ENGINEERING COUNCIL EXAM MATHEMATICS C101
SOLUTIONS TO EXAM PAPER 2004

Q6

(a) An electric charge q_1 is moving with velocity v_1, produces a magnetic induction given by:

$$B = \alpha q_1 \frac{v_1 \times r}{r^2}$$

Where α is a constant and r is the position vector of the charge. The magnetic force exerted on a second charge q_2 moving with velocity v_2 is given by:

$$F = q_2 v_2 \times B$$

Find F if $v_1 = i + j - k$, $v_2 = 3i + 2j$, $r = 2i - j + 3k$ and $q_1 = q_2 = 1$

(b) With respect to a fixed origin O the lines ℓ_1 and ℓ_2 are given by

$$\ell_1 : r_1 = i + j + \lambda(-6i + 8k); \quad \ell_2 : r_2 = -2i + j + 4k + \mu(i + 2j - 4k);$$

Where λ and μ are scalar parameters. Find the point of intersection of the lines and the acute angle between their directions.

SOLUTION

$$\underline{A} \times \underline{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

(a) First find B

The cross product $v_1 \times r$ is the determinant of the matrix shown

$v_1 = i + j - k$,

$r = 2i - j + 3k$

$\mathbf{v_1} \times \mathbf{r} = i(a_2 b_3 - a_3 b_2) + j(a_3 b_1 - a_1 b_3) + k(a_1 b_2 - a_2 b_1)$

$\mathbf{v_1} \times \mathbf{r} = i\{(1)(3) - (-1)(-1)\} + j\{(-1)(-2) - (1)(3)\} + k\{(1)(-1) - (1)(2)\}$

$\mathbf{v_1} \times \mathbf{r} = 2i - 5j - 3k$

using Pythagoras find $r^2 = (2)^2 + (-1)^2 + (3)^2 = 14$

$$B = \alpha q_1 \frac{v_1 \times r}{r^2} = \frac{\alpha(1)}{14}\left(2i - 5j - 3k\right)$$

Now find $F = q_2 v_2 \times B$

The cross product $v_2 \times B$ is the determinant of the matrix.

$v_2 = 3i + 2j$,

$B = (\alpha/14)(2i - 5j - 3k)$

$\mathbf{V_2} \times \mathbf{B} = (\alpha/14)[\, i(a_2 b_3 - a_3 b_2) + j(a_3 b_1 - a_1 b_3) + k(a_1 b_2 - a_2 b_1)]$

$\mathbf{V_2} \times \mathbf{B} = (\alpha/14)\,[\{i(2)(-3) - (0)(-5)\} + j\{(0)(2) - (3)(-3)\} + k\{(3)(-5) - (2)(2)\}]$

$\mathbf{V_2} \times \mathbf{B} = (\alpha/14)\,[-6i + 9j - 19k]$

$$F = q_2 v_2 \times B = (1)\frac{\alpha}{14}\left(-6i + 9j - 19k\right)$$

(b) A line must have two sets of coordinates to define the ends but only one is given. If the line extends from the origin to the point given, then they must intersect at the origin. If anyone understands the question please let me know.

With no explanation of the meaning of terms like ℓ_1:r_1 I assume that ℓ is the length or magnitude and r is the coordinates of the tip.

$r_1 = i + j + \lambda(-6i + 8k) = i + j - 6\lambda i + 8\lambda k$
$r_1 = i(1 - 6\lambda) + j + k(8\lambda)$
$r_1 = i(a_1) + j(b_1) + k(c_1)$

$r_2 = -2i + j + 4k + \mu(i + 2j - 4k) = -2i + j + 4k + \mu i + 2\mu j - 4\mu k$
$r_2 = i(\mu - 2) + j(1 + 2\mu) + k(4 - 4\mu)$
$r_2 = i(a_2) + j(b_2) + k(c_2)$

First find the dot product.
$\underline{\mathbf{r_1}} \bullet \underline{\mathbf{r_2}} = (a_1 a_2 + b_1 b_2 + c_1 c_2)$
$\underline{\mathbf{r_1}} \bullet \underline{\mathbf{r_2}} = (1 - 6\lambda)(\mu - 2) + (1)(1 + 2\mu) + (8\lambda)(4 - 4\mu)$

The magnitudes are found from Pythagoras
$\ell_1 = \{(1 - 6\lambda)^2 + 1^2 + (8\lambda)^2\}^{1/2}$
$\ell_2 = \{(\mu - 2)^2 + (1 + 2\mu)^2 + (4 - 4\mu)^2\}^{1/2}$

The angle between them is found from $\underline{\mathbf{r_1}} \bullet \underline{\mathbf{r_2}} = \ell_1 \ell_2 \cos\theta$
$\cos\theta = (\underline{\mathbf{r_1}} \bullet \underline{\mathbf{r_2}}) / \ell_1 \ell_2$

$$\cos\theta = \frac{(1-6\lambda)(\mu-2)+(1+2\mu)+(8\lambda)(4-4\mu)}{\{(1-6\lambda)^2+1+(8\lambda)^2\}^{\frac{1}{2}}+\{(\mu-2)^2+(1+2\mu)^2+(4-4\mu)^2\}^{\frac{1}{2}}}$$

The angle obviously depends on the values of λ and μ.