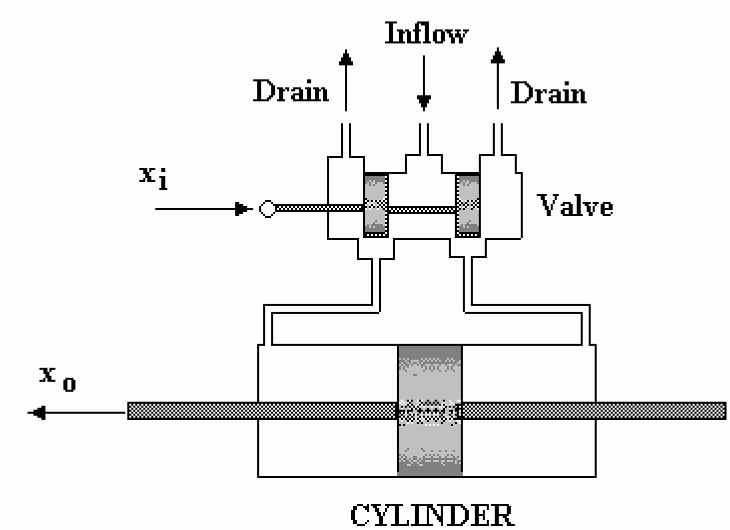


Figure 19

The input to the system is the movement of the valve x_i. This allows a flow of oil into the cylinder of Q m^3/s which makes the cylinder move a distance x_o.

Making a big assumption that for a constant supply pressure the flow rate is directly proportional to the valve position we may say $Q = k_v\, x_i$

k_v is the valve constant and examining its units we find they are m^2/s

The area of the piston is A m^2.

The velocity of the actuator is $v = dx_o/dt$ and this is related to the flow and the piston area by the law of continuity such that $Q = k_v x_i = A \dfrac{dx_o}{dt}$

Changing to a function of s this becomes $k_v\, x_i = Asx_o$

Expressed as a transfer function we have $G(s) = \dfrac{x_o}{x_i}(s) = \dfrac{1}{(A/k_v)s}$

The units of A/k_v are seconds and we deduce this is yet another time constant T.

$$G(s) = \frac{x_o}{x_i}(s) = \frac{1}{Ts}$$

Note that this is not quite the standard first order equation 1/{Ts+1} and the difference is that the output will keep changing for a given input, unlike the previous examples where a limit is imposed on the output.

If the actuator is a motor instead of a cylinder the equation is similar but the output is angle instead of linear motion.

WORKED EXAMPLE No.5

A hydraulic cylinder has bore of 90 mm and is controlled with a valve with a constant $k_v = 0.2\ m^2/s$

Calculate the time constant T. Given that x_i and x_o are zero when t = 0, calculate the velocity of the piston and the output position after 0.1 seconds when the input is changed suddenly to 5 mm.

SOLUTION

$A = \pi D^2/4 = 6.362 \times 10^{-3}\ m^2$

$T = A/k_v = 6.362 \times 10^{-3} / 0.2 = 0.032$ seconds

$$G(s) = \frac{x_o}{x_i}(s) = \frac{1}{Ts}$$

$$Tsx_o = x_i$$

$$T\frac{dx_o}{dt} = x_i \qquad \frac{dx_o}{dt} = \text{velocity} = \frac{x_i}{T} = \frac{0.005\ \text{m}}{0.032\ \text{s}} = 0.156\ \text{m/s}$$

Velocity = distance /time distance = x_o = v t = 0.156 x 0.1 = 0.0156 or 15.6 mm assuming the velocity is constant.

SELF ASSESSMENT EXERCISE No.4

1. A hydraulic motor has a nominal displacement of 5 cm^3/radian. Calculate the torque produced at a pressure of 120 bar.
   (60 N m)

2. A hydraulic cylinder has bore of 50 mm and is controlled with a valve with a constant k_v = 0.05 m^2/s

   Calculate the time constant T. Given that x_i and x_o are zero when t = 0, calculate the velocity of the piston and the output position after 0.2 seconds when the input is changed suddenly to 4 mm.

   (0.039 s, 0.102 m/s and 20 mm)

7.4 ADVANCED HYDRAULIC MODEL

Consider the same system but this time let the actuator move a mass M kg and have to overcome a damping force. Further suppose that the valve now meters the pressure and not the flow rate such that the pressure applied to the cylinder is $p = k_v x_i$. Consider the free body diagram of the actuator.

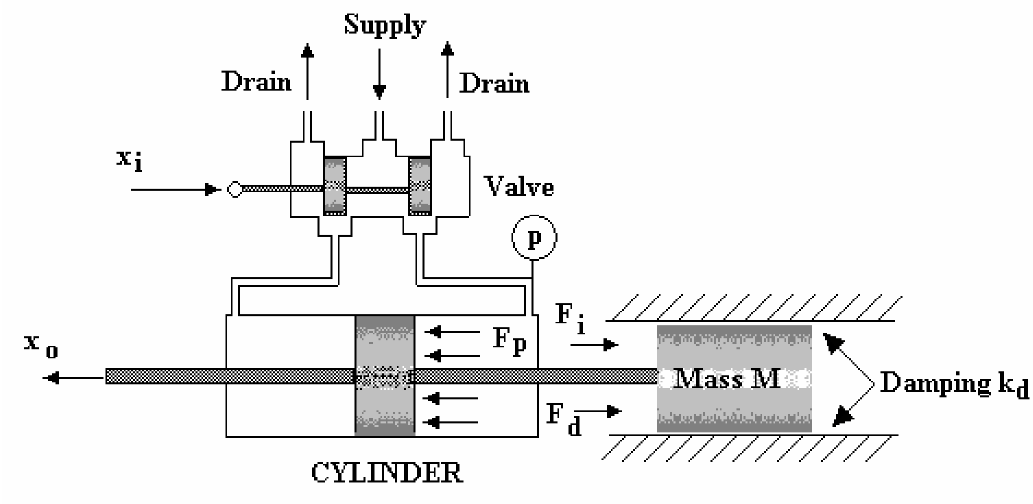


Figure 20

The applied force is due to pressure F_d and this is determined by the pressure acting on the area A such that:
$Fp = pA$.

The applied force is opposed by the inertia force F_i and the damping force F_d.

$F_i = M\, d^2x_o/dt^2$ and $F_d = k_d\, dx_o/dt$.

Balancing forces gives $$pA = M\frac{d^2x_o}{dt^2} + k_d\frac{dx_o}{dt}$$

Substituting $p = k_v x_i$. we have $$k_v x_i A = M\frac{d^2x_o}{dt^2} + k_d\frac{dx_o}{dt}$$

In Laplace form we have $$k_v x_i A = Ms^2 x_o + k_d s x_o$$

Rearranging it into a transfer function. $$G(s) = \frac{x_o}{x_i}(s) = \frac{1}{(M/Ak_v)s^2 + (k_d/Ak_v)s}$$

If we examine the units we find $M/Ak_v = T^2$ where T is a time constant.
The critical damping coefficient is $Cc = \sqrt{(4\,M\,A\,k_v)}$ and the damping ratio is $\delta\ k_d/Cc$

The transfer function becomes: $$G(s) = \frac{x_o}{x_i}(s) = \frac{1}{T^2s^2 + 2\delta Ts}$$

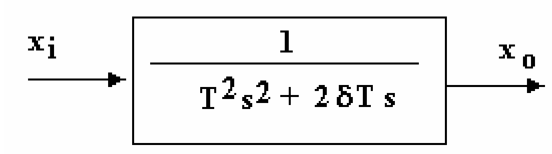


Figure 21

Note the similarity with the standard 2nd order equation $1/\{T^2s^2 + 2\delta Ts + 1\}$. The difference is again due to there being no limitation on the output. If the actuator is a motor instead of a cylinder, the transfer function is similar but the output is angle and angular quantities are used instead of linear quantities.

WORKED EXAMPLE No.6

A hydraulic cylinder has bore of 90 mm and moves a mass of 80 kg. It is controlled with a valve with a constant k_v = 20000 Pa/m. The damping coefficient is 180 Ns/m.
Calculate the time constant T, Cc and δ.
Given that x_i and x_o are zero when t = 0, calculate the initial acceleration of the mass when the input is changed suddenly to 5 mm.
Calculate the acceleration when the velocity reaches 2 mm/s.
Calculate the velocity when the acceleration is zero.

SOLUTION

$A = \pi D^2/4 = 6.362 \times 10^{-3}\ m^2$
$T = \sqrt{(M/Ak_v)} = \sqrt{\{80/(20000 \times 6.362 \times 10^{-3})\}} = 0.793$ seconds
$Cc = \sqrt{(4\ M\ A\ k_v)} = 201.78$ Ns/m
$\delta = k_d/Cc = 0.892$

$$G(s) = \frac{x_o}{x_i}(s) = \frac{1}{T^2 s^2 + 2\delta Ts}$$ or in terms of time

$x_i = (T^2 \times \text{acceleration}) + (2\delta T \times \text{velocity})$

The initial velocity is zero. $0.005 = 0.793^2\ a + 0$ $a = 7.952 \times 10^{-3}\ m/s^2$
When v = 0.002
$0.005 = 0.793^2\ a + (2 \times 0.892 \times 0.793 \times 0.002)$
$a = \{0.005 - (2 \times 0.892 \times 0.793 \times 0.002)\}/0.793^2 = 3.452 \times 10^{-3}\ m/s^2$

The system initially accelerates and will eventually settle down to a constant velocity with no acceleration. Put a = 0.
$0.005 = 0 + (2 \times 0.892 \times 0.793) \times$ velocity
velocity = 0.00353 m/s or 3.53 mm/s.

SELF ASSESSMENT EXERCISE No.5

A hydraulic cylinder has bore of 50 mm and moves a mass of 10 kg. It is controlled with a valve with a constant k_v = 80 Pa/m. The damping coefficient is 2 Ns/m.

Calculate the time constant T, Cc and δ. (7.98 s, 2.5.7 Ns/m and 0.798)

Given that x_i and x_o are zero when t = 0, calculate the initial acceleration of the mass when the input is changed suddenly to 10 mm. (0.157 mm/s^2)

Calculate the acceleration when the velocity reaches 0.1 mm/s. (0.137 mm/s^2)

Calculate the velocity when the acceleration is zero. (0.785 mm/s)

8. ELECTRIC SYSTEM ELEMENTS MODELS

8.1 RESISTANCE

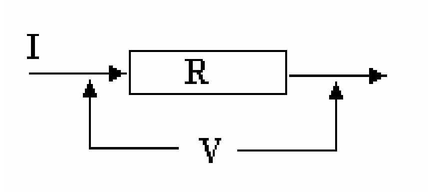
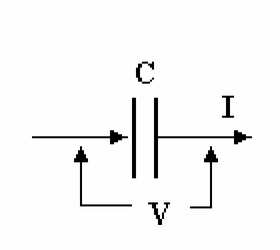


Figure 22

Applying Ohm's Law we have V = I R   V/I = R
This may be a function of time or of s.
The equation may be expressed in terms of charge Q.
Since I = dQ/dt I(s) = sQ hence $G(s) = \frac{V}{Q}(s) = sR$
This is similar to the model for the damper.

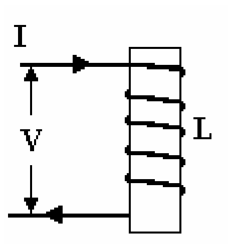
8.2 CAPACITANCE



The law of a capacitor is Q = C V V/Q = 1/C
This is similar to the model for spring.
Differentiating with respect to time we have dQ/dt = C dV/dt
dQ/dt is current I so the equation may be expressed as I = C dV/dt
As a function of s this becomes I (s) = C sV
The transfer function is $G(s) = \frac{V}{I}(s) = \frac{1}{sC}$

Figure 23

8.3 INDUCTANCE



Faraday's Law gives us $V = L\frac{dI}{dt} = L\frac{d^2Q}{dt^2}$
This is similar to the model for a mass and can be either a first or 2nd order equation as required.
Expressed as a function of s we have V (s) = L sI or Ls^2Q
$G(s) = \frac{V}{I}(s) = sL \text{ or } s^2Q$

Figure 24

8.4 POTENTIOMETER

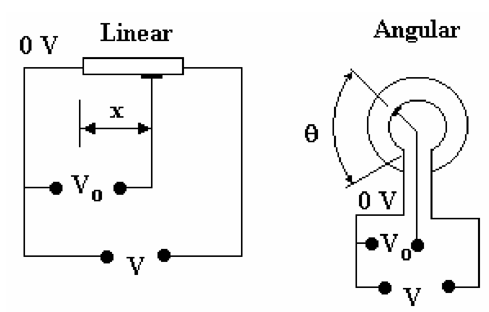


Figure 25

f the supply voltage is constant and the current is negligible, the output voltage V is directly proportional to the position x or angle θ so a simple transfer function is obtained.

$$G(s) = \frac{V_o}{x}(s) = \text{constant} = k_p \text{ (linear)}$$

$$G(s) = \frac{V_o}{\theta}(s) = \text{constant} = k_p \text{ (angular)}$$

8.5 R -C SERIES CIRCUIT

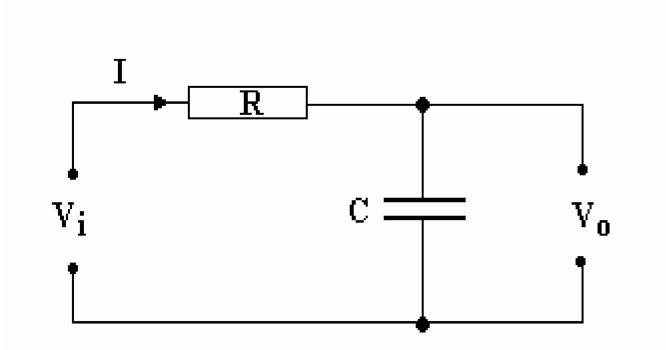


Figure 26

The input voltage V_i is the sum of the voltage over the resistor and the capacitor so

$V_i = I R + I/Cs$

$V_i = I (R + 1/Cs)$

The output is the voltage over the capacitor so

$V_o = I/Cs$

The transfer function is then $G(s) = \frac{V_o}{V_i}(s) = \frac{I/Cs}{I(R + 1/Cs)} = \frac{1}{RCs + 1}$

The units of RC are seconds and this is another electrical time constant T. The transfer function may be written as

$$G(s) = \frac{V_o}{V_i}(s) = \frac{1}{Ts + 1}$$

This is the standard first order equation and is the same as both the spring and damper and the thermal example.

8.6 L - C - R in SERIES

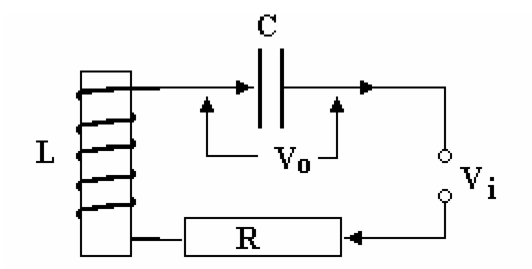


Figure 27

This is 3 sub-systems in series. In this case we will take the output as the voltage on the capacitor and the input as the voltage across the series circuit.

The input voltage is the sum of all three voltages and is found by adding them up.

$$V_i = I R + I sL + I/sC$$

The output voltage is $V_o = I/sC$ The transfer function is then

$$G(s) = \frac{V_o}{V_i}(s) = \frac{I/Cs}{I(R + sL + 1/Cs)} = \frac{1}{RCs + CLs^2 + 1}$$

$$G(s) = \frac{V_o}{V_i}(s) = \frac{1}{s^2CL + sRC + 1}$$

If we examine the units of CL we find it is seconds^2 and we have yet another time constant defined as $T^2 = CL$ and the equation may be rewritten as:

$$G(s) = \frac{V_o}{V_i}(s) = \frac{1}{T^2s^2 + 2\delta Ts + 1}$$

The damping ratio δ is defined as $\delta = \frac{R}{\sqrt{4\frac{L}{C}}}$ and $\sqrt{4\frac{L}{C}}$ is called the critical damping value.

Note this is the standard 2nd order equation identical to the mass-spring-damper system.

WORKED EXAMPLE No.7

A Capacitance of 200 µF is connected in series with a resistor of 20 kΩ as shown in figure 26. The transfer function is $\frac{\theta_o}{\theta_i}(s) = \frac{1}{(Ts + 1)}$

The voltage across the resistor is suddenly changed from 3V to 10V.

Calculate the time constant T and derive formulae for how the voltage across the capacitor varies with time. Sketch the graph.

SOLUTION

$T = RC = 20 \times 10^3 \times 200 \times 10^{-6} = 4$ seconds

The derivation is identical to that in example 3 simply changing θ to V we get the result.

$$V_o = V_i - \Delta V e^{-\frac{t}{T}}$$

Put in the values $T = 4$ $\quad \Delta V = 10 - 3 = 8$ $\quad V_i = 10$

$\mathbf{V_o = 10 - 7e^{-t/4}}$ Evaluating and plotting produces the result below. It is an exponential growth.

t secs	V_o
0	3
2	5.754
4	7.425
6	8.438
8	9.053
10	9.425
12	9.651
14	9.789
16	9.872
18	9.922
20	9.953

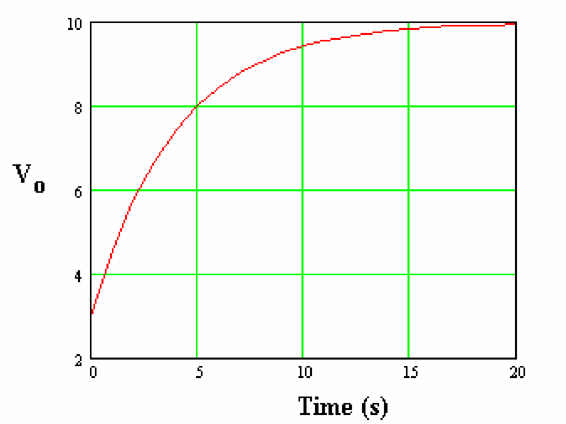


Figure 28

SELF ASSESSMENT EXERCISE No.6

1. Calculate the time constant for an RC circuit with a resistance of 220 Ω and capacitance of 470 nF.
(103 µs)

2. Calculate the second order time constant and the damping ratio for a R-L-C circuit with L = 5 µH, C= 60 µF and R=6.8 Ω.
(17.3 µs and 11.8)

9 ELECTRIC MOTORS

This is covered in greater depth in tutorial 2. Here are the basic models.

9.1 FIELD CONTROLLED MOTOR,

The main theory of electric motors is covered in another tutorial. It can be shown that for a d.c. servo motor with field control

$$T = k_f i_f$$

If the motor drives an inertial load and has damping the dynamic equation becomes

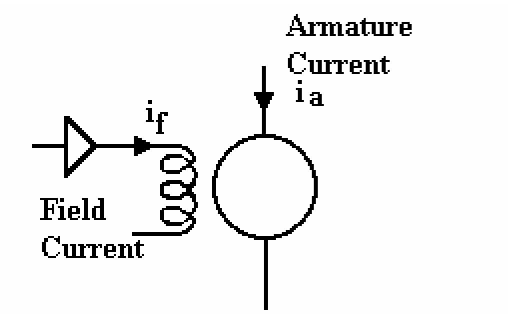


Figure 29

$$T = I\frac{d^2\theta}{dt^2} + k_d\frac{d\theta}{dt}$$

$$T = Is^2\theta + k_d s\theta = \theta(Is^2 + k_d s) = k_f i_f$$

$$G(s) = \frac{\theta}{i_f}(s) = \frac{k_f}{Is^2 + sk_d}$$

This models the relationship between the shaft angle and the control current.

9.2 ARMATURE CONTROLLED MOTOR

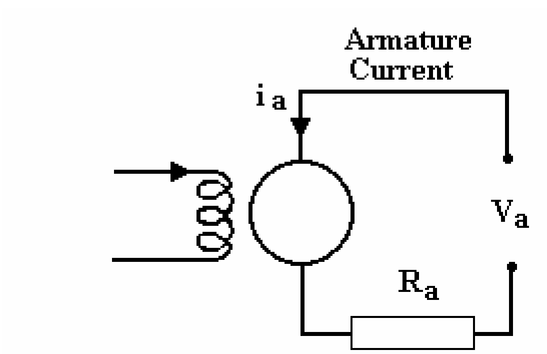


Figure 30

It can be shown that the torque is related to armature voltage and resistance by the formula $T = \left(V_a - k\frac{d\theta}{dt}\right)\frac{k}{R_a}$

The torque must overcome inertia and damping as before so

$$T = Is^2\theta + k_d s\theta$$

Equating we get

$$T = \theta(Is^2 + k_d s) = (V_a - ks\theta)\frac{k}{R_a} = \frac{k}{R_a}V_a - \frac{k^2 s\theta}{R_a}$$

$$\theta\left(Is^2 + k_d s + \frac{k^2 s}{R_a}\right) = \frac{k}{R_a}V_a$$

With rearrangement we find

$$G(s) = \frac{\theta}{V_a}(s) = \frac{(k/R_a)}{Is^2 + sk_d - k^2 s/R_a}$$

This models the relationship between the angle of the shaft and the control voltage.

Worked examples and self assessment exercise for this section may be found in tutorial 2.

10. CLOSED LOOP SYSTEMS TRANSFER FUNCTION WITH UNITARY FEED BACK.

Consider a simple system with an input θ_i and output θ_o related by the transfer function G(s). If the system is to be a controlled system in which we require the output to change and match the value of the input (set value), we must make the input the error θ_e instead of the set value. The error is obtained by comparing the output value with the input value with the signal summing device. This produces the result $\theta_e = \theta_i - \theta_o$ and because θ_o is subtracted, this idea is called NEGATIVE FEED BACK. The block diagram shows that the signal passes around a closed loop hence the name CLOSED LOOP SYSTEM.

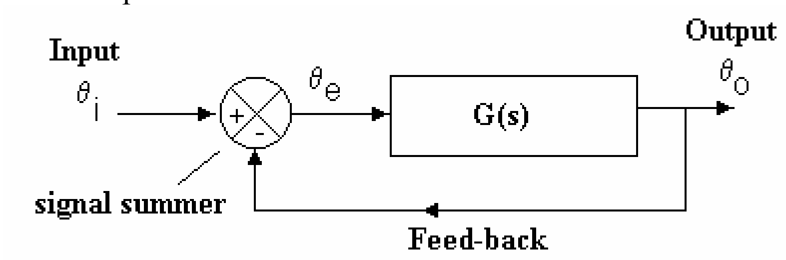


Figure 31

$$G(s) = \frac{\theta_o}{\theta_e} \quad \text{substitute } \theta_e = \theta_i - \theta_o$$

$$G(s) = \frac{\theta_o}{\theta_i - \theta_o} \quad \text{divide the bottom line by } \theta_o$$

$$G(s) = \frac{1}{\dfrac{\theta_i}{\theta_o} - 1} \quad \text{rearrange} \quad \frac{\theta_i}{\theta_o} - 1 = \frac{1}{G(s)}$$

$$\frac{\theta_i}{\theta_o} = \frac{1}{G(s)} + 1 = \frac{1 + G(s)}{G(s)} \quad \text{invert}$$

$$\frac{\theta_o}{\theta_i} = \frac{G(s)}{1 + G(s)} \quad \text{or} \quad \frac{1}{1/G(s) + 1}$$

The transfer function for the closed system is hence $\dfrac{\theta_o}{\theta_i} = \dfrac{1}{1/G(s) + 1}$

G(s) is the transfer function of the open loop system.

Let's revisit the hydraulic open loop transfer functions derived previously.

When the hydraulic valve and actuator is turned into a closed loop system, the two transfer functions become:

$G(c.l) = \dfrac{1}{Ts + 1}$ for the first order version and $G(c.l) = \dfrac{1}{T^2 s^2 + 2\delta Ts + 1}$ for the second order version.

These models are mathematically identical to the transfer function of the mass-spring- damper and the L-C-R circuits.

Note that for any system with an open loop transfer function G(s) the closed loop transfer function with unit feedback is $G_{cl} = \dfrac{1}{1/G(s) + 1}$

WORKED EXAMPLE No.8

An open loop system has a transfer function $G(s) = 2/(s^2 + 2s + 1)$. Derive the closed loop function when unit feedback is used.

SOLUTION

$$G_{cl} = \frac{1}{1/G(s) + 1} = \frac{1}{\dfrac{s^2 + 2s + 1}{2} + 1} = \frac{2}{s^2 + 2s + 1 + 2} = \frac{2}{s^2 + 2s + 3}$$

SELF ASSESSMENT EXERCISE No.7

1. An open loop system has a transfer function $G(s) = 5/(4s^2 + 2s + 2)$. Derive the closed loop function when unit feedback is used.
   $G_{cl} = 5/(4s^2 + 2s + 7)$

2. An open loop system has a transfer function $G(s) = 10/(s^3 + 5s)$. Derive the closed loop function when unit feedback is used.
   $G_{cl} = 10/(s^3 + 5s + 10)$