MECHANICS OF SOLIDS
DEGREE YEAR 1 LEVEL NVQ 5/6
MECHANICAL AND STRUCTURAL ENGINEERING C105

TUTORIAL 9 - TORSION

You should judge your progress by completing the self assessment exercises.

On completion of this tutorial you should be able to do the following.

- Derive the torsion equation
- Derive polar second moment of area.
- Solve problems involving torque, shear stress and angle of twist.
- Derive the formula for the power transmitted by a shaft
- Relate power transmission to torsion.
- Outline the method of solution for rectangular cross sections.
- Solve problems with shafts of rectangular cross section.

Contents

1. *Torsion of Shafts*

Torsion occurs when any shaft is subjected to a torque. This is true whether the shaft is rotating (such as drive shafts on engines, motors and turbines) or stationary (such as with a bolt or screw). The torque makes the shaft twist and one end rotates relative to the other inducing shear stress on any cross section. Failure might occur due to shear alone or because the shear is accompanied by stretching or bending.

1.1. Torsion Equation

The diagram shows a shaft fixed at one end and twisted at the other end due to the action of a torque T.

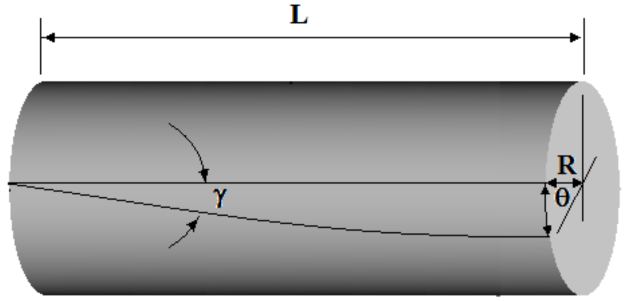

Figure 1

The radius of the shaft is R and the length is L.

Imagine a horizontal radial line drawn on the end face. When the right hand end is twisted relative to the other end, the line rotates through an angle θ. The length of the arc produced is $R\theta$.

Now consider a line drawn along the length of the shaft. When twisted, the line moves through an angle γ. If this angle is very small the length of the arc $L\gamma$ is almost exactly the same as arc $R\theta$.

It follows that $R\theta = L\gamma$
Hence by equating $L\gamma = R\theta$ we get

$$\gamma = \frac{R\theta}{L} \ldots\ldots (1A)$$

If you refer to basic stress and strain theory, you will appreciate that γ is the shear strain on the outer surface of the shaft. The relationship between shear strain and shear stress is

$$G = \frac{\tau}{\gamma} \ldots\ldots (1B)$$

τ is the shear stress and G the modulus of rigidity. G is one of the elastic constants of a material. The equation is only true so long as the material remains elastic. Substituting (1A) into (1B) we get

$$\frac{G\theta}{L} = \frac{\tau_{max}}{R} \ldots\ldots (1C)$$

Since the derivation could be applied to any radius, it follows that shear stress is directly proportional to radius 'r' and is a maximum on the surface. Equation (1C) could be written as

$$\frac{G\theta}{L} = \frac{\tau}{r} \ldots\ldots (1D)$$

Now let's consider how the applied torque 'T' is balanced by the internal stresses of the material.

Consider an elementary ring of material with a shear stress τ acting on it at radius r.

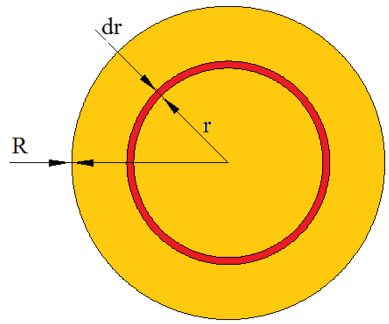

Figure 2

The area of the ring is $dA = 2\pi\, r\, dr$

The shear force acting on it tangential is $dF = \tau\, dA = \tau\, 2\pi r\, dr$

This force acts at radius r so the torque produced is $dT = \tau\, 2\pi r^2\, dr$

From equation (1D) then

$$\tau = \frac{G\theta r}{L} \text{ and } dT = \frac{G\theta}{L} \times 2\pi r^3 dr$$

The torque on the whole cross section resulting from the shear stress is

$$T = \frac{G\theta}{L} 2\pi \int_0^R r^3 dr$$

The Polar 2nd Moment of Area is defined as:

$$J = 2\pi \int_0^R r^3 dr$$

The Torque equation reduces to

$$T = \frac{G\theta J}{L}$$

This is usually written as

$$\frac{T}{J} = \frac{G\theta}{L} \dots\dots (1E)$$

Combining (1D) and (1E) we get the torsion equation

$$\frac{T}{J} = \frac{G\theta}{L} = \frac{\tau}{r} \dots\dots (1F)$$

1.2 Polar 2nd Moments of Area

This tutorial only covers circular sections. The formula for J is found by carrying out the integration or may be found in standard tables. For a shaft of diameter D the formula is

$$J = \frac{\pi D^4}{32}$$

This is not to be confused with the second moment of area about a diameter, used in bending of beams (I) but it should be noted that J = 2 I.

WORKED EXAMPLE No. 1

A shaft 50 mm diameter and 0.7 m long is subjected to a torque of 1 200 N m. Calculate the shear stress and the angle of twist. Take G = 90 GPa.

SOLUTION

Important values to use are D = 0.05 m, L = 0.7 m, T = 1 200 Nm, G = 90 $\times 10^9$ Pa

$$J = \frac{\pi D^4}{32} = \frac{\pi \times 0.05^4}{32} = 613.59 \times 10^{-9}\ \text{m}^4$$

$$\tau_{max} = \frac{TR}{J} = \frac{1\ 200 \times 0.025}{613.59 \times 10^{-9}} = 48.89 \times 10^6\ \text{Pa or } 48.89\ \text{MPa}$$

$$\theta = \frac{TL}{GJ} = \frac{1\ 200 \times 0.7}{90 \times 10^9 \times 613.59 \times 10^{-9}} = 0.0152\ \text{radian}$$

Alternatively

$$\theta = \frac{\tau_{max} L}{GR} = \frac{48.89 \times 10^6 \times 0.7}{90 \times 10^9 \times 0.025} = 0.0152\ \text{radian}$$

Converting to degrees

$$\theta = 0.0152 \times \frac{180}{\pi} = 0.871^0$$

1.3 Hollow Shafts

Since the shear stress is small near the middle, then if there is no other stress considerations other than torsion, a hollow shaft may be used to reduce the weight.

The formula for the polar second moment of area is

$$J = \frac{\pi(D^4 - d^4)}{32}$$

D is the outside diameter and d the inside diameter.

WORKED EXAMPLE No. 2

Repeat the previous problem but this time the shaft is hollow with an internal diameter of 30 mm.

SOLUTION

Important values to use are D = 0.05 m, L = 0.7 m, T = 1 200 Nm, G = 90 $\times 10^9$ Pa

$$J = \frac{\pi(D^4 - D^4)}{32} = \frac{\pi(0.05^4 - 0.03^4)}{32} = 534.07 \times 10^{-9}\ \text{m}^4$$

$$\tau_{max} = \frac{TR}{J} = \frac{1\ 200 \times 0.025}{534.07 \times 10^{-9}} = 56.17 \times 10^6 \text{ Pa or } 56.17 \text{ MPa}$$

$$\theta = \frac{TL}{GJ} = \frac{1\ 200 \times 0.7}{90 \times 10^9 \times 534.07 \times 10^{-9}} = 0.0175 \text{ radian}$$

Alternatively

$$\theta = \frac{\tau_{max} L}{GR} = \frac{56.17 \times 10^6 \times 0.7}{90 \times 10^9 \times 0.025} = 0.0175 \text{ radian}$$

Converting to degrees

$$\theta = 0.0175 \times \frac{180}{\pi} = 1^0$$

Note that the answers are nearly the same even though there is much less material in the shaft.

WORKED EXAMPLE No. 3

A shaft 40 mm diameter is made from steel and the maximum allowable shear stress for the material is 50 MPa. Calculate the maximum torque that can be safely transmitted. Take G = 90 GPa.

SOLUTION

Important values to use are:
D = 0.04 m, R = 0.02 m, $\tau = 50 \times 10^6$ Pa and G = 90×10^9 Pa

$$J = \frac{\pi D^4}{32} = \frac{\pi 0.04^4}{32} = 251.32 \times 10^{-9}\ \mathrm{m^4}$$

From the torsion equation

$$\frac{T}{J} = \frac{\tau_{max}}{R}$$

$$T = \frac{\tau_{max} J}{R} = \frac{50 \times 10^6 \times 251.32 \times 10^{-9}}{0.02} = 628.3\ \mathrm{N\ m}$$

SELF ASSESSMENT EXERCISE No. 1

1. A shaft is made from tube 25 mm outer diameter and 20 mm inner diameter. The shear stress must not exceed 150 MPa. Calculate the maximum torque that should be placed on it.
   (Ans. 271.69 Nm).

2. A shaft is made of solid round bar 30 mm diameter and 0.5 m long.
   The shear stress must not exceed 200 MPa. Calculate the following.

i. The maximum torque that should be transmitted.
ii. The angle of twist which will occur.

Take G = 90 GPa.
(Ans. 1 060 Nm and 4.2^o)

1.4 Mechanical Power Transmission by a Shaft

In this section you will derive the formula for the power transmitted by a shaft and combine it with torsion theory.

Mechanical power is defined as work done per second. Work done is defined as force times distance moved. Hence

$$P = \frac{Fx}{t} = Fv \ldots \ldots (2A)$$

P is the Power
F is the force
x is distance moved.
t is the time taken.
v is the velocity.

When a force rotates at radius R it travels one circumference in the time of one revolution. Hence the distance moved in one revolution is $x = 2\pi R$

If the speed is N rev/second then the time of one revolution is 1/N seconds. The mechanical power is hence

$$P = F \times \frac{2\pi R}{\frac{1}{N}} = F \times 2\pi RN$$

Since FR is the torque produced by the force this reduces to **$P = 2\pi NT$**... ... (2B)

Since $2\pi N$ is the angular velocity ω radians/s it further reduces to **$P = \omega T$**... ... (2C)

Note that equations (2C) is the angular equivalent of equation (2A) and all three equations should be remembered.

WORKED EXAMPLE No. 4

A shaft is made from tube. The ratio of the inside diameter to the outside diameter is 0.6. The material must not experience a shear stress greater than 500 kPa. The shaft must transmit 1.5 MW of mechanical power at 1 500 rev/min. Calculate the shaft diameters.

SOLUTION

The important quantities are P = 1.5×10^6 Watts, $\tau = 500 \times 10^3$ Pa, N = 1 500 rev/min and d = 0.6D.

$$N = \frac{1\,500}{60} = 25\text{rev/s} \quad P = 2\pi NT$$

$$T = \frac{P}{2\pi N} = \frac{1.5 \times 10^6}{2\pi \times 25} = 9\,549.3 \text{ N m}$$

$$J = \frac{\pi(D^4 - D^4)}{32} = \frac{\pi(D^4 - (0.6D)^4)}{32} = \frac{\pi(D^4 - 0.1296D^4)}{32} = \frac{\pi(0.8704D^4)}{32} = 0.08545D^4$$

$$\frac{T}{J} = \frac{\tau_{max}}{R} = \frac{2\tau}{D}$$

$$\frac{9\,549.3}{0.08545D^4} = \frac{2 \times 500 \times 10^3}{D}$$

$$\frac{9\,549.3}{0.08545 \times 2 \times 500 \times 10^3} = 0.11175 = D^3$$

$$D = \sqrt[3]{0.11175} = 0.4816 \text{ m or } 481.6 \text{ mm}$$

d = 0.6D = 289 mm

SELF ASSESSMENT EXERCISE No. 2

1. A shaft is made from tube 25 mm outer diameter and 20 mm inner diameter. The shear stress must not exceed 150 MPa. Calculate the maximum power that should be transmitted at 500 rev/min.
   (Ans. 14.226 kW)

2. A shaft must transmit 20 kW of power at 300 rev/min. The shear stress must not exceed 150 MPa. Calculate a suitable diameter.
   (Ans.27.8 mm)

3. A steel shaft 5 m long, has a diameter of 50 mm. It transmits power at 600 rev/min. If the maximum shear stress is limited to 60 MN/m^2. Determine the following.

   (i) The maximum power that can be transmitted. (92.5 kW)
   (ii) The corresponding angle of twist. (8.59^o)
   Assume the modulus of rigidity for steel is 80 GN/m^2.

4. A hollow steel shaft with a diameter ratio of 0.75 and a length of 4 m is required to transmit 1 MW at 120 rev/min. The maximum shear stress is not to exceed 70 MN/m^2 nor is the overall angle of twist to exceed 1.75^o. Determine the following.

(i) The necessary outside diameter of the shaft so that both of the above limitations are satisfied. (210 mm)

(ii) The actual maximum shear stress and the actual angle of twist. (64.1 MPa and 1.75^o)

Assume the modulus of rigidity for steel is 80 GPa