ENGINEERING COUNCIL EXAM MATHEMATICS C101
SOLUTIONS TO EXAM PAPER 2004

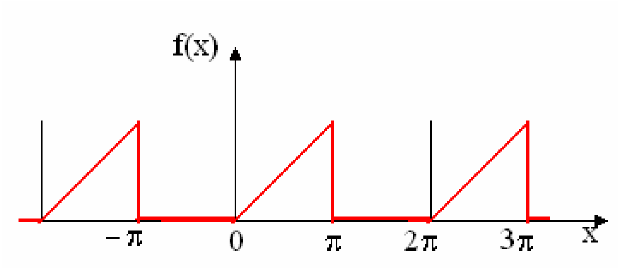
Q5a

Find the Fourier series for the function defined by

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 \leq x \leq \pi \end{cases} \qquad f(x + 2\pi) = f(x)$$

SOLUTION

I interpret this function as the wave form shown.



The Fourier series is given by: $f(x) = \frac{1}{2}a_o + \sum_{n=1}^{\infty}\{a_n \cos(nx) + b_n \sin(nx)\}$

Where $a_n = \frac{1}{\pi}\int_0^{\pi} f(x)\cos(nx)dx$ and $b_n = \frac{1}{\pi}\int_0^{\pi} f(x)\sin(nx)dx$

Note the simplification cos(nπ) = $(-1)^n$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} x\cos(0)dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} x\cos(nx)dx = \frac{1}{\pi}\left[\frac{x}{n}\sin(nx) + \frac{1}{n^2}\cos(nx)\right]_0^{\pi}$$

$$a_n = \frac{1}{\pi}\left[\left\{\frac{\pi}{n}\sin(n\pi) + \frac{1}{n^2}\cos(n\pi)\right\} - \left\{\frac{\pi}{n}\sin(0) + \frac{1}{n^2}\cos(0)\right\}\right]$$

$$a_n = \frac{1}{\pi}\left[\left\{0 + \frac{1}{n^2}(-1)^n\right\} - \left\{0 + \frac{1}{n^2}\right\}\right]$$

$$a_n = \frac{1}{\pi n^2}\left[(-1)^n - 1\right]$$

This has a value of 0 for even values of n and $-2/\pi n^2$ for odd values of n.

$$b_n = \frac{1}{\pi}\int_0^{\pi} x\sin(nx)dx = \frac{1}{\pi}\left[-\frac{x}{n}\cos(nx) + \frac{1}{n^2}\sin(nx)\right]_0^{\pi}$$

$$b_n = \frac{1}{\pi}\left[\left\{-\frac{\pi}{n}\cos(n\pi) + \frac{1}{n^2}\sin(n\pi)\right\} - \left\{-\frac{0}{n}\cos(0) + \frac{1}{n^2}\sin(0)\right\}\right]$$

$$b_n = \frac{1}{\pi}\left[\left\{-\frac{\pi}{n}(-1)^n + 0\right\} - \{0\}\right]$$

$$b_n = -\frac{1}{n}(-1)^n$$

This has values of 1/n for odd numbers and -1/n for even numbers

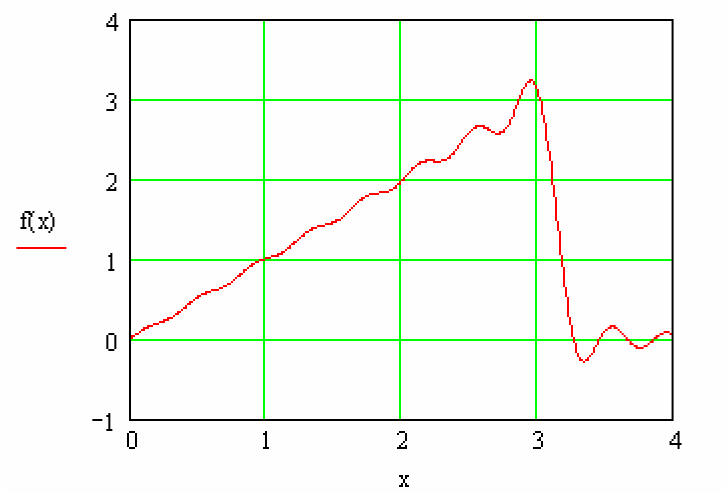
$$f(x) = \frac{1}{2}a_o + \sum_{n=1}^{\infty}\{a_n \cos(nx) + b_n \sin(nx)\}$$

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left\{\frac{1}{\pi n^2}\left[(-1)^n - 1\right]\cos(nx) - \frac{1}{n}(-1)^n \sin(nx)\right\}$$

$$f(x) = \frac{\pi}{4} + \left\{\frac{-2}{\pi}\cos(x) + \sin(x)\right\} + \left\{-\frac{1}{2}\sin(2x)\right\} + \left\{\frac{-2}{9\pi}\cos(3x) + \frac{1}{3}\sin(3x)\right\} + ...$$

$$f(x) = \frac{\pi}{4} + \left\{\sin(x) - \frac{1}{2}\sin(2x) + \frac{1}{3}\sin(3x) - .....\right\} - \frac{2}{\pi}\left\{\cos(x) + \frac{1}{9}\cos(3x) + ...\right\}$$

Evaluating with 15 terms yields the graph below so the result is correct for the range 0 to π



NOTE the same result is obtained more easily by working out the series for –π to + π which yields the following.

$a_o = 0$, $a_n = 0$ and $b_n = -(2/\pi)\cos(nx)$ and hence

$f(x) = 2\left\{\sin(x) - \frac{1}{2}\sin(2x) + \frac{1}{3}\sin(3x) - .....\right\}$ and the plot with 15 terms is as shown. This is the same as before with a negative image but over the range 0 to π the result is the same.